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SOME LOCAL CONVEXITY THEOREMS FOR THE ZETA-FUNCTION-LIKE ANALYTIC FUNCTIONS

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§ 1. INTRODUCTION. Suppose $f(s)$ is an analytic function of $s = \sigma + it$ defined in the rectangle $R = \{a \leq \sigma \leq b, t_0 - H \leq t \leq t_0 + H\}$ where a and b are constants with $a < b$. We assume that $| f(s) | \leq M$ (with $M \geq 2$ sometimes we assume implicitly that M exceeds a large positive constant) throughout R. We are interested in the question of lower bounds for

$$I(\sigma) = \int_{|v| \leq H} | f(\sigma + it_0 + iv) |^k \, dv \tag{1}$$

where $k > 0$ is a real constant. The method we employ is very much related to the methods of our paper [1] in combination with the results in the appendix to [5]. Some of our results regarding (1) are improvements of some lemmas in Ivić's book [3] (see page 172 of this book). It should be mentioned that our results (though of some interest in themselves) do not give any new important applications except (5),(6) and (7). In § 2 we prove a general result on local-convexity from which the following theorem is a consequence. (All our constants will be effective and we do not state this fact every time).

THEOREM 1. *Suppose there exists a constant d such that $a < d < b$ and that in $d \leq \sigma \leq b, | f(s) |$ is bounded below and above by β and β^{-1} where $\beta \leq 1$ is a positive constant, (it is enough to assume this condition for*

$I(\sigma)$ with H replaced by an arbitrary quantity lying between $H/2$ and H, in place of $| f(s) |$). Let $\varepsilon > 0$ be any constant. Then for $H = D$ where D is a certain positive constant depending only on ε and other constants mentioned before, we have, for $a \leq \sigma \leq d$,

$$I(\sigma) \gg M^{-\varepsilon}. \tag{2}$$

We next prove a δ-convexity theorem which is simply this.

THEOREM 2. *Let A_0, σ_1 and δ be any three constants satisfying $A_0 > 0, a < \sigma_1 < b$ and $\delta > 0$ and let $H = \delta$. Then we have, with $\sigma = \sigma_1 + (\log M)^{-1}$, the inequality*

$$| f(\sigma_1 + it_0) |^k \ll M^{-A_0} + I(\sigma) \log M. \tag{3}$$

Also, we have,

$$\int_{|u| \leq \delta/2} | f(\sigma_1 + it_0 + iu) |^k \, du \ll M^{-A_0} + I(\sigma) \log\log M. \tag{3'}$$

we have also similar results with $\sigma^* = \sigma_1 - (\log M)^{-1}$ in place of σ.

The third of our theorems is

THEOREM 3. *Let σ_1 be a constant satisfying $a < \sigma_1 < b$ and H be a large constant depending on other constants. Then we have (with any large positive constant A_0 and any constant $k > 0$)*

$$| f(\sigma_1 + it_0) |^k \ll M^{-A_0} + I(\sigma_1) \log M \qquad (*)$$

and because of Theorem 1, we have, (assuming on $f(s)$ the conditions of Theorem 1) the inequality

$$| f(\sigma_1 + it_0) |^k \ll I(\sigma_1) \log M. \qquad (*)$$

For a remark on equations marked with an asterisk see the post-script at the end.

Theorems 1 to 3 have immediate applications to $\zeta(s)$ and L-functions we can take for example $f(s) = \zeta(s)$ and we obtain the following theorem as a corollary. We state only two applications of each of these theorems.

