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ON THE EQUATION

$a(x^m - 1) / (x - 1) = b (y^n - 1) / (y - 1)$ (II)

By

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§ 1. Let $m > 1$, $n > 1$, $x > 1$, $y > 1$ and a, b with $(a,b) = 1$ be positive integers satisfying $a(y-1) \neq b(x-1)$. This equation of Goormaghtigh arose from the question whether an integer has all the digits identically equal in their expansions to two distinct bases. It follows from Baker's effective version [1] of Thue's theorem [6] that the equation

$$(1) \qquad a \frac{x^m - 1}{x - 1} = b \frac{y^n - 1}{y - 1}$$

implies that max (m, n) is bounded by an effectively computable number depending only on a, b, x and y. Further Balasubramanian and the author [3] applied the theory of linear forms in logarithms to generalise this result by showing that equation (1) implies that max (a, b, x, y, m, n) is bounded by an effectively computable number depending only on the greatest prime factor of abxy. In this paper, we apply the theory of linear forms in logarithms to obtain the following generalisations. We shall always write $z = \max(x, y)$.

Theorem 1.

Let $0 \leq \theta < 1$ *and* $F > 1$. *If positive integers* $m \geq n > 1$, $x > 1$, $y > 1$, a *and* b *with* $a < x$, $b < y$, $a(y-1) \neq b(x-1)$ *and*

$$(2) \qquad |x - y| \leq \max(F, z^{\theta})$$

satisfy (1), *then* m *is bounded by an effectively computable number depending only on* θ *and* F.

For an account of earlier results in the direction of equation (1), see [3]. We combine theorem 1 with an elementary argument to obtain the following result.

Theorem 2.

Let $F_1 > 1$. *There exists an effectively computable absolute constant* $C > 0$ *and an effectively computable number* $C_1 > 0$ *depending only on* F_1 *such that equation* (1) *in positive integers* $m > n > 1$, $m > 2$, $x > 1$, $y > 1$, a *and* b *with* $(a, b) = 1$, $a < x$, $b < y$, $a(y-1) \neq b(x-1)$ *and*

$$|x - y| < \max(F_1, (\log z)^C) \tag{3}$$

implies that

$$\max(m, n, x, y, a, b) < C_1.$$

Combining theorem 1 with lemma 1 and an estimate on p-adic linear forms in logarithms, we have

Theorem 3.

If $m > 1$, $n > 1$, $x > 1$ *and* $y > 1$ *with* $(x, y) = 1$ *satisfy*

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}, \tag{4}$$

then

$$\max(m, n, x, y) < C_2$$

where $C_2 > 0$ *is an effectively computable number depending only on the greatest prime factor of* $x(y - x)$.

If a and b are fixed, the restriction (3) in theorem 2 can be relaxed considerably.

Theorem 4.

Let a *and* b *be positive integers. Let* $F_2 > 1$. *There exists an effectively computable number* $C_3 > 0$ *depending only on* a,b *and* F_2 *such that equation* (1) *in positive integers* $m > n > 1$, $x > 1$, $y > 1$ *with* $a(y-1) \neq b(x-1)$ *and*

$$(5) \qquad |x-y| < F_2\, z/(\log z)^2 (\log \log z)^3$$

implies that

$$\max(m, n, x, y) < C_3.$$

§ 2. The proof of these results depend on the following application of a theorem of Baker [2] on linear forms in logarithms.

Lemma 1.

Let $F_3 > 1$. *Let* m, n, $x > 1$, $y > 1$, $a < F_3 x$, *and* $b < F_3 y$ *with* $a(y-1) \neq b(x-1)$ *satisfy* (1). *Put* $z = \max(x, y)$ *and* $z_1 = \min(x, y)$.

Then

$$\max(m,n) \leq C_4 \ (\log z)^3 (\log \log z)^2 / (\log z_1)$$

where $C_4 > 0$ *is an effectively computable number depending only on* F_3.

Proof of Lemma 1.

We may assume that $m > n$. Denote by $C_1 > 0$ and $C_2 > 0$ effectively computable numbers depending only on F_3.

By equation (1), we have

$$(6) \qquad 0 \neq \left| \frac{ax^m}{x-1} - \frac{by^n}{y-1} \right| < C_1$$

By an estimate of Baker [2], the left hand side of inequality (6) exceeds $x^m \exp(-C_2 (\log m) (\log z)^3 (\log \log z))$.

Now the lemma follows immediately by combining these estimates.

