Hardy-Ramanujan Journal
Vol.11 (1988) 13-29

LARGE VALUES OF SOME ZETA-FUNCTIONS NEAR THE LINE $\sigma = 1$

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§ 1. Introduction.

To determine the order of $\zeta(1+it)$ is one of the central problems in the theory of the Riemann zeta-function $\zeta(s)$. The best known upper bound at present is

$$\zeta(1+it) \ll \log^{2/3} t. \tag{1.1}$$

It is obtained by an application of the estimate

$$\sum_{N<n\leq N'} n^{it} \ll N\, exp(-\frac{C \log^3 N}{\log^2 t}) \quad (C = 10^{-5}, N < N' \leq 2N, 1 \ll N \ll t)$$

which is a consequence of Vinogradov's method (see [4], Chapter 6) for the estimation of exponential sums. Here and later $f \ll g$ and $f = O(g)$ both mean $| f(x) | \leq Cg(x)$ for some $C > 0$ and $x \geq x_0$. On the other hand, it is known (see [14], Chapter 9) that on the Riemann hypothesis

$$e^{\gamma} \leq \limsup_{t\to\infty} \frac{|\zeta(1+it)|}{\log\log t} \leq 2e^{\gamma},$$

where $\gamma = 0,577...$ is Euler's constant. Thus it seems of interest to investigate the occurrence of large values of $\zeta(1+it)$, where "large" means roughly of the order not less than $\log\log t$. An interesting result in this direction was obtained recently by K. Ramachandra [11] : Let $X = exp(\log\log T/\log\log\log T)$, and cover $[T, T+e^X]$ with intervals of length $1/X$ (the last interval may be shorter). If $0 < \varepsilon < 1$ is an arbitrary constant, then $|\log\zeta(1+it)| \geq \varepsilon \log\log T$ for t in all of these intervals, except in at most K of them, where

$K = K(\varepsilon)$ is a constant. Ramachandra obtained his result by shrewdly applying an elementary inequality for complex numbers, and using complex integration to evaluate a certain sum over primes. His theorem is "local" in nature, in the sense that $X = o_\delta(\log^\delta T)$ as $T \to \infty$ for any $\delta > 0$. It seems also interesting to consider the "global" problem of the estimation of R, the number of points t_r in $[T, 2T]$ such that $| t_r - t_s | \geq 1$ for $r \neq s$, $| \log \zeta(1 + it_r) | \geq \varepsilon \log\log T$ (or $| \zeta(1 + it_r) | \geq (\log T)^\varepsilon$, where the method of [11] furnishes the same bound as in the former case). Breaking the interval $[T, 2T]$ into subintervals of length e^X and applying Ramachandra's estimate to each of these intervals one easily obtains

$$R \ll_\varepsilon Te^{-\frac{1}{2}X}. \tag{1.2}$$

We could also suppose that $| t_r - t_s | \geq X^{-1}$ for $r \neq s$, but this would only have the effect that the bound in (1.2) is multiplied by X.

A non-trivial result on large values of $\zeta(\sigma + it)$ when σ is close to 1 follows from Corollary 1 of K. Ramachandra [9] : Let R denote the number of points t_r in $[1, T]$ such that $| t_r - t_s | \geq 1$ for $r \neq s$ and $| \zeta(\sigma + it_r) | \geq V$, where $\frac{1}{2} \leq \sigma \leq 1$ and

$$\exp(C(\log T \log\log T)^{\frac{1}{2}}) \ll V \ll T^{100(1-\sigma)^{3/2}} \log^{2/3} T \tag{1.3}$$

for a suitable constant $C > 0$. Then uniformly

$$R \ll TV^{-A(1-\sigma)^{-3/2}} \quad (A = 1/300\sqrt{2} = 0,002357...). \tag{1.4}$$

