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SRINIVASA RAMANUJAN
(THE INVENTOR OF THE CIRCLE METHOD)
(22.12.1887 TO 26.4.1920)

INAUGURAL ADDRESS

K. RAMACHANDRA

Dear President, Vice-Chancellor, Professor K.S. Padmanabhan, Professor E. Sampathkumar, other office bearers of the Ramanujan Mathematical Society, Ladies and Gentlemen,

I thank you very much for the honour you have done me by asking me to inaugurate the first conference of the Ramanujan Mathematical Society. The 99th birthday of Srinivasa Ramanujan falls on 22 December 1986 and we are approaching the Centenary. I am glad that you have started Ramanujan Mathematical Society to perpetuate the memory of this genius. This was a long felt need. Hardy's Centenary was on 7.2.1977. Any worthwhile account of Ramanujan should start with the correspondence between Ramanujan and Hardy and their glorious collaboration in Cambridge University, U.K.. Hardy was the closest friend of Ramanujan and also a teacher who taught him many things. As a close friend he cleared up what appeared to the world as superstitious religious feelings of Ramanujan and published the true image of Ramanujan as an admirer of the broad principles of all religions of the world. As a teacher he brought out the best in Ramanujan without doing any injustice to Ramanujan. He goes to the extent of saying (in a certain context) that he learnt from Ramanujan much more than what he taught him. Which Professor in the present day says this? It would be considered beneath one's dignity and a lowering of one's status in the eyes of others. He has tried to build up a rational picture of Ramanujan's mathematical genius and not as a inspired mystic from India. To do all this especially to an unknown poor clerk of Madras port trust and to collaborate with him on equal terms is the great quality of the great teacher Hardy. Of course Ramanujan was great. But he would have perhaps gone to oblivion without the help of Hardy. We remember him today because Hardy brought him to the lime light of the world. One of the great discoveries of Ramanujan and Hardy is a development of a problem stated by Ramanujan from Madras port trust in one of his letters to Hardy namely 'an asymptotic formula for the coefficient $q(n)$ of x^n in $(\sum_{m=-\infty}^{\infty}(-1)^m x^{m^2})^{-1}$. The glorious method developed by Ramanujan and Hardy came to be known as circle method. This was developed further by Hardy, Littlewood, Vinogradov, Davenport and others. Thanks to the circle method. Not only has it given birth to large sieve, Dispersion method and so on but it has solved some very difficult problems which a common man would like to pose. For example the ternary Goldbach Conjecture and the progress on binary Goldbach Conjecture. Another example is Waring's problem. All these problems appeared to be inaccessible for a number of years.

§1. GOLDBACH CONJECTURE (1742).

Goldbach conjectured that every even number ≥ 4 is the sum of two primes. He wrote this conjecture in a letter to L. Euler and Euler attached his name to this conjecture. This is called the binary Goldbach conjecture. There is the ternary Goldbach conjecture which says that every odd number ≥ 9 is the sum of three odd primes. Hardy-Littlewood developed the Ramanujan-Hardy Circle method to show on GRH (which I will explain)

(1) All sufficiently large odd numbers are the sum of three prime numbers. (Only $\theta \leq \frac{3}{4}$ was assumed here).

(2) The number of exceptions to binary Goldbach Conjecture for $2n \leq x$ is $O(x^{1/2+\epsilon})$.

In 1937 Vinogradov developed this method further to establish their result (1) without any hypothesis.

In 1975 H.L. Montgomery and R.C. Vaughan [11] proved the fundamental result that the number of binary Goldbach exceptions $2n \leq x$ is $0(x^{\delta})$ with a $\delta < 1$. In 1980 Chen-Jing-Run and Pan-Cheng-Dong [4] showed that δ could be taken $= 0.99$. In 1982 both of them independently showed (see [5]) that δ could be taken to be $0.96 = \frac{24}{25}$.

§2. WARING'S PROBLEM (1770).