Proof of theorem 1.

Denote by $C_3, C_4, \ldots, C_9$ effectively computable positive numbers depending only on θ and F. Suppose that the assumptions of theorem 1 are satisfied. Then, by lemma 1 and (2), we have

$$(7) \qquad m \leq C_3 (\log x)^3.$$

In view of (7) and (2), we may assume that

$$(8) \qquad \min(x, y) > C_4$$

with C_4 sufficiently large. Further, by (1), we have

$$ax^{m-1} < by^{n-1}\left(1 + \frac{2}{y}\right),$$

$$(9) \qquad by^{n-1} < ax^{m-1}\left(1 + \frac{2}{x}\right)$$

Now it follows from (9), (2), (7) and (8) that either $m = n$ or $m = n+1$.

Let $m = n$. It follows from (9), (2) and (7) that

$$\left| \log \frac{a}{b} \right| \leq C_5 x^{-1+\theta} (\log x)^3.$$

Further, by (2), we have

$$\max\left(\left| \log \frac{x}{y} \right| , \left| \log\left(\frac{x-1}{y-1}\right) \right| \right) < C_6\, x^{-1+\theta}.$$

Consequently

$$\left| \log \frac{a(y-1)}{b(x-1)} \right| < (C_5 + C_6)\, x^{-1+\theta} (\log x)^3.$$

Now it follows from an estimate of Waldschmidt [7] or Ramachandra and the author [5] (in the latter reference, the arguments allow to prove the estimate without the restriction on the multiplicative independence of α_1 and α_2) on linear forms in logarithms that the left hand side of inequality (6), with $m = n$ and $F_3 = 1$, exceeds $x^{m - C_7}$.

Thus, by (6), $x^{m-C_7} < C_1$ which implies that $m \leq C_8$, since $x > 1$.

Let $m = n + 1$. Re-write equation (1) as

$$ax \frac{x^n - 1}{x - 1} = b \frac{y^n - 1}{y - 1} - a.$$

Now argue as in the case $m = n$ to conclude that $m < C_9$. This completes the proof of theorem 1.

Proof of theorem 2.

We shall choose, later, an effectively computable absolute constant C satisfying $0 < C < 1$.

Case I :

$F_1 \geq (\log z)^C$. Then $|x - y| \leq F_1$. Hence, by theorem 1, we see that m is bounded by an effectively computable number depending only on F_1. Now the assertion of theorem 2 follows from (1), since $(a, b) = 1$.

Case II :

$F_1 < (\log z)^C$. Then, by (3), observe that

$$|x - y| \leq (\log z)^C < \log z \leq \max\,(27, z^{\frac{1}{4}}).$$

Now apply theorem 1 to conclude that m is bounded by an effectively computable absolute constant C_{10}. Let $2 < m \leq C_{10}$ be given. Denote by $C_{11}, C_{12}, \ldots \ldots, C_{16}$ effectively computable positive numbers depending only on m. We may assume that $\min(x, y) > C_{11}$ with C_{11} sufficiently large, otherwise the assertion of theorem 2 follows from (1) and $(a, b) = 1$. Then equation (1) implies that $m = n$ or $m = n + 1$.

If $x = y$, then $a \neq b$ and equation (1) implies that

$$x^n(ax^{m-n} - b) = a - b$$

which, since $n > 1$, is not possible if C_{11} is sufficiently large. Thus we may assume that $x \neq y$.

Re-write equation (1) as

$$a\left(\frac{P_m(x)}{d}\right) = b\left(\frac{P_n(y)}{d}\right)$$

where

$$P_m(X) = \frac{X^m - 1}{X - 1}, \quad P_n(Y) = \frac{Y^n - 1}{Y - 1}$$

and d is the greatest common divisor of $P_m(x)$ and $P_n(y)$. Thus

$$P_m(x)\,d^{-1} \leq b \leq 2x.$$

Put

$$\rho_m = e^{2\pi i/m},\ \rho_n = e^{2\pi i/n},\ K = Q(\rho_m, \rho_n).$$

For a prime p dividing d, let $\mathrm{ord}_p(d) = \alpha_p$. Let $\wp$ be a prime ideal in the ring of integers of K dividing p. Then $\wp$ divides an ideal

(11) $$[x - y - \rho_m^r + \rho_n^s]$$

for some positive integers $r < m$ and $s < n$.