The upper bound for V in (1.3) follows from the best known upper bound for $\zeta(\sigma + it)$ (see Chapter 6 of [4]). From (1.3) it follows that

$$\sigma \leq 1 - C(\log\log T/\log T)^{1/3}$$

must hold with some suitable $C > 0$, hence (1.4) holds only if σ is not too close to 1. From (1.4) one deduces easily

$$m(\sigma) \geq A(1 - \sigma)^{-3/2} \quad (A = 1/300\sqrt{2}) \tag{1.5}$$

for $\frac{1}{2} < \sigma \leq \sigma_0$, where $\sigma_0 < 1$ is fixed, and $m(\sigma)$ is the infimum of numbers m such that for a given σ and any fixed $\varepsilon > 0$

$$\int_1^T | \zeta(\sigma + it) |^m \, dt \ll T^{1+\varepsilon}.$$

The bound (1.5) is mentioned in the Notes of Chapter 8 of [4], without any specific value of the constant A. It is superseded by Theorem 8.4 of [4] for $\sigma \le \sigma_1$, where σ_1 can be explicitly evaluated, but for σ close to 1 (1.5) remains the best existing lower bound for $m(\sigma)$. Some other relevant results on large values of $\zeta(\sigma + it)$ and related topics may be found in K. Ramachandra [10].

The aim of this paper is to provide estimates which improve (1.2) and (1.4) when σ is sufficiently close to 1, and V lies in a certain range. This will be expounded in the next section. The method of approach is fairly general, and can be used to furnish analogous results for a class of zeta-functions that are similar to $\zeta(s)$. To this class belong the zeta-functions associated with the Fourier coefficients of cusp forms and the Dedekind zeta-function of algebraic number fields. These zeta-functions will be dealt with in § 3 and § 4, respectively.

Acknowledgement. I wish to thank M. Jutila and K. Ramachandra for valuable remarks and the Mathematical Institute of Belgrade for financing this research.

§ 2. **Large values of $\zeta(\sigma + it)$.**

The basic analytic principle of our approach is simple. In suitable horizontal strips, free of zeros of $\zeta(s)$, $\frac{\zeta'}{\zeta}(s)$ can be estimated and found to be small. Integration shows that $log\ \zeta(s)$ is then small, too. On the other hand, the number of well-spaced points t_r for which $\zeta(\sigma + it_r)$ lies near a zero of $\zeta(s)$ may be estimated satisfactorily by zero-density estimates.

We proceed now to give the details of this method. Henceforth suppose that t is given, w is a complex variable and

(2.1)

$$\zeta(w) \ne 0 \text{ for } Re\, w > \alpha_0 (\frac{1}{2} \le \alpha_0 < 1), |\, Im\, w - t\,| \le log^2 T, T \le t \le 2T, T \ge T_0.$$

We shall bound $|\, \zeta(\sigma + it)\,|$ for $\alpha < \sigma \le 1, \alpha > \alpha_0$, where α and α_0 will be specified later. The starting point is the inversion formula ($s = \sigma + it$)

$$(2.2) \qquad \sum_{n=1}^{\infty} \Lambda(n) e^{-n/Y} n^{-s} = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s+w)\Gamma(w)Y^w dw.$$

This follows from the Mellin integral (see (A.7) of [4])

$$e^{-x} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w) x^{-w} dw \qquad (x > 0)$$

on setting $x = n/Y$, multiplying by $\Lambda(n)n^{-s}$ and summing over n, since

$$\sum_{n=1}^{\infty}\Lambda(n)n^{-s} = -\frac{\zeta'(s)}{\zeta(s)} \qquad (\sigma = Re\ s > 1).$$

In (2.2) $\Lambda(n)$ is the familiar von Mangoldt function ($\Lambda(n) = log\ p$ if $n = p^m, p$ prime, and $\Lambda(n) = 0$ otherwise), and Y is a suitable parameter which satisfies $1 \ll Y \ll log^2 T$. In (2.2) we replace the line of integration $Re\ w = 2$ by the contour consisting of $[\alpha - \sigma - \frac{1}{2}\ ilog^2T, \alpha - \sigma + \frac{1}{2}\ ilog^2T], [\alpha - \sigma \pm \frac{1}{2}\ ilog^2T, 2 \pm \frac{1}{2}\ ilog^2T], [2 \pm \frac{1}{2}\ ilog^2T, 2 \pm i\infty]$. In view of (2.1) it is seen that $s + w$ will stay in a region free of zeros of the zeta-function, hence $\frac{\zeta'}{\zeta}(s + w)$ will be regular as a function of w. The only pole of the integrand will be $w = 0$, which yields the residue $-\frac{\zeta'}{\zeta}(s)$. We shall use the bound

$$\Gamma(w) \ll \frac{e^{-|Im\ w|}}{|\ w\ |}, \tag{2.3}$$

which is a weak form of Stirling's formula, to estimate the integrals in question. In this way we obtain from (2.2)