In 1770 Waring made the following conjecture. I explain the conjecture by starting with squares. Let us express the positive integers as sums of squares.

$$
\begin{aligned}
1 &= 1^2+0^2+0^2+0^2\\
2 &= 1^2+1^2+0^2+0^2\\
3 &= 1^2+1^2+1^2+0^2\\
4 &= 2^2+0^2+0^2+0^2\\
5 &= 2^2+1^2+0^2+0^2\\
6 &= 2^2+1^2+1^2+0^2\\
7 &= 2^2+1^2+1^2+1^2
\end{aligned}
$$

Bachet (1621) conjectured that four squares are sufficient for all integers.

This was proved by Lagrange after Waring made his conjecture. We write this result as $g(2) = 4$. Wiefrich and Kempner independently proved that $g(3) = 9$. Waring made the general conjecture

$$g(k) = 2^k + [(\frac{3}{2})^k] - 2.$$

Building on the work of I.M. Vinogradov, S.S. Pillai proved this for all $k, 6 \leq k \leq 100$ and also for all k satisfying a certain property. Some of these results were proved independently also by L.E. Dickson. This property namely $k \geq 100000$ and $|(\frac{3}{2})^k - I_k| \geq (\frac{7}{8})^k$ (where I_k is the integer nearest to $(\frac{3}{2})^k$) was proved by K. Mahler [10] to be true for all but finitely many k. Chen-Jing-Run [6] proved 21 years ago that $g(5) = 37$. R. Balasubramanian, J.M. Deshouillers and F. Dress [2], [3] finally solved the 200 year old problem $g(4) = 19$ for fourth powers, only last year.

GRH means Generalised Riemann Hypothesis. Define $\lambda(n)$ $(n = 1, 2, ...)$ as follows. $\lambda(1) = 1$ and $\lambda(n) = (-1)^{\sum a}$ where $n = \pi p^a$ is the decomposition of n into prime powers. Let θ be the positive constant which is the greatest lower bound of numbers α such that for every pair of integers ℓ and k, with $(\ell, k) = 1$,

$$\frac{1}{x^\alpha} \sum_{n \leq x,\ n \equiv \ell (mod k)} \lambda(n) \to 0.$$

Then $\theta = \frac{1}{2}$ is GRH.

Asymtotic formula of Ramanujan mentioned on the previous page is

$$q(n) \sim \frac{1}{4n}(Cosh(\pi\sqrt{n})) - \frac{Sinh(\pi\sqrt{n})}{\pi\sqrt{n}})....$$

This is equation no. (1.14) of Hardy's book "RAMANUJAN".

§3. MORE ABOUT $q(n)$ AND $p(n)$.

The coefficients $q(n)$ are generated by

$$(\sum_{m=-\infty}^{\infty}(-1)^m x^{m^2})^{-1} = \prod_{h=1}^{\infty}\{(1-x^{2h})(1-x^{2h-1})^2\}^{-1} = \sum_{n=0}^{\infty} q(n)x^n.$$

It is clear that $q(n)$ are non-negative. They do not have a simple arithmetical interpretation. But historically the genesis of the circle method lies in the discovery of Ramanujan that $q(n)$ is the integer nearest to

$$\frac{1}{4n}(Cosh(\pi\sqrt{n}) - \frac{Sinh(\pi\sqrt{n})}{\pi\sqrt{n}}).$$

This is written in his first letter to Professor G.H. Hardy (see equation number (1.14) of "RAMANUJAN"). Ramanujan was sure that the approximation given by him was very much more intimately corrected with $q(n)$ than a mere asymptotic formula

$$\frac{1}{8n}Exp(\pi\sqrt{n})(1+0(n^{-\frac{1}{2}})).$$

Even this asymptotic formula is far from being obvious. Just like his assertion on the divisor function, here assertion regarding $q(n)$ is a formal principle (see page XXIV of collected papers of Ramanujan, equation (3) where he writes $d(1)+d(2)+\ldots+d(n) = n(\log n + 2\gamma - 1) + \frac{1}{2}d(n)$, which we can easily disprove by finding a suitable value of n. In fact G.H. Hardy showed that $\frac{1}{2}d(n)$ has to be replaced by $\Omega_{\pm}(n^{\frac{1}{4}})$ and much more), rather than an absolutely indisputable statement. He must have had some ingenious proof whether intutive or rigourous. But we have nowhere any record of how he obtained this result. After he collaborated with Hardy on the Asymptotic Theory of partitions (they considered instead of $q(n)$ the coefficients $p(n)$ defined by

