Hardy-Ramanujan Journal
vol. 10 (1987) 2-8

A REMARK ON $\zeta(1+it)$

K. RAMACHANDRA

§1. INTRODUCTION.

It is well-known that $\zeta(\sigma+it)\neq 0$ for $\sigma\geq 1$. Let us consider $log\ \zeta(1+it)$ for $t\geq 1000$. The object of this note is to prove the following theorem.

THEOREM. *Let $T\geq 1000$. Put $X=Exp(\frac{log\ log\ T}{log\ log\ log\ T})$. Consider any set of disjoint open intervals I each of length $\frac{1}{X}$ all contained in the interval $T\leq t\leq T+e^{X}$. Let ε be any positive constant not exceeding 1. Then with the exception of K intervals I (where K depends only on ε) we have*

$$|\ log\ \zeta(1+it)\ |\leq \varepsilon log\ log\ T.$$

REMARK 1. This note has its origin in the concluding result in the Appendix to [1].

REMARK 2. The proof depends on the inequalities $|\ a+\sum_{\alpha}p^{-i\alpha}\ |^2\geq 0$ and $|\ ai+\sum_{\alpha}p^{-i\alpha}\ |^2\geq 0$, where a is any real number and α runs over a finite set of distinct real numbers. By replacing $\sum_{\alpha}p^{-i\alpha}$ by $\chi(p)\sum_{\alpha}p^{-i\alpha}$ where χ is a residue class character we can work out the analogues of the theorem stated above for $log\ L(1+it,\chi)$, where $L(s,\chi)$ denotes the L-function defined with respect to χ. In this case we can (instead of $T\leq t\leq T+e^{X}$) consider for example the interval $0\leq t\leq 1$. However we reserve these analogues and other generalisations to number fields and so on for another paper.

REMARK 3. Letting α run over complex numbers $\alpha+i\beta$ with $\beta\leq\beta_0=(log\ T)^{-\mu}$ where μ is a constant $>\frac{2}{3}$, we can prove the following result. Let ε be a constant satisfying $0<\varepsilon<1$. Let $T\geq T_0(\mu,\varepsilon)$. Let X be as in the theorem and I as in the theorem. Let $J=[1-\beta_0,\infty)\times I$ be the Cartesian product of the σ interval $[1-\beta_0,\infty)$ and the t interval I. Then with the exception of K rectangles J (where K depends only on μ and ε) we have, for s in J

$$|\ log\ \zeta(s)\ |\leq \varepsilon log\ log\ T.$$

We postpone details of this result and refinements to a later paper. * We can take $\beta_0 = A(\log T)^{-\mu}(\log\log T)^{-2\mu}$ where $\mu = \frac{2}{3}$ and A is any positive constant.

§2. **PROOF OF THE THEOREM.** We begin by remarking that in the course of the proof we give in some ways better results for $|\log|\zeta(1+it)||$ and $|\arg\zeta(1+it)|$. We combine these two results to get an upper bound for $|\log\zeta(1+it)|$. Throughout this section k will be a fixed positive integer; α's will denote k distinct real numbers satisfying $T \le \alpha \le T + e^X$. We write $Y = Exp(10^{10}(\log T)^{\lambda}(\log\log T)^{10})$. Here $\lambda = 1$ if we assume the trivial zero-free region $\sigma \ge 1 - \frac{c}{\log t}$ and $\lambda = \frac{2}{3}$ if we assume the Vinogradov zero-free region $\sigma \ge 1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}}$, $(t \ge 1000)$; p will denote a prime and a a real constant to be chosen later. We prove the theorem by a series of lemmas.

LEMMA 1. *We have,*

$$0 \le \left| a + \sum_{\alpha} p^{-i\alpha} \right|^2 = a^2 + k + 2a\sum_{\alpha} Re(p^{-i\alpha}) + \sum\sum_{\alpha \ne \alpha'} p^{-i(\alpha-\alpha')} \tag{1}$$

and

$$0 \le \left| ai + \sum_{\alpha} p^{-i\alpha} \right|^2 = a^2 + k + 2a\sum_{\alpha} Im(p^{-i\alpha}) + \sum\sum_{\alpha \ne \alpha'} p^{-i(\alpha-\alpha')} \tag{2}$$

PROOF. Trivial.

LEMMA 2. *Let*

$$S_1 = \sum_{p} \frac{1}{p} e^{-\frac{p}{Y}}, S_2 = \sum_{p} p^{-i\alpha-1} e^{-\frac{p}{Y}}$$

and

$$S_3 = \sum_{p} p^{-i(\alpha-\alpha')-1} e^{-\frac{p}{Y}}. Then$$

$$0 \le (a^2 + k)S_1 + 2a\sum_{\alpha} Re(S_2) + \sum\sum_{\alpha \ne \alpha'} S_3. \tag{3}$$

PROOF. Follows from the first part of Lemma 1.