Put

$$\sigma = e^{2\pi i/6},\ T_1 = \{\sigma, \sigma^5\},\ T_2 = \{\sigma^2, \sigma^4\}.$$

Suppose that (11) is a zero ideal. Then, since $x \neq y$ and $1 \leqslant r < m$, $1 \leqslant s < n$, we see that $|x-y| = 1$. Then

$$\cos\left(\frac{2\pi r}{m}\right) = \pm 1 + \cos\left(\frac{2\pi s}{n}\right),\ \sin\left(\frac{2\pi r}{m}\right) = \sin\left(\frac{2\pi s}{n}\right).$$

These equations imply that either $\rho_m^r \in T_1$, $\rho_n^s \in T_2$ or $\rho_m^r \in T_2$, $\rho_n^s \in T_1$. Thus $m \neq n+1$, since m and n are divisible by 3. Therefore $m = n$. Then m is even, since m is divisible by 6. If $x - y = 1$, then equation (1) with $m = n$ implies that $x = y+1$ divides $a(x^{m-1} + \ldots + 1)$. Therefore x divides a, which is not possible, since $a < x$. Similarly if $y - x = 1$ and m even, equation (1) with $m = n$ has no solution. Thus we may assume that (11) is a non-zero ideal.

Put

$$\mathfrak{C}' = (2m^2)^{-1}.$$

Then, since $\wp$ divides a non-zero ideal (11), we obtain by taking norms,

$$p \leqslant C_{12} (\log x)^{\frac{1}{2}}.$$

Infact either $\alpha_p \leqslant C_{13}$ or $\wp^{\alpha_p - C_{13}}$ divides a non-zero ideal of the form (11). Therefore

$$p^{\alpha_p} \leqslant (\log x)^{C_{14}}.$$

Hence we conclude $q \leqslant x^{\frac{1}{2}}$ which, together with (10), implies that

$$x^{m-1} < P_m(x) \leqslant 2x^{3/2}.$$

Then, since $m > 2$, we conclude that $x \leqslant C_{15}$. Then $y \leqslant 2x \leqslant 2C_{15}$. Further, by (1) and $(a, b) = 1$, we see that $\max(a, b) \leqslant C_{16}$. This completes the proof of theorem 2.

Proof of Theorem 3.

We may assume that $m \neq n$, otherwise equation (4) has no solution, since $x \neq y$. Denote by $C_{17}, C_{18}, \ldots, C_{22}$ effectively computable positive numbers depending only on the greatest prime factor of $x(y-x)$. Put $y - x = k$. Then it follows from equation (4) and $(x, y) = 1$ that k divides

$$(x^{|m-n|} - 1) / (x-1).$$

Thus, for a prime p dividing k, we have

$$\mathrm{ord}_p(k) \leqslant \mathrm{ord}_p(x^{|m-n|} - 1).$$

Now we apply an estimate of van der Poorten [4] on p-adic linear forms in logarithms to obtain

$$\mathrm{ord}_p(k) \leqslant C_{17} (\log |m+n| + \log \log x)^2$$

Thus

$$\log |k| \leq C_{18} (\log |m+n| + \log \log x)^2 .$$

By lemma 1, we have

$$\max (m, n) \leq C_{19} (\log z)^4 .$$

Therefore

(12) $$\log |k| \leq C_{20} (\log \log z)^2 .$$

Now we apply theorem 1 to conclude that $\max (m, n) \leq C_{21}$. If $\max (x, y) > C_{22}$ with C_{22} sufficiently large, then it follows from (12) and (4) that $m = n$. This is not possible, since $x \neq y$. This completes the proof of theorem 3.

Proof of theorem 4.

Suppose that the assumptions of theorem 4 are satisfied. Denote by $C_{23}, C_{24}, \ldots$ effectively computable positive numbers depending only on a, b and F_2. By lemma 1 and (5) we have

(13) $$m \leq C_{23} ((\log x) (\log \log x))^2 .$$

In view of (13) and (5), we may assume that $\max (x, y) > C_{24}$ with C_{24} sufficiently large. Then we use inequality (9) to conclude that equation (1) implies that $m = n$. Therefore we may assume that $a \neq b$, otherwise $x \neq y$ and equation (1) with $m = n$ has no solution. Then, by again applying (9), we see from (5) and (13) that

$$C_{25} < \left| \log \frac{a}{b} \right| < C_{26} (\log \log x)^{-1}$$

which implies that $x \leq C_{27}$. Hence, by (5) and (13), we conclude that $\max (m, x, y) \leq C_{28}$. This completes the proof of theorem 4.

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