(2.4)

$$\sum_{n=1}^{\infty}\Lambda(n)e^{-n/Y}n^{-s} = -\frac{\zeta'}{\zeta}(s) + o(1) - \frac{1}{2\pi i}\int_L \frac{\zeta'}{\zeta}(s+w)\Gamma(w)Y^w dw \ \ (T \to \infty),$$

where L denotes the segment $[\alpha - \sigma - \frac{1}{2}\ ilog^2T, \alpha - \sigma + \frac{1}{2}\ ilog^2T]$. To estimate the last integral we note that for $z = s + w, |\ Im\ w\ | \le \frac{1}{2}\ log^2T, s = \sigma + it$ we have (see (1.52) of [4])

$$\frac{\zeta'(z)}{\zeta(z)} = \sum_{\rho:\zeta(\rho)=0, |Im\ \rho - Im\ z| \le 1}\frac{1}{z-\rho} + O(log\ T) \ll \frac{log\ T}{\alpha - \alpha_0}, \tag{2.5}$$

since (2.1) holds and there are $\ll\ log\ T$ zeros ρ in every horizontal strip of unit width. Using again (2.3) it follows from (2.4) that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty}\Lambda(n)e^{-n/Y}n^{-s} + O(1) + O(\frac{Y^{\alpha-\sigma}log\ T}{(\alpha-\alpha_0)(\sigma-\alpha)}) \tag{2.6}$$

holds uniformly for $s = \sigma + it, \alpha < \sigma \le 1$, if (2.1) is true. Set now in (2.6) $\sigma = \theta$ and integrate over θ for $\sigma \le \theta \le 2, \alpha < \sigma \le 1$. If we define

$$\Lambda_1(n) = \begin{cases} 0, & n = 1, n \ne p^m, \\ \frac{1}{m}, & n = p^m, \end{cases}$$

where p denotes primes, we obtain that uniformly

$$(2.7)\quad \log \zeta(s) = \sum_{n=1}^{\infty} \Lambda_1(n) e^{-n/Y} n^{-s} + O(1) + O\left(\frac{Y^{\alpha-\sigma}}{(\alpha-\alpha_0)} \log T \log \frac{1}{\sigma-\alpha}\right),$$

if $s = \sigma + it, \alpha < \sigma \le 1$, and (2.1) holds.

We shall first examine some consequences of (2.7). Suppose α_0 in (2.1) is fixed, and take $\sigma = 1, \alpha = \frac{1}{2}(1+\alpha_0) < 1$. Then from (2.7) we infer that

$$\log \zeta(1+it) = \sum_{n \le Y} \Lambda_1(n) e^{-n/Y} n^{-1-it} + O(1) + O_{\alpha_0}(Y^{\alpha-1} \log T),$$

where the subscript in the last O-term means that the constant in question depends on α_0. With the choice $Y = (\log T)^{1/(1-\alpha)}$ this gives

$$\log | \zeta(1+it) | \le | \log \zeta(1+it) | \le \sum_{n \le Y} \Lambda_1(n) n^{-1} + O_{\alpha_0}(1)$$

$$= \sum_{p \le Y} p^{-1} + O_{\alpha_0}(1) = \log\log Y + O_{\alpha_0}(1) = \log\log\log T + O_{\alpha_0}(1).$$

Therefore

$$(2.8)\qquad | \zeta(1+it) | \le C(\alpha_0) \log\log t \quad (t \ge t_0(\alpha_0))$$

for some constant $C(\alpha_0) > 0$ if (2.1) holds for a fixed α_0. In case the Riemann hypothesis that all complex zeros of $\zeta(s)$ satisfy $Re\ s = \frac{1}{2}$ is true, then (2.1) holds with $\alpha_0 = \frac{1}{2}$, and as mentioned in § 1 one may take $C(\frac{1}{2}) = 2e^{\gamma} + \varepsilon$ in that case.