THEOREM 4. *Let k and ε be any two positive constants, A_0 any large* *positive constant, σ a constant satisfying $1/2 \leq \sigma \leq 1$. Then, we have, for $t_0 \geq 10$ and D a certain large positive constant,*

$$\int_{|v|\leq D} |\zeta(\sigma + it_0 + iv)|^k \, dv \gg t_0^{-\epsilon}. \tag{5}$$

Combining this with the functional equation we obtain,

$$\int_{|v|\leq D} |\zeta(\sigma - 1/2 + it_0 + iv)|^k \, dv \gg t_0^{k(1-\sigma)-\epsilon}. \tag{6}$$

Also

$$\int_{|v|\leq D} |\zeta(1 + it_0 + iv)|^k \, dv \gg (\log t_0)^{-\epsilon} \tag{7}$$

and here we can (if we assume Riemann Hypothesis) replace RHS by $(\log\log t_0)^{-\epsilon}$

Next, we have

$$|\zeta(\sigma + it_0)|^k \ll t_0^{-A_0} + (\log t_0)\int_{|v|\leq\delta} |\zeta(\sigma + (\log t_0)^{-1} + it_0 + iv)|^k \, dv \tag{8}$$

and the same inequality holds if we replace on the RHS $(\log t_0)^{-1}$ by $-(\log t_0)^{-1}$. Finally

$$|\zeta(\sigma + it_0)|^k \ll (\log t_0)\int_{|v|\leq D} |\zeta(\sigma + it_0 + iv)|^k \, dv. \qquad (9)(*)$$

Next we can apply our method to other problems such as proving that the large values of $|\zeta(s)|$ are "rare" in a certain sense providing an alternative approach to some results of A. Ivić [4]. More specifically we prove

THEOREM 5. *Let α_0 be a small positive constant and let t_0 exceed a sufficiently large positive constant and $\alpha_0 \geq \delta \geq (\log t_0^{-3/4}$. Let $s_1 = 1 + it_0, s_0 = \sigma_0 + it_1$ where $0 \leq 1 - \sigma_0 \leq \delta$ and $|t_1 - t_0| \leq \delta$. Suppose that for $|s_1 - s| \leq 20\delta$ we have $\zeta(s) \neq 0$. Then*

$$|\zeta(\sigma_0 + it_1)| \ll Exp(Z^{1-\sigma_0}(\log\log Z + 1)) \tag{10}$$

where

$$Z = Exp(\frac{10}{\delta}\log\log t_0 + \frac{13}{\delta}). \tag{11}$$

REMARK. This theorem can be generalised very much by our method for example to ζ-functions and L-functions of number fields.

Before concluding this section we make a few remarks. In a series of papers the second of us started with the kernel function $Exp((Sinz)^2)$ and made extensive use of this kernel proving a number of convexity results over short intervals. Some of them are (with $H \gg loglog\ M$ with a suitable implied constant) that we can replace the RHS in (2) by H. Also he proved (H subject to the same condition) things like [6]

$$\frac{1}{H}\int_{|v|\le H} |\zeta(\frac{1}{2}+it_0+iv)|^{2k}\,dv \gg (log H)^{k^2}$$

where $k > 0$ is any rational constant. (With $k = \frac{1}{2}$ this was first proved by Ramachandra [7]. Next Heath-Brown proved this with $k > 0$ any rational constant and $H = t_0$ [2]. Next Ramachandra extended Heath-Brown's result to $t_0 \ge H \gg loglog\ t_0$. For positive irrational constants k Ramachandra proved [8], subject to the same conditions on H, the result

$$\frac{1}{H}\int_{|v|\le H} |\zeta(\frac{1}{2}+it_0+iv)|^{2k}\,dv \gg (\frac{log\ H}{loglog\ H})^{k^2}.$$

Later he proved a stronger result [9] where RHS here was replaced by a bigger function depending on the simple continued fraction expansion of k).

§ 2. A GENERAL RESULT ON LOCAL CONVEXITY.

First of all a remark about the real constant $k > 0$. We will (for technical simplicity) assume that k is an integer. To prove the general case we have to proceed as we do in this section and to use the Riemann mapping theorem (with zero cancelling factors for a certain rectangle i.e. $(\theta(w))^k$ suitably) as given in the appendix to [5] (see Lemmas 2,3 and 4 of the appendix). If k is an integer we can consider $f(s)$ in place of $(f(s))^k$ without loss of generality.