$$1+\sum_{n=1}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}(\frac{1}{1-x^n})$$

since $p(n)$ has a nice arithmetical interpretation namely the number of solutions of $n = a_1 + a_2 + \ldots + a_k, (1 \leq a_1 \leq a_2 \leq \ldots \leq a_k)$ where $a_1, \ldots, a_k$ are positive integers and k is also an unrestricted integer) he writes

$$q(n) = \frac{d}{dn}(\frac{Cosh(\pi\sqrt{n})-1}{2\pi\sqrt{n}}) + 2\sqrt{3}Cos(\frac{2n\pi}{3}-\frac{\pi}{6})\frac{d}{dn}(\frac{Cosh(\frac{\pi\sqrt{n}}{3})-1}{2\pi\sqrt{n}}) + \ldots.$$

(see p.73 of note book vol. 1 held in Trinity College, Cambridge. Equation number (1.14) is also to be found on p.178 of note book number 2 (published by TIFR, edited by K. Chandrasekharan)). See also page 304 of collected papers for a remark concerning $q(n)$ that the method of Hardy and Ramanujan is applicable for $q(n)$ and by taking $[\alpha n^{\frac{1}{2}}]$ terms one can calculate $q(n)$ exactly since the error is $0(n^{-\frac{1}{4}})$. However the analogue of the Hardy-Ramanujan-Rademacher formula was worked out for $q(n)$ in 1981 by L.A. Goldberg [7] (a student of Bruce C. Berndt) and the result runs as follows: Let $((x)) = 0$ if x is an integer and $x - [x] - \frac{1}{2}$ otherwise. Let

$$s(h,k) = \sum_{j=1}^{k}(-1)^{j+[\frac{hj}{k}]}((\frac{j}{k})).$$

Put

$$A_k(n) = \sum_{\substack{0 \le h < 2k \\ (h,2k)=1}} Exp\{-\frac{ni\pi h}{k} - \frac{i\pi}{2}S(h,k)\}.$$

Then

$$q(n) = \frac{(-1)^n}{2\pi} \sum_{k=1,\ k odd}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}(\frac{Sinh(\frac{\pi\sqrt{n}}{k})}{\sqrt{n}}).$$

The first term of this series viz. $k = 1$ is precisely Ramanujan's expression for $q(n)$. So in a way the genesis for Rademacher's improvement of Hardy-Ramanujan formula (whether for partitions or $q(n)$) lies in equation (1.14) of the book "RAMANUJAN" by G.H. Hardy. (Here we may also refer to equation (7) of page XXVII of collected papers, for the first letter written to Professor Hardy by Ramanujan). The coefficients in the expansion of $(\sum_{n=-\infty}^{\infty} x^{n^2})^{-1}$ must be $(-1)^k q(k)$ as can be seen by replacing x by $-x$. Hence we may concentrate on the coefficients of the latter expansion. Although Cauchy's theorem was not known to Ramanujan it is plain that he knew that if $y > 0$ is any fixed number then

$$2(-1)^k q(k) = \int_{-1}^{1}(\sum_{n=-\infty}^{\infty} e^{i\pi n^2(x+iy)})^{-1} e^{-i\pi k(x+iy)}dx.$$

Ramanujan knew that, for all $x > 0$,

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty} e^{-\pi n^2/x}.$$

From this it follows that if $z = x + iy$, with $y > 0$, then

$$\sum_{n=-\infty}^{\infty} e^{-i\pi n^2 z} = \left(\frac{z}{i}\right)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-i\pi n^2/z}.$$

From this Ramanujan must have concluded (heuristically) that *if y is linked with k in a suitable manner* then

$$2(-1)^k q(k) \sim \int_{-1}^{1} \left(\frac{z}{i}\right)^{-\frac{1}{2}} e^{-i\pi k z} dx.$$