Suppose now again that α_0 in (2.1) is fixed, and take $\alpha = \alpha_0 + \varepsilon, \alpha_0 + 2\varepsilon \le \sigma \le \sigma_0$, where $\sigma_0 < 1$ is fixed. For $N, Y \gg 1$ note that

$$\sum_{p > N} p^{-\sigma} e^{-p/Y} \ll \int_N^{\infty} t^{-\sigma} e^{-t/Y} (\log t)^{-1} dt \ll \frac{N^{-\sigma}}{\log N} \int_N^{\infty} e^{-t/Y} dt \ll \frac{Y N^{-\sigma}}{\log N},$$

where p denotes primes. Also using the prime number theorem we have

$$\sum_{p \le Y} p^{-\sigma} = \frac{Y^{1-\sigma}}{(1-\sigma)\log Y} + O\left(\frac{Y^{1-\sigma}}{\log^2 Y}\right).$$

Integrating (2.6) we obtain then

$$\log \zeta(s) = \sum_{n=1}^{\infty} \Lambda_1(n) e^{-n/Y} n^{-s} + O(1) + O_\varepsilon\Big(Y^{\alpha-\sigma}\frac{\log T}{\log Y}\Big),$$

which gives

$$\log |\zeta(\sigma+it)| \le |\log \zeta(\sigma+it)| \le \sum_{n\le Y} \Lambda_1(n) e^{-n/Y} n^{-\sigma} + O\Big(\frac{Y^{1-\sigma}}{\log Y}\Big)$$

$$+O_\varepsilon\Big(Y^{\alpha-\sigma}\frac{\log T}{\log Y}\Big) \ll_\varepsilon \frac{Y^{1-\sigma}}{\log Y} + Y^{\alpha-\sigma}\frac{\log T}{\log Y} \ll_\varepsilon \frac{(\log T)^{(1-\sigma)/(1-\alpha)}}{\log\log T}$$

on choosing $Y = (\log T)^{1/(1-\alpha)}$. This means that

$$|\zeta(\sigma+it)| \le \exp\Big(\frac{D(\log t)^{\frac{1-\sigma}{1-\alpha}}}{\log\log t}\Big) \quad (t \ge t_0(\varepsilon), D > 0) \tag{2.9}$$

if $\alpha = \alpha_0 + \varepsilon, \alpha_0 + 2\varepsilon \le \sigma \le \sigma_0, \alpha_0$ and σ_0 are fixed, $\varepsilon > 0$ is a small, positive number, $D = D(\varepsilon)$ and (2.1) holds. If the Riemann hypothesis is true, then one has (2.9) with $\alpha = \frac{1}{2}$ for $t \ge t_0$ by Theorem 14.5 of [14], and with more care the foregoing proof could be adapted to give this result also.

Now we shall give an upper bound for $|\zeta(\sigma+it)|$ in the whole range $\alpha_0 < \alpha < \sigma \le 1$, choosing $\alpha_0 = 1 - \frac{3A}{\log\log T}, \alpha = 1 - \frac{2A}{\log\log T}, 1 - \frac{A}{\log\log T} \le \sigma \le 1$, where $A > 0$ is an absolute constant. In that case we obtain from (2.6)

$$\log \zeta(s) = \sum_{n\le Y} \Lambda_1(n) e^{-n/Y} n^{-s} + O\Big(\frac{Y^{1-\sigma}}{\log Y}\Big) + O\Big(Y^{-A/\log\log T}\frac{\log T}{\log Y}(\log\log T)^2\Big),$$

$$\log |\zeta(\sigma+it)| \le |\log \zeta(\sigma+it)| \le \sum_{p\le Y} p^{-\sigma} + O(1) + O\Big(\frac{Y^{1-\sigma}}{\log Y}\Big)$$

$$+O\Big(Y^{-A/\log\log T}\frac{\log T}{\log Y}(\log\log T)^2\Big) \le Y^{1-\sigma}(\log\log Y + O(1))$$

$$+O\Big(Y^{\frac{-A}{\log\log T}}\frac{\log T}{\log Y}(\log\log T)^2\Big).$$

Choose now

$$Y = exp(\frac{2}{A}(loglog\ T)^2),$$

so that the last O-term above is bounded (but it depends on A). We obtain

$$log \mid \zeta(\sigma + it) \mid \leq exp(\frac{2-2\sigma}{A}(loglog\ T)^2) log(C(A)(loglog\ T)^2),$$

that is

$$(2.10) \quad \mid \zeta(\sigma + it) \mid \leq exp\{exp(\frac{2-2\sigma}{A}(loglog\ T)^2) \cdot log(C(A)(loglog\ T)^2)\}.$$

This bound is valid for $1 - \frac{A}{loglog\ T} \leq \sigma \leq 1, C(A) > 0$ a constant depending on A (whose value could be made explicit), if (2.1) holds with $T \geq T_0(A)$.