Let $a \le \sigma_0 < \sigma_1 < \sigma_2 \le b, 0 < D \le H, s_1 = \sigma_1 + it_0$ and let P denote the contour $P_1P_2P_3P_4P_1$ where $P_1 = -(\sigma_1-\sigma_0) - iD, P_2 = \sigma_2 - \sigma_1 - iD, P_3 = \sigma_2 - \sigma_1 + iD$ and $P_4 = -(\sigma_1 - \sigma_0) + iD$. Let $w = u + iv$ be a complex variable. We have

$$2\pi i f(s_1) = \int_P f(s_1+w)X^w\frac{dw}{w} \text{ where } X > 0. \qquad (12)$$

We put

$$X = Exp(Y + u_1 + u_2 + \ldots + u_r) \qquad (13)$$

where $Y \geq 0$ and $(u_1, u_2, ..., u_r)$ is any point belonging to the r-dimensional cube $[0,C]\times[0,C]\times...\times[0,C]$, C being a positive constant to be chosen later. The contour P consists of the two vertical lines $-V_0$ and V_2 respectively given by P_4P_1 and P_2P_3 and two horizontal lines $Q_1, -Q_2$ respectively given by P_1P_2 and P_3P_4. Averaging the equation (12) over the cube we get

$$2\pi i f(s_1) = C^{-r}\int_0^C ... \int_0^C \int_P f(s_1+w)\frac{X^w}{w}dw\, du_1...du_r. \qquad (14)$$

Over V_0 and V_2 we do not do the averaging. But over Q_1 and Q_2 we do average and replace the integrand by its absolute value. We obtain

$$\begin{aligned} &| 2\pi f(s_1) | \leq Exp(-Y(\sigma_1-\sigma_0))\textstyle\int_{V_0} | f(s_1+w)\frac{dw}{w} | \\ &+Exp((Y+Cr)(\sigma_2-\sigma_1))\textstyle\int_{V_2} | f(s_1+w)\frac{dw}{w} | \\ &+\tfrac{2^{r+1}}{C^rD^r}Exp((Y+Cr)(\sigma_2-\sigma_1))(max_{w\in Q_1\cup Q_2} | f(s_1+w) |)(\sigma_2-\sigma_0) \end{aligned}$$

and thus

$$\begin{aligned} &| 2\pi f(s_1) | \leq (Exp(-Y(\sigma_1-\sigma_0)))I_0 \\ &+(Exp(Cr(\sigma_2-\sigma_1)))(Exp(Y(\sigma_2-\sigma_1)))(I_2+M^{-A}) \\ &+2M(\sigma_2-\sigma_0)(Exp(Y(\sigma_2-\sigma_1)))(\tfrac{2\, Exp(C(\sigma_2-\sigma_1))}{CD})^r \qquad (15) \end{aligned}$$

where A is any positive constant and

$$I_0 = \int_{V_0} | f(s_1+w)\frac{dw}{w} | \text{ and } I_2 = \int_{V_2} | f(s_1+w)\frac{dw}{w} | . \qquad (16)$$

Choosing Y to equalise the first two terms on the RHS of (15), i.e. choose Y by

$$Exp(Y(\sigma_2-\sigma_0)) = (\frac{I_0}{I_2+M^{-A}})Exp(-Cr(\sigma_2-\sigma_1))$$

i.e.

$$Exp(Y(\sigma_2-\sigma_1)) = (\frac{I_0}{I_2+M^{-A}})^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}}Exp(-Cr(\sigma_2-\sigma_1)^2(\sigma_2-\sigma_0)^{-1})$$

and noting that

$$(\sigma_2-\sigma_1)-(\sigma_2-\sigma_1)^2(\sigma_2-\sigma_0)^{-1} = (\sigma_2-\sigma_1)(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)^{-1},$$

we obtain

$$\begin{aligned} &| 2\pi f(s_1) | \leq 2\{I_0^{\sigma_2-\sigma_1}(I_2+M^{-A})^{\sigma_1-\sigma_0}\}^{(\sigma_2-\sigma_0)^{-1}}\{Exp(\tfrac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0})\}^r \\ &+2M(\sigma_2-\sigma_0)(\tfrac{I_0}{I_2+M^{-A}})^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}}\{\tfrac{2}{CD}Exp(\tfrac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0})\}^r. \quad (17) \end{aligned}$$

Collecting we state the following convexity theorem.