This seems to be the plausible method (which is heuristic) of Ramanujan. Ramanujan being an expert on definite integrals could handle the right hand side of the last asymptotic relation. (This method is very well explained in a rigourous fashion by C.L. Siegel in his lectures on analytic number theory given at New York University in the spring term of 1945. The notes of these lectures was taken down by B. Friedman. I do not know whether it is published or not; but a typed (xerox) copy is available at the TIFR library). This looks like a plausible explanation of how Ramanujan arrived at equation (1.14) of Hardy's book "RAMANUJAN". The method christened as "CIRCLE METHOD" by G.H. Hardy was applicable to a whole class of problems as later papers of G.H. Hardy and J.E. Lttlewood show. We do not need a transformation formula for the generating function. An asymptotic estimate and an error term as $y \to 0$ and $x = \frac{p}{q}$ (where p and q are positive integers and $(p, q) = 1$) will do. Thus we had solutions of Waring's problem and also ternary Goldbach conjecture. The latter investigation needed the hypothesis $\theta \leq \frac{3}{4}$ and in the former the number of kth power summands was around $k2^k$ or some such thing. But the Russian Mathematician I.M. Vinogradov continued the investigations of Hardy and Littlewood and showed that ternary Goldbach conjecture is true without any hypothesis. Not only that; he showed [17] that in Waring's problems the number of nth power summands could be reduced to not more than $n(2\log n + 4n\log\log n + 2\log\log\log n + 13)$ if we exclude a finite set of integers. Recently A.A. Karatsuba [8] (his student) reduced it to not more than $n(2\log n + 2\log\log n + 12)$.

But before proceeding further it is better to give a brief account of the historic collaboration of Ramanujan and Hardy on the asymptotic theory of partitions $p(n)$. This magnificient work which had a tremendous influence on the later work by almost every great mathematician is best described in the words of the great master G.H. Hardy. It (both the joint work and its account in "RAMANUJAN") is like a running commentary of the spectacular discovery. I closely follow Professor Hardy in what follows. For any fixed integer $r > 1$ the coefficients $p_r(n)$ defined by

$$1 + \sum_{n=1}^{\infty} p_r(n)x^n = \{(1-x)(1-x^2)...(1-x^r)\}^{-1}$$

has been studied by many mathematicians like Cayley, Sylvester, Glaisher (see p. 276-277 of Collected papers of Ramanujan). For example

$$p_3(n) = \frac{1}{12}(n+3)^2 - \frac{7}{72} + \frac{1}{8}(-1)^n + \frac{2}{9}cos\frac{2n\pi}{3}$$

as can be seen by partial fraction decomposition. Other expressions are given on page 277. We may quote $p_3(n)$ = integer nearest to $\frac{1}{12}(n+3)^2$. These results are, from a certain point of view, of a very trivial character. The interest which they possess is algebraical. The coefficient $p(n)$ given by the relation

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{h=1}^{\infty}(\frac{1}{1-x^h}) \equiv F(x), \text{ say}$$

as already remarked can be interpreted as the number of unrestricted partitions of n. It is easy to prove that if $x > 0$,

$$\frac{1}{1-x}\sum_{m=1}^{\infty}\frac{x^m}{m^2} < log\ F(x) < \frac{1}{1-x}\sum_{m=1}^{\infty}\frac{x}{m^2}.$$

From the latter part we get

$$log\ (p(n)e^{-un}) < (\frac{\pi^2}{6}\frac{e^{-u}}{1-e^{-u}}) < (\frac{\pi^2}{6u})$$

for all $u > 0$. From this it follows that if $K = \pi\sqrt{\frac{2}{3}}$, then $p(n) < e^{K\sqrt{n}}$, $(n = 1, 2, 3, ...)$. It is actually true that $p(n) \sim \frac{1}{4n\sqrt{3}}e^{K\sqrt{n}}$ as $n \to \infty$, but we should use more powerful methods. By Cauchy's theorem

$$p(n) = \frac{1}{2\pi i}\int_C \frac{F(x)}{x^{n+1}}dx \tag{1}$$

where C is the circle $|x| = R$ with radius $R = 1 - \frac{\beta}{n}$ where β is a suitable constant. $F(x)$ has a functional equation

$$F(x) = \frac{x^{1/24}}{\sqrt{2\pi}}(log\frac{1}{x})^{\frac{1}{2}}Exp\{\frac{\pi^2}{6log\ (\frac{1}{x})}\}F(x')$$

where

$$log\ \frac{1}{x} log\ \frac{1}{x'} = 4\pi^2, \text{ i.e. } x' = Exp\{-\frac{4\pi^2}{log\ (\frac{1}{x})}\}.$$