Consider now the region

$$\mathcal{D} = \{s \in C : Re\ s \geq 1 - \frac{A}{loglog\ T}, T \leq Im\ s \leq 2T\}$$

and divide it into subregions

$$\mathcal{D}_k = \{s \in \mathcal{D} : T + (k-1)log^2 T \leq Im\ s < T + k\ log^2 T\},$$

where $k = 1, 2, ..., [T/log^2 T]$. Using the zero-density estimate

$$(2.11) \qquad N(\sigma, T) = \sum_{\rho:\zeta(\rho)=0, Re\ \rho\geq\sigma, |Im\ \rho|\leq T} 1 \ll T^{1600(1-\sigma)^{3/2}} log^{15} T$$

(see Theorem 11.3 of [4]), it is seen that there are

$$\ll exp(\frac{1600 A^{3/2} log\ T}{(loglog\ T)^{3/2}}) log^{15} T \ll exp(\frac{1700 A^{3/2} log\ T}{(loglog\ T)^{3/2}})$$

zeros of $\zeta(s)$ in $\mathcal{D}$. Hence there are at most

$$exp(\frac{1800 A^{3/2} log\ T}{(loglog\ T)^{3/2}})$$

values of k such that $\mathcal{E}_k = \mathcal{D}_{k-1} \cup \mathcal{D}_k \cup \mathcal{D}_{k+1}$ contains a zero of $\zeta(s)$. Thus if we construct R arbitrary points $t_1,...,t_R$ belonging to $[T,2T]$ such that $| t_r - t_s | \geq 1$ for $r \neq s$ and (2.10) fails to hold for $t = t_1,...,t_R$ and a suitable $C(A)$, then each point $\sigma + it_1,...,\sigma + it_R$ must fall into some $\mathcal{E}_k$ which contains a zero of $\zeta(s)$. This provides us with an upper bound for R, contained in

THEOREM 1. Let $1 - \frac{A}{loglog\ T} \leq \sigma \leq 1$ for a constant $A > 0$, and let R be the number of points $t_r \in [T,2T]$ such that $| t_r - t_s | \geq 1$ for $r \neq s$ and

$$| \zeta(\sigma + it_r) | \geq exp\{exp(\frac{2-2\sigma}{A}(loglog\ T)^2) \cdot log(C(A)(loglog\ T)^2)\}$$

for some suitable $C(A) > 0$. Then for $T \geq T_0(A)$

$$R \leq exp(\frac{2000A^{3/2}\ log\ T}{(loglog\ T)^{3/2}}). \tag{2.12}$$

Theorem 1 thus provides a large values estimate in the region

$$1 - \frac{A}{loglog\ T} \leq \sigma \leq 1,$$

which in a sense complements Ramachandra's bound (1.4). Of course it is possible to obtain a similar type of result for a somewhat different region, but this one is of a particularly simple shape. Moreover in this region we have the bound (2.12), which is much stronger than just $R \ll_\varepsilon T^\varepsilon$. Note also that $exp(\frac{2-2\sigma}{A}(loglog\ T)^2) \leq e$ for $\sigma \geq 1 - \frac{A}{2(loglog\ T)^2}$, so that in the last region $| \zeta(\sigma + it) | \leq B(A)(loglog\ t)^{2e}$ except for a relatively few points.

For $\sigma = 1$ we have that (2.12) holds for the number of points t_r for which $| \zeta(1 + it_r) | \geq C(A)(loglog\ T)^2$ for some $C(A) > 0$. The same method of proof gives also that $R \ll_\varepsilon T^\varepsilon$ holds for any fixed $\varepsilon > 0$ if $| \zeta(1+it_r) | \geq D(\varepsilon) loglog\ T$ for a suitable constant $D(\varepsilon) > 0$. No information seems obtainable by our method in the case when $| \zeta(1 + it_r) | \geq f(T)$ and $f(T) > 0$ is a function which satisfies $f(T) = o(loglog\ T)$ as $T \to \infty$.