THEOREM 6. *Suppose $f(s)$ is an analytic function of $s = \sigma + it$ defined in the rectangle $R : \{a \le \sigma \le b, t_0 - H \le t \le t_0 + H\}$ where a and b are constants with $a < b$. Let the maximum of $| f(s) |$ taken over R be $\le M$. Let $a \le \sigma_0 < \sigma_1 < \sigma_2 \le b$ and let A be any large positive constant. Let r be any positive integer, $0 < D \le H$ and $s_1 = \sigma_1 + it_0$. Then for any positive constant C, we have,*

$$
\begin{aligned}
&| 2\pi f(s_1) | \le 2\{I_0^{\sigma_2-\sigma_1}(I_2 + M^{-A})^{\sigma_1-\sigma_0}\}^{(\sigma_2-\sigma_0)^{-1}}\{Exp(\tfrac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0})\}^r \\
&+2M^{A+2}(\sigma_2-\sigma_0)(2(1+(log(\tfrac{D}{\sigma_1-\sigma_0}))^*)^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}} \times \\
&\times\{\tfrac{2}{CD}Exp(\tfrac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0})\}^r \qquad (18)
\end{aligned}
$$

where

$$I_0 = \int_{|v|\le D} | f(\sigma_0 + it_0 + iv)\frac{dv}{\sigma_0 - \sigma_1 + iv} | \qquad (19)$$

and

$$I_2 = \int_{|v|\le D} | f(\sigma_2 + it_0 + iv)\frac{dv}{\sigma_2 - \sigma_1 + iv} |, \qquad (20)$$

and we have written $(x)^* = max(0, x)$ for any real number x.

PROOF. We have used $I_2 + M^{-A} \ge M^{-A}$ and $(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_0)^{-1} \le 1$ and if $D \ge \sigma_1 - \sigma_0$,

$$I_0 \le M\int_{|v|\le D} | \frac{dv}{\sigma_0 - \sigma_1 + iv} | \le 2M\{\int_0^{\sigma_1-\sigma_0}\frac{dv}{\sigma_1-\sigma_0} + \int_{\sigma_1-\sigma_0}^{D}\frac{dv}{v}\}.$$

This completes the proof of Theorem 6.

In (18) we replace t_0 by $t_0 + \alpha$ and integrate with respect to α in the range $| \alpha | \le D$, where now $2D \le H$. LHS is now $I(\sigma_1)$ defined by

$$J(\sigma_1) = 2\pi\int_{|\alpha|\le D} | f(\sigma_1 + it_0 + i\alpha) | d\alpha. \qquad (21)$$

Next

$$
\begin{aligned}
&\textstyle\int_{|\alpha|\le D}(I_0^{\sigma_2-\sigma_1}(I_2 + M^{-A})^{\sigma_1-\sigma_0})^{(\sigma_2-\sigma_0)^{-1}} \\
&\le (\textstyle\int_{|\alpha|\le D} I_0\, d\alpha)^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}}(\textstyle\int_{|\alpha|\le D}(I_2 + M^{-A})d\alpha)^{(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)^{-1}}.
\end{aligned}
$$