Roughly since the radius is close to 1 we can replace $F(x')$ by 1 since $|x'|$ is very small in the formula (1) after applying the functional equation (the error being of order $e^{Hn^{1/2}}$ with $H < K$). The integrand involves only elementary functions and can be calculated very precisely. The net result is the formula

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} (\frac{e^{K\lambda_n}}{\lambda_n}) + 0(e^{Hn^{1/2}}) \tag{2}$$

where $\lambda_n = \sqrt{(n - \frac{1}{24})}$ and $H < K$. The singularity $x = 1$ is in some sense the heaviest. There are other singularities of $F(x)$. In fact $x_{p,q}$ defined by $exp(\frac{2\pi ip}{q})$ (p, q positive integers with $(p, q) = 1$) are all singularities. As $x \to x_{p,q}$ along the radius, $F(n)$ behaves roughly like

$$Exp\{\frac{\pi^2}{6q^2(1-|x|)}\}.$$

Moving the contour C close to these singularities we expect

$$p(n) = P_1(n) + P_2(n) + ... + P_Q(n) + R(n) \tag{3}$$

where $P_1(n)$ is the dominant terms in (2) and

$$P_q(n) = L_q(n)\phi_q(n) \text{ where } \phi_q(n) = \frac{q^{1/2}}{2\pi\sqrt{2}} \frac{d}{dn} (\frac{e^{K\lambda_n/q}}{\lambda_n})$$

and

$$L_q(n) = \sum_p w_{p,q} e^{-2ni\pi p/q}, R(n) = 0(e^{H_Q n^{1/2}}) \tag{4}$$

where $w_{p,q}$ are certain $(24q)$th roots of unity not depending on n and p runs through positive integers less than and prime to q; and $H_Q < K/Q$. This expected formula is in fact true and is proved in their famous collaboration. I now quote from Hardy's book "RAMANUJAN".

"At this point we might have stopped had it not been for Major MacMahon's love of calculation. MacMahon was a practised and enthusiastic computer and made us a table of $p(n)$ up to $n = 200$. In particular he found that

$$p(200) \quad = \quad 39729 \quad 99029 \quad 388. \tag{5}$$

and we naturally took this value as a test for our asymptotic formula. We expected a good result with an error of perhaps one or two figures, but we had never dared to hope for such a result as we found. Actually 8 terms of our formula gave $p(200)$ with an error of 0.004. We were inevitably led to ask whether the formula could not be used to calculate $p(n)$ *exactly* for any large n".

Thus they proceeded to make Q in (3) a function of n. The result was

$$p(n) = \sum_{q \leq \alpha n^{1/2}} P_q(n) + 0(n^{-1/4}), \tag{6}$$

so that the sum on the RHS of (6) gives $p(n)$ with an error less than 1/2. However D.H. Lehmer showed that the infinite series obtained in (6) by letting $q = 1$ to ∞, is divergent. Hence in order to find out $p(n)$ exactly it was necessary to find out α and the O-constant in (6). In the meantime Lehmer used 21 terms of the series in (6) to get a plausible value for $p(721)$ namely

$$16106 \quad 17557 \quad 50279 \quad 47763 \quad 55347 \quad 62.0041, \tag{7}$$

but not still conclusive. The gap was filled by H. Rademacher who (trying to simplify their work) arrived at an identity as follows. Ramanujan and Hardy worked not exactly with

$$\phi(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{K\lambda_n}}{\lambda_n}\right)$$

but with the nearly equivalent function

$$\frac{1}{\pi\sqrt{2}}\frac{d}{dn}\left(\frac{Cosh(K\lambda_n) - 1}{\lambda_n}\right),$$