If $\sigma_0 \leq \sigma \leq \sigma_1 < 1$ is fixed, then it follows by the method of proof of Theorem 1 that

$$R \ll_\varepsilon T^{1600(1-\sigma)^{3/2}+2\varepsilon}$$

for the number of points $t_r \in [T, 2T]$ such that $|\, t_r - t_s \,| \geq 1$ for $r \neq s$ and $\zeta(\sigma + it_r) \gg T^{\epsilon}$. Hence using the bound (see Chapter 6 of [4])

$$\zeta(\sigma + it) \ll t^{100(1-\sigma)^{3/2}} \log^{2/3} t \quad (\frac{1}{2} \leq \sigma \leq 1)$$

it follows that

$$m(\sigma) \geq \frac{1}{100}(1-\sigma)^{-3/2} - 16 \quad (\sigma_0 \leq \sigma \leq \sigma_1 < 1) \tag{2.13}$$

for any fixed $\sigma, \frac{1}{2} < \sigma_0 < \sigma < 1$. Since, for $1 - 3200^{-2/3} \leq \sigma \leq 1$, we have

$$\frac{1}{100}(1-\sigma)^{-3/2} - 16 \geq \frac{1}{200}(1-\sigma)^{-3/2},$$

it is seen that we obtain an alternative proof of (1.5), with a better value of A.

§ 3. Large values of zeta-functions of cusp forms near $\sigma = 1$.

There are several classes of zeta-functions besides $\zeta(s)$ to which the result of Theorem 1 can be generalized, with appropriate modifications. A property, essential that such a generalization may be made, is the existence of a simple Euler product representation for the zeta-function in question in the region $\sigma = Re\ s > 1$. One such class is given by Dirichlet functions $L(s, \chi)$, where the generalization is obvious and straightforward. More interesting examples appear to be the Dedekind zeta-functions, and the zeta-functions associated with Fourier coefficients of cusp forms, which will be treated in this section. A classical representative is

$$T(s) = \sum_{n=1}^{\infty} \tau(n) n^{-11/2-s} \quad (Re\ s > 1),$$

where $\tau(n)$ is Ramanujan's function, defined by

$$\sum_{n=1}^{\infty} \tau(n) x^n = x\{(1-x)(1-x^2)(1-x^3)...\}^{24} \quad (|\, x \,| < 1).$$

More generally, let $a(n)$ be the Fourier coefficients (see e.g. T.M. Apostol

[1]) of a normalized Hecke eigenform (cusp form) of weight κ for the full modular group. Let $\tilde{a}(n) = a(n)n^{-\frac{1}{2}(\kappa-1)}$, and let

$$(3.1) \quad F(s) = \sum_{n=1}^{\infty} \tilde{a}(n)n^{-s} = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\overline{\alpha}_p p^{-s})^{-1} \quad (Re\ s > 1)$$

be the zeta-function associated with $a(n)$. The zeta-function $F(s)$ seems more natural than the zeta-function associated with $a(n)$ directly (i.e. if we had in (3.1) $a(n)$ and not $\tilde{a}(n)$), whose "critical strip" is $\frac{1}{2}(\kappa-1) < Re\ s < \frac{1}{2}(\kappa+1)$. On the other hand, the critical strip for $F(s)$ is $0 < Re\ s < 1$ as in the case of $\zeta(s)$, and the Riemann hypothesis for $F(s)$ is that all complex zeros ρ of $F(s)$ satisfy $Re\ \rho = \frac{1}{2}$. The zeta-function $F(s)$ is in many ways similar to $\zeta^2(s)$, which is an analogy that is often exploited (see M. Jutila [5]). It is known that the numbers α_p in (3.1) are of the form $\alpha_p = e^{i\theta(p)}$, and $\theta(p)$ is real by a deep result of P. Deligne [2]. It is precisely the Euler product representation which is important in our problem, namely the investigation of values of $F(\sigma+it)$ for σ close to 1. Taking the logarithmic derivative in (3.1) we obtain

$$-\frac{F'(s)}{F(s)} = \sum_{n=1}^{\infty} \Lambda_F(n)n^{-s} = \sum_p \sum_{m=1}^{\infty} (\alpha_p^m + \overline{\alpha}_p^m)\frac{log\ p}{p^{ms}} \quad (Re\ s > 1).$$