Now

$$\int_{|\alpha|\le D} I_0\, d\alpha = \int_{|v|\le D}\int_{|\alpha|\le D} \mid f(\sigma_0+it_0+i\alpha+iv)\frac{d\alpha\, dv}{\sigma_0-\sigma_1+iv} \mid$$
$$\le \left(\int_{|v|\le 2D} \mid f(\sigma_0+it_0+iv)\mid dv\right)\int_{|v|\le D} \mid \frac{dv}{\sigma_0-\sigma_1+iv} \mid$$
$$\le 2(1+(log(\frac{D}{\sigma_1-\sigma_0}))^*)I(\sigma_0) \quad (22)$$

where

$$I(\sigma_0) = \int_{|v|\le 2D} \mid f(\sigma_0+it_0+iv)\mid dv. \quad (23)$$

Proceeding similarly, with

$$I(\sigma_2) = \int_{|v|\le 2D} \mid f(\sigma_2+it_0+iv)\mid dv \quad (24)$$

we have

$$\int_{|\alpha|\le D}(I_2+M^{-A})d\alpha \le 2DM^{-A} + 2(1+(log(\frac{D}{\sigma_2-\sigma_1}))^*)I(\sigma_2). \quad (25)$$

Thus we have the following corollary.

THEOREM 7. *In addition to the conditions of Theorem 6, let $2D \le H$ and let $J(\sigma_1), I(\sigma_0)$ and $I(\sigma_2)$ be defined by (21),(23) and (24). Then, we have,*

$$2\pi J(\sigma_1) \le 4\{I(\sigma_0)(1+(log(\frac{D}{\sigma_1-\sigma_0}))^*)\}^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}}\times$$
$$\times\{I(\sigma_2)(1+(log(\frac{D}{\sigma_2-\sigma_1}))^*)+M^{-A}\}^{(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)^{-1}}\times$$
$$\times\{Exp(\frac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0})\}^r + 4M^{A+2}(\sigma_2-\sigma_0)\{1+(log(\frac{D}{\sigma_1-\sigma_0}))^*\}^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}}\times$$
$$\times\{\frac{2}{CD}Exp(\frac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0})\}^r. \quad (26)$$

§ 3. **PROOF OF THEOREM 1.** In Theorem 7 replace D by $D/2$ and assume that $J(\sigma_1)$ is bounded below (by $\frac{1}{2}\beta D$) and $I(\sigma_2)$ is bounded above by $\beta^{-1}D$ (these conditions are implied by the conditions of Theorem 1). Put $C=1, r=[\varepsilon\ log\ M]+1$ and $D=Exp(\varepsilon^{-1}E)$ where E is a large constant. Let σ_0,σ_1 and σ_2 be constants satisfying $a\le\sigma=\sigma_0<\sigma_1<\sigma_2\le b$. We see that the second term on the RHS of (26) is $\le M^{-A}$ so that

$$\frac{1}{2}\beta D \le J(\sigma_1) \le 4\{I(\sigma_0)(1+(log(\frac{D}{\sigma_1-\sigma_0}))^*)\}^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}}\times$$
$$\times\{\beta^{-1}D(1+(log(\frac{D}{\sigma_2-\sigma_1}))^*)\}^{(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)^{-1}}\{Exp(\frac{(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0})\}^r. \quad (27)$$

This proves Theorem 1.

§ 4. **δ-CONVEXITY AND PROOF OF THEOREM 2.** In theorems 6 and 7 choose $\mathcal{D} = \delta$ (δ any positive constant), $\sigma_2 - \sigma_1 = (\log M)^{-1}, \sigma_2 = \sigma, \sigma_0 = a, r = [\log M]$, $\mathcal{C}=$ a large constant times δ^{-1}. We obtain the first part of Theorem 2 namely (3) and (3′). To obtain the second part we argue as in the proof of Theorems 6 and 7 but now with

$$\frac{1}{2\pi i}\int f(s_1+w)X^{-w}\frac{dw}{w}$$

along the same contour P with the same X as before (note X^{-w} in the present integrand). The rest of the details are similar.