(and afterwards discarding the less important parts of the function). Rademach worked with

$$\psi(n) = \frac{1}{\pi\sqrt{2}}\frac{d}{dn}\left(\frac{Sinh(K\lambda_n)}{\lambda_n}\right).$$

The net result was

$$p(n) = \sum_{q=1}^{\infty} L_q(n)\psi_q(n) \tag{8}$$

where

$$\psi_q(n) = \frac{q^{1/2}}{\pi\sqrt{2}}\frac{d}{dn}\left\{\frac{Sinh(K\lambda_{n/q})}{\lambda_n}\right\}. \tag{9}$$

Since $|L_q(n)| \leq q$ it follows that in (8) the remainder after Q terms is in absolute value less than

$$CQ^{-1/2} + D\left(\frac{Q}{n}\right)^{1/2} Sinh\left(\frac{Kn^{1/2}}{Q}\right). \tag{10}$$

where C and D are certain positive constants.

Thus Rademacher showed that 21 terms of Hardy-Ramanujan formula gave $p(721)$ with an error less than 0.38. A cruder estimate would also suffice since Ramanujan showed that $p(721) \equiv 0(mod 11^2)$. Thus the exact value of $p(721)$ was obtained and checked that (as predicted by Ramanujan) $p(721) \equiv 0(mod 11^3)$. Next they wanted to test Ramanujan's conjecture that $p(14031)$was $\equiv 0(mod 11^4)$. Thus Lehmer obtained (by Hardy-Ramanujan-Rademacher formula) that

$$\begin{aligned} p(14031) = 92\ & 85303\ 04759\ 09931\ 69434\ 85156\ 67127 \\ & 75089\ 29160\ 56358\ 46500\ 54568\ 28164 \\ & 58081\ 50403\ 46756\ 75123\ 95895\ 59113 \\ & 47418\ 88383\ 22063\ 43272\ 91599\ 91345 \\ & 00745. \end{aligned} \tag{11}$$

and verified Ramanujan's conjecture.

Rademacher's expression for $w_{p,q}$ is $e^{i\pi s_{p,q}}$ where $s_{p,q} = \frac{1}{q}\sum_{\mu=1}^{q-1} \mu\left(\frac{\mu p}{q} - \left[\frac{\mu p}{q}\right] - \frac{1}{2}\right)$. D.H. Lehmer proved that $L_q(n)$ is multiplicative as a function of q. H. Gupta verified Lehmer's and Rademacher's calculations of $p(599)$ by direct computation.

§4. SUMS OF SQUARES, RAMANUJAN-HARDY-LITTLEWOOD-VINOGRADOV CIRCLE METHOD, COMMENTS FROM THE BOOK "SELECTED WORKS OF I.M. VINOGRADOV".

Ramanujan applied his methods to study the coefficients of $\left(\sum_{n=-\infty}^{\infty} x^{n^2}\right)^{2s}$ where s is a positive integer. He found that the coefficient $r_{2s}(n)$ of x^n satisfies

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n) \tag{12}$$

where $\delta_{2s}(n)$ is a certain divisor function of n and $e_{2s}(n)$ is of much smaller order compared with $\delta_{2s}(n)$, so that as $n \to \infty, r_{2s}(n) \sim \delta_{2s}(n)$. In fact Ramanujan proved things like $r_8(n) = \delta_8(n)$ through he was (here) anticipated by earlier mathematicians like Jacobi. We now describe $\delta_{2s}(n)$. Put

$$\sigma_\nu^*(n) = \begin{cases} \sigma_\nu(n) & \text{if } n \text{ is odd} \\ \sigma_\nu^e(n) - \sigma_\nu^0(n) & \text{if } n \text{ is even} \end{cases} \tag{13}$$

where $\sigma_\nu^e(n)$ and $\sigma_\nu^0(n)$ denote the sums of νth powers of the even and odd divisors of n. Ramanujan proved that

$$\delta_{2s}(n) = \frac{\pi^s}{\Gamma(2s)(1-2^{-s})\zeta(s)}\sigma_{s-1}^*(n). \tag{14}$$