LEMMA 3. *We have*

$$S_1 = \log\log Y + 0(1). \tag{4}$$

PROOF. Using

$$e^{-\frac{p}{Y}} \begin{cases} \leq \frac{Y}{p} & \text{for } p \geq Y \\ = 1 + O(\frac{p}{Y}) & \text{for } p \leq Y \end{cases}$$

the lemma follows on using the well-known result $\sum_{p \leq Y} \frac{1}{p} = \log \log Y + O(1)$.

LEMMA 4. *We have,*

$$S_2 = \log \zeta(1 + i\alpha) + 0(1). \tag{5}$$

PROOF. Since, $\sigma \geq \frac{3}{4}$, the prime powers contribute $O(1)$ to the series for $\log \zeta(s)$, we have

$$S_2 = \frac{1}{2\pi i} \int_{Re(w)=2} \log \zeta(1 + i\alpha + w) Y^w \Gamma(w) dw + 0(1). \tag{6}$$

Put $L = 2(\log T)c^{-1}$ or $2(\log T)^{2/3}(\log \log T)^{1/3}c^{-1}$. We assume the zero free-region $\sigma \geq 1 - \frac{1}{L}, |t| \leq 2T$ (which comes from the two zero-free regions already referred to). By a simple application of the Borel-Carathéodory theorem (see p.282 of [2]; see also p. 174 of [3]) we have $\log \zeta(s) = 0((\log T)^2)$ in $\sigma \geq 1 - \frac{1}{2L}, T - (\log T)^2 \leq t \leq T + e^X + (\log T)^2$. (In fact we can get better estimates, but we do not need these). We deform the Contour $Re(w) = 2$ as follows. Let $\delta = \frac{1}{2L}, Q_1 = 2 - i\infty, Q_2 = 2 - i(T - (\log T)^2), Q_3 = -\delta - i(T - (\log T)^2), Q_4 = -\delta + i(T + e^X + (\log T)^2), Q_5 = 2 + i(T + e^X + (\log T)^2), Q_6 = 2 + i\infty$. We deform the Contour $Re(w) = 2$ to be the path formed by the straight line segments joining the points $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ in this order. The pole $w = 0$ gives the residue $\log \zeta(1 + i\alpha)$. Since $\Gamma(w) = 0(e^{-|Im w|})$ and $Y^\delta > (\log T)^{30}$, Lemma 4 follows by taking the upper bound for the integrals along the new path, obtained by replacing the integrand by its absolute value.

LEMMA 5. *Let $\tau = \alpha - \alpha'$. Then, we have,*

(a) $S_3 = \log \zeta(1 + i\tau) + 0(1)$, *if* $|\tau| \geq (\log T)^{10}$; (7)

(b) $|S_3| \leq \log \log Z + 0(1) + 0(\frac{1}{|\tau| \log Z})$ *otherwise, where Z is any number satisfying* $Exp(100(\log \log T)^2) \leq Z \leq Y$. (8)

PROOF. The part (a) follows by an argument similar to that of Lemma 4. We prove (b) as follows. By using the inequalities used in the proof of Lemma 3 we have

$$S_3 = \sum_{p \leq Y} p^{-i\tau - 1} + 0(1) = \sum_{p \leq Z} p^{-i\tau - 1} + \sum_{Z < p \leq Y} p^{-i\tau - 1} + 0(1). = S_4 + S_5 + 0(1).$$

say.

Clearly $|S_4| \leq \log\log Z + 0(1)$, and

$$S_5 = \int_Z^Y u^{-i\tau-1} d\pi(u) = I_1 + I_2$$

where

$$I_1 = \int_Z^Y u^{-i\tau-1}\frac{du}{\log u} \text{ and } I_2 = \int_Z^Y u^{-i\tau-1} dE(u)$$

where $E(u) = \pi(u) - \int_2^u \frac{du}{\log u}$. Using the mean value theorem for integrals we see that $I_1 = 0(\frac{1}{|\tau|\log Z})$. Now

$$\begin{aligned} I_2 &= u^{-i\tau-1}E(u)]_Z^Y + \int_Z^Y (i\tau+1)u^{-i\tau-2}E(u)du \\ &= 0(1) + 0((\log T)^{10}\int_Z^Y u^{-1}e^{-2\sqrt{\log u}}du) \end{aligned}$$

by using the fact that $E(u) = 0(ue^{-2\sqrt{\log u}})$. This completes the proof of (b).