§ 5. **PROOF OF THEOREM 3.** We follow closely the notation of § 2. We now put $\sigma_0 = a, \sigma_2 = b$. Now consider the integral

$$\frac{1}{2\pi i}\int_{Re\ w=b-\sigma_1, |Im\ w|\leq H} f(s_1+w)X^{-w}\ \frac{dw}{w}. \tag{28}$$

This is $O(M^{-A})$ provided X exceeds a suitable power of M. On the other hand, the integral

$$\frac{1}{2\pi i}\int f(s_1+w)X^{w}\frac{dw}{w} \tag{29}$$

over the same path equals

$$f(s_1)+\frac{1}{2\pi i}\left(\int_{H_1}+\int_{H_2}+\int_{V}\right)f(s_1+w)X^{w}\frac{dw}{w} \tag{30}$$

where s_1+w runs over the paths H_1, H_2 and V where H_1 and H_2 are the horizontal sides of R and V is the left vertical line boundary of R. (The contribution from V is $O(M^{-A})$ if X is a large power of M). Hence

$$\frac{1}{2\pi i}\int f(s_1+w)(X^{w}-X^{-w})\frac{dw}{w} \tag{31}$$

taken over the path in (28) (same as for (29) also) equals the integral (29) plus $O(M^{-A})$. Now move the line of integration to $Re\ w = 0$ and do the averaging as before writing $X = Exp(Y+u_1+\ldots+u_r)$ and we are led to the proof of Theorem 3.

§ 6. **ANOTHER APPLICATION OF OUR METHOD.** In this section we prove Theorem 5 (our proof is quite general and goes through for zeta and L-functions of algebraic number fields and so on). We need from

Titchmarsh's book [10] (see § 5.5, pages 174-175) the following theorem (we state it in the notation of this book).

THEOREM 8 (BOREL-CARATHÉODORY). *Let $f(z)$ be analytic in $|z| \leq R$ and $M(r)$ denote the maximum of $|f(z)|$ in $|z| \leq r < R$. Let $A(R)$ denote the maximum of the real part of $f(z)$ on $|z| = R$. Then*

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|.$$

We now take $f(z)$ to be $\log \zeta(s_1 + z)$. We will assume that t_0 exceeds a large constant. It is not hard to prove that $\log \zeta(s_1) = O(\log\log t_0)$. Taking $R = 20\delta, r = 19\delta$ we see that in $|z| \leq 19\delta$ we have $|\log \zeta(s_1 + z)| \leq 800\, \delta \log t_0$. Let now $|t_0 - t_1| \leq \delta$ and $1 - \delta \leq \sigma_0 \leq 1$. (We now use a notation not to be confused with the earlier one). Let $P_1 = 1 + \frac{1}{2} i t_0, P_2 = s_1 - 4i\delta, P_3 = 1 - 4\delta + it_0 - 4i\delta, P_4 = 1 - 4\delta + it_0 + 4i\delta, P_5 = s_1 + 4i\delta, P_6 = 1 + 2it_0$. Let P denote the contour $P_1P_2P_3P_4P_5P_6P_1$. Put $X = Exp(Y + u_1 + u_2 + ... + u_r)$ where $(u_1, ..., u_r)$ is as before with the earlier notation. Averaging over the cube we have

$$|\log \zeta(s_0)| \leq \left| \frac{1}{2\pi i} \int ... \int \int \log \zeta(s_0 + w) \frac{X^w}{w} dw\, du_1 ... du_r \right| C^{-r}$$
$$+ C^{-r} \int ... \int \sum_p \frac{1}{p^{\sigma_0}} Exp(-\frac{p}{X}) du_1 ... du_r + 10.$$