We next give an example of explicit results discovered by Ramanujan. Let $\tau(n)$ be defined by

$$g(x) = x\{(1-x)(1-x^2)....\}^{24} = \sum_{n=1}^{\infty}\tau(n)x^n.$$

Then Ramanujan discovered the formula

$$(\sum_{n=-\infty}^{\infty} x^{n^2})^{24} - 1 = \frac{16}{691}\sum_{n=1}^{\infty}\sigma_{11}^*(n)x^n - \frac{33152}{691}g(-x) - \frac{65536}{691}g(x^2) \tag{15}$$

a formula which gives $r_{24}(n)$ in terms of divisor functions and $\tau(n)$. Ramanujan expanded the main term $\delta_{2s}(n)$ in terms of "Ramanujan sums" defined by

$$C_s(n) = \sum_{\lambda} cos\frac{2\pi\lambda n}{s} \tag{16}$$

where λ runs through numbers prime to s and not exceeding s. Whereas (15) gives his work explicitly in the form of "Einsenstein series plus a cusp form" for the representation of a number by sums of an even number of squares (those of an odd number of squares was developed by Hardy and others) the second expression in terms of (16) for such representations has a strong connection with the circle method (discovered with respect to $p(n)$, see also equation (1.14) of Hardy's book "RAMANUJAN") of Ramanujan and Hardy developed further by Hardy and Littlewood and later by Vinogradov. In view of these Ramanujan is the inventor of circle method (at least in a naive form) and it is fully justified to refer to circle method as "RAMANUJAN-HARDY-LITTLEWOOD-VINOGRADOV CIRCLE METHOD". We end this section by an extract from page 388 of the book "I.M. VINOGRADOV, SELECTED WORKS, (SPRINGER-VERLAG, (1985))".

In such additive problems as those of Waring, Goldbach and others the principal term is investigated with the help of a method similar to the circle method of Hardy, Littlewood and Ramanujan (at present this method is known as "The circle method of Hardy, Littlewood and Ramanujan in the form of Vinogradov's trigonometric sums").

I.M. Vinogradov

A brief outline of his life and works

by K.K. Mardzhanishvili

§5. SOME OTHER RESULTS OF RAMANUJAN. (1) CONGRUENCE PROPERTIES OF PARTITIONS.

Ramanujan discovered two beautiful identities

$$p(4) + p(9)x + p(14)x^2 + \ldots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^5}{\{(1-x)(1-x^2))(1-x^3)\ldots\}^6} \tag{17}$$

and

$$\begin{aligned} p(5) + p(12)x + p(19)x^2 + \ldots &= 7\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^3}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^4} \\ &\quad +49\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^7}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^8} \end{aligned} \tag{18}$$

This makes obvious the congruence to moduli 5 and 7. He went on to prove congruences to moduli $5, 7, 11, 5^2, 7^2, 11^2$, that to the modulus 5^2 being $p(25m+24) \equiv 0(mod 5^2)$. He put forward a general conjecture : if $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1(mod\delta)$ then $p(m\delta + \lambda) \equiv 0(mod\delta)$ for every positive integer m. It would be sufficient to prove congruences to moduli 5^a, 7^b and 11^c. The general case would be a consequence. The congruences to modulus 5^a were proved by G.N. Watson. In extending the MacMahon's table of partitions from $p(200)$ to $p(300)$ H. Gupta found that

$$p(243) = 13397 \quad 82593 \quad 44888.$$

Since $24.243 \equiv 1(mod 7^3)$ and $p(243) \not\equiv 0(mod 7^3)$ S. Chowla found that Ramanujan's conjecture for powers of 7 needs a modification. G.N. Watson proved this with a modification. He proved that if $b > 1$ and $24n \equiv$

$1(mod7^{2b-2})$ then $p(n) \equiv 0(mod7^b)$. The congruences to modulus 11^3 was proved by J. Lehner. Following this method and with a good deal of complicated work A.O.L. Atkin [1] proved the congruences $(mod 11^c)$. Thus if $24m \equiv 1(mod 5^a 7^b 11^c)$ then $p(m) \equiv 0(mod 5^a 7^\beta 11^c)$ where $\beta = [(b+2)/2]$. The references to previous works are given in Atkin's paper.