LEMMA 6. *Let* $100(\log\log T)^2 \leq X \leq \log Y, Z = e^X$, *and* $X^{-1} \leq |\tau| \leq e^X$. *Then*

$$0 \leq (a^2+k)\log\log Y + 2a\sum_\alpha Re(\log\zeta(1+i\alpha)) + 0(\log X). \qquad (9)$$

PROOF. Follows from Lemmas 2 to 5 on observing that $\log\zeta(1+it) = 0(\log\log t)$ for $t \geq 100$.

We now come to the proof of the theorem.

LEMMA 7. *In the theorem consider the set of alternate intervals. Out of these fix* $k-1$ *intervals* I *for which minimum of* $Re(\log\zeta(1+i\alpha))$ *are successively as small as possible. If* J_1 *denotes* $Re(\log\zeta(1+i\alpha))$ *for* α *in any one of the remaining intervals then, we have with* $a > 0$,

$$0 \leq (a^2+k)\log\log Y + 2akJ_1 + 0(\log X) \qquad (10)$$

and so

$$J_1 \geq -\frac{1}{2k}(a+\frac{k}{a})\log\log Y + 0(\log X). \qquad (11)$$

COROLLARY. *With the exception of* $2(k-1)$ *intervals, we have,*

$$J_1 \geq (-k^{-1/2}-\varepsilon)\log\log Y \qquad (12)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Lemma 7 follows from Lemma 6 and the Corollary follows by taking $a = k^{1/2}$ and considering the other set of alternate intervals also.

LEMMA 8. *In the theorem consider one set of alternate intervals. Out of these fix* $k-1$ *intervals* I *for which the maximum of* $Re(\log \zeta(1+i\alpha))$ *are successively as large as possible. If* J_2 *denotes* $Re(\log \zeta(1+i\alpha))$ *for* α *in any one of the remaining intervals then, we have, with* $a < 0$,

$$-2akJ_2 \leq (a^2 + k)\log\log Y + 0(\log X) \qquad (13)$$

and so

$$J_2 \leq \frac{1}{2k}(-a - \frac{k}{a})\log\log Y + 0(\log X). \qquad (14)$$

COROLLARY. *With the exception of* $2(k-1)$ *intervals, we have,*

$$J_2 \leq (k^{-1/2} + \epsilon)\log\log Y \qquad (15)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Lemma 8 follows from Lemma 6, and the Corollary follows by taking $a = -k^{1/2}$ and considering the other set of alternate intervals also.

LEMMA 9. *With the exception of* $4(k-1)$ *intervals* I *we have,*

$$| Re(\log \zeta(1+it)) | \leq (k^{-1/2} + \epsilon)\log\log Y \qquad (16)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Follows from the Corollaries to Lemmas 7 and 8.

LEMMA 10. *With the exception of* $4(k-1)$ *intervals* I *we have,*

$$| Im(\log \zeta(1+it)) | \leq (k^{-1/2} + \epsilon)\log\log Y \qquad (17)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Just as we deduced Lemma 9 starting from the inequality (1) by a sequence of lemmas, we can deduce Lemma 10 from the inequality (2).

LEMMA 11. *With the exception of* $8(k-1)$ *intervals* I *we have,*

$$| \log \zeta(1+it) | \leq (\sqrt{\frac{2}{k}} + \epsilon)\log\log Y \qquad (18)$$

for $T \geq T_0(k, \epsilon)$.

PROOF. Follows from Lemmas 9 and 10.

Since k is an arbitrary positive integer constant and ϵ any positive constant, Lemma 11 completes the proof of the theorem stated in the introduction.

ACKNOWLEDGEMENT.

I had explained the results of this paper nearly an year ago to my colleague Professor R. Balasubramanian. I am thankful to him for checking the manuscript and encouragement.

REFERENCES

(1) K. Ramachandra, *Mean-value of the Riemann zeta-function and other Remarks-II*, International conference on analytic number theory (Vinogradov's 90th birthday celebrations), Steklov Institute, Moscow (14th to 19th September 1981); Trudy Mats Inst. Steklova 163 (1984), 200-204; Proceedings of the Steklov Institute of Mathematics (1985) Issue 4, p. 233-237.

(2) E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford, Clarendon Press 1951.

(3) E.C. Titchmarsh, *The theory of functions*, Oxford University Press, (1939).

ADDRESS OF THE AUTHOR

Professor K. Ramachandra
Tata Institute of Fundamental Research
Homi Bhabha Road
Colaba
Bombay 400 005
INDIA