The contribution (to the integral involving $\zeta(s_0 + w)$) from P_3P_4 has absolute value ≤ 10 if

$$Exp(Y\delta) \geq 8000 \log t_0.$$

To satisfy this we put $Y = \frac{1}{\delta} (\log\log t_0 + 13)$. The average over other parts of the same integral has absolute value

$$\leq 800(\delta \log t_0)(\frac{2}{C\delta})^r\ Exp((Y + Cr)(1 - \sigma_0)).$$

We put $C = \frac{9}{\delta}$ and $r = [\log\log t_0]$, so that this expression is

$$\leq \varepsilon\ Exp((\tfrac{1}{\delta}(\log\log t_0 + 13) + \tfrac{9}{\delta} \log\log t_0)(1 - \sigma_0))$$
$$= \varepsilon\ Exp((\tfrac{1-\sigma_0}{\delta})(10 \log\log t_0 + 13))$$

The average of the sum over primes is

$$\leq S = \sum_p p^{-\sigma_0}\ Exp(-\frac{p}{Z})$$

where $Z = Exp(\frac{1}{8}(10\ loglog\ t_0 + 13))$, since $Z \geq$ maximum value of X during the averaging. Now

$$S \leq \sum_{p \leq Z} p^{-\sigma_0} + \sum_{p > Z} Z p^{-1-\sigma_0} = S_1 + S_2 \text{ say.}$$

Here

$$S_1 \leq Z^{1-\sigma_0} \sum_{p \leq Z} \frac{1}{p} \text{ and } S_2 \leq \varepsilon Z^{1-\sigma_0},$$

where $\varepsilon > 0$ is an arbitrary constant. We have

$$\sum_{p \leq Z} \frac{1}{p} \leq loglog\ Z + \gamma - \sum_{p,m \geq 2} (mp^m)^{-1} + \varepsilon$$

where $\varepsilon > 0$ is an arbitrary constant (see [11], p.58, Equation (3.14.5)) and γ is the Euler's constant. Hence

$$|\ log\ \zeta(s_0)\ | \leq O(1) + \varepsilon Z^{1-\sigma_0} + Z^{1-\sigma_0}(loglog\ Z + \gamma - \alpha + \varepsilon) \text{ where } \alpha = \sum_{p,m \geq 2} (mp^m)^{-1}$$

Therefore

$$|\ \zeta(s_0)\ | \ll Exp(Z^{1-\sigma_0}(loglog\ Z + 1)).$$

This completes the proof of Theorem 5.

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If $k > 0$ is an integer then by the method of § 5 the following improvement of equation (3) can be proved under the only assumptions $0 < \delta \leq 1$, $|\ f(s_1 + w)\ | \leq M, (M \geq 6)$ in $|\ w\ | \leq 10\delta$, and $|\ \sigma - \sigma_1\ | \leq \delta(log\ M)^{-1}$. Then $I(\sigma) log\ M$ on the RHS can be replaced by $\delta^{-1} I(\sigma) log\ M$. The implied constant in $\ll$ would then be independent of δ. Professor A. Ivić pointed out that the method of § 5 allows one to take $H = 1 + \delta$ in Theorem 3. The authors then succeeded in proving the above improvement.

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POST-SCRIPT. In all the equations marked with an asterisk there is no change if k is an integer. But if k is not an integer there will be a loss of a *loglog* factor. To see this we apply (3) and (3′) with the last remark in Theorem 2. Thus if k is not an integer, we have for any constant σ in $a < \sigma < b$ (and 2δ in place of δ),

$$| f(\sigma + it) |^{k} \ll M^{-A_0} + I(\sigma) \log M \log\log M.$$

(But if k is an integer *loglog* M on RHS can be dropped provided in place of δ we choose a large constant). It will be nice if we can remove *loglog* M even if k is not an integer.

The reference [8] is to be taken with the following.

K. RAMACHANDRA, Mean-value of the Riemann zeta-function and other remarks-II, International Conference on Analytic Number Theory (VINOGRADOV'S 90th BIRTHDAY CELEBRATIONS) STEKLOV INSTITUTE, MOSCOW, 14th-19th September (1981), Trudy Mats. Inst. Steklova 163 (1984), 200-204.