(2) RESULTS ON RAMANUJAN'S FUNCTION $\tau(n)$.

(a) Ramanujan conjectured that whenever $(m,n) = 1$ (m,n positive integers) we have $\tau(mn) = \tau(m)\tau(n)$. This was proved by L.J. Mordell. Ramanujan also conjectured that

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p (1 - \frac{\tau(p)}{p^s} + \frac{p^{11}}{p^{2s}})^{-1} \tag{19}$$

This was also proved by Mordell.

(b) Ramanujan observed that for $p = 2,3,5,7,23$ we have $\tau(p) \equiv 0(modp)$ and so $\tau(pn) \equiv 0(modp)$ for these primes for every positive integer n. Ramanujan also discovered $\tau(7m+k) \equiv 0(mod7)$ for $k = 3,5,6$ and also for $k = 0$. He also had the congruence $\tau(23m+k) \equiv 0(mod23)$ where k is any quadratic non-residue of 23. Ramanujan conjectured $\tau(n) \equiv 0(mod5)$ for almost all n. Similarly to the modulus 691. The later was deduced by G.N. Watson from Ramanujan's result $\tau(n) \equiv \sigma_{11}(n)(mod691)$. A lot of work has been done by J.-P. Serre [15], [16] who proved various results on congruence properties of $\tau(n)$. In particular "$\tau(n)$ is divisible by m for almost all n" is true for almost all m. There is a conjecture due to D.H. Lehmer [9] that $\tau(n)$ is never zero (he verifies this for $n < 3316779$ by some theory and computation). Serre has some results in this direction. V. Kumar Murty [12] and M. Ram Murty and V. Kumar Murty [14] have got some extensions of these results. For the earlier works in these direction see the paper of V. Kumar Murty and that of Serre [16]. Actually it is conjectured by Atkin and Serre that $|\tau(p)| \gg_\delta p^{-\delta+9/2}$.

(c) Ramanujan conjectured that $|\tau(n)| \leq n^{11/2}d(n)$. He knew that this would follow from $|\tau(p)| \leq 2p^{11/2}$ for primes p. Ramanujan proved $\tau(n) = 0(n^7)$ by elementary methods. Hardy proved that $\sum_{n \leq x} \tau^2(n)$ lies between Ax^{12} and Bx^{12} where A and B are certain constants. Rankin has proved that this quantity is $\alpha x^{12} + 0(x^{12-2/5})$ using this it is obvious that $\tau(n) = 0(n^{6-1/5})$. The conjecture of Ramanujan was finally solved by P. Deligne. By way of lower bounds (infinitely often) R. Balasubramanian and Ram

Murty have proved much beyond $|\tau(n)| \gg n^{11/2}$ ($\gg$ means greater than a constant times) for infinitely many n. The latest result in this direction is due to Ram Murty [13] who proves $\tau(n) = \Omega(n^{11/2} Exp(\frac{(\log n)^{2/3}}{(\log \log n)^A}))$ for some constant $A > 0$. (Ω means $\gg$ for infinity of n). The earlier results are due to R.A. Rankin and H. Joris and for references to these see Ram Murty's paper cited above. The reference to the paper of P. Deligne is also to be found in this paper.

§6. CONCLUDING REMARKS.

The present paper is the development of my address at the first meeting of the Ramanujan Mathematical Society which I gave as a chief guest. Listed at the end are seventeen references; most of the references which are implicit in the text and not listed among the seventeen are to the three fundamental works of Hardy and Wright, Hardy and Ramanujan namely

(a) G.H. Hardy and E.M. Wright, *Introduction to the theory of numbers*, Clarendon Press, Oxford (1954).

(b) G.H. Hardy, *RAMANUJAN, twelve lectures on subjects suggested by his life and work* (Chapters 8,9 and 10), Chelsea Publishing Company, N.Y. (printed in U.S.A.) originally by Cambridge Univ. Press (1940).

(c) S. Ramanujan, Collected papers (edited by G.H. Hardy, P.V. Seshu Iyer and B.M. Wilson) (papers 18,21,25 and 36), Chelsea Publishing Company, N.Y. (printed in U.S.A.) (1962).

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