Hence equating coefficients it follows that

$$\Lambda_F(n) = \begin{cases} 0, & n = 1, n \neq p^m, \\ (\alpha_p^m + \overline{\alpha}_p^m) log\ p, & n = p^m, \end{cases}$$

so that $\Lambda_F(n)$ is the analogue of the von Mangoldt function $\Lambda(n)$ for $F(s)$. Proceeding as in the case of $\zeta(s)$ we have

$$(3.2) \quad log\ F(s) = \sum_{n=1}^{\infty} \Lambda_{1,F}(n)e^{-n/Y}n^{-s} + O(1) + O(\frac{Y^{\alpha-\sigma} log\ T}{(\alpha-\alpha_0)} log \frac{1}{\sigma-\alpha})$$

uniformly for $s = \sigma + it, \alpha_0 < \alpha < \sigma \le 1$, provided that (2.1) holds with $F(s)$ in place of $\zeta(s)$, where

$$\Lambda_{1,F}(n) = \begin{cases} 0, & n = 1, n \neq p^m, \\ (\alpha_p^m + \overline{\alpha}_p^{-m})m^1, & n = p^m. \end{cases}$$

In the course of the proof one needs the fact that the analogue of (2.5) holds for $F(s)$, which follows e.g. from Lemma 3.4 of C.J. Moreno [7]. In the special case when $\sigma = 1, \alpha_0$ is fixed $\alpha = \frac{1}{2}(1+\alpha_0)$, we obtain from (3.2)

$$log\ F(1+it) = \sum_{n=1}^{\infty} \Lambda_{1,F}(n) e^{-n/Y} n^{-1-it} + O(1) + O_{\alpha_0}(Y^{\alpha-1} \frac{log\ T}{log\ Y}).$$

Hence for $Y = (log\ T)^{1/(1-\alpha)}$

$$log \mid F(1+it) \mid \leq \mid log\ F(1+it) \mid \leq \sum_{n \leq Y} \mid \Lambda_{1,F}(n) \mid n^{-1} + O_{\alpha_0}(1)$$

$$= \sum_{p \leq Y} \mid \tilde{a}(p) \mid p^{-1} + O_{\alpha_0}(1),$$

since by (3.1)

$$\tilde{a}(p) = \alpha_p + \overline{\alpha}_p = \Lambda_{1,F}(p)$$

and clearly

$$\sum_{n \leq Y, n \neq p} \mid \Lambda_{1,F}(n) \mid n^{-1} \ll 1.$$

Now we use the asymptotic formula

$$\sum_{n \leq x} \mid a(n) \mid^2 \Lambda(n) = x^{\kappa} + O(x^{\kappa} exp(-c\sqrt{log\ x})) \quad (c > 0),$$

proved by A. Perelli [8] (we could also use e.g. Lemma 2 of M. Ram Murty [12]; this would give (3.3) with *loglog t* replaced by $(loglog\ t)^{1+\epsilon}$). This is the analogue of the prime number theorem for modular forms, and gives by partial summation

$$\sum_{p \leq Y} \mid \tilde{a}(p) \mid^2 p^{-1} = \sum_{p \leq Y} \mid a(p) \mid^2 p^{-\kappa} = loglog\ Y + O(1).$$

Hence by the Cauchy-Schwarz inequality

$$\sum_{p \leq Y} \mid \tilde{a}(p) \mid p^{-1} \leq (\sum_{p \leq Y} \mid \tilde{a}(p) \mid^2 p^{-1})^{\frac{1}{2}} (\sum_{p \leq Y} p^{-1})^{\frac{1}{2}} = loglog\ Y + O(1),$$

since

$$\sum_{p \leq Y} p^{-1} = loglog\ Y + O(1).$$

Therefore

$$log \mid F(1+it) \mid \le logloglog\, T + O_{\alpha_0}(1) \quad (T \le t \le 2T),$$

$$F(1+it) \ll_{\alpha_0} loglog\, t, \tag{3.3}$$

provided that (2.1) holds for $F(s)$. In particular, (3.3) is then true if the Riemann hypothesis for $F(s)$ holds.

The arguments that yield Theorem 1 will work also in the case of $F(s)$. The analogue of (2.11) can be obtained for $F(s)$, but the sketch of proof of this result would lead us too much astray. Thus the only noteworthy change in the proof is that we shall use the zero-density estimate

$$N_F(\sigma,T) = \sum_{\rho: F(\rho)=0, Re\ \rho \ge \sigma, |Im\ \rho| \le T} 1 \ll T^{3-3\sigma} log^C T \tag{3.4}$$

which certainly holds for $\frac{19}{20} \le \sigma \le 1$ and some $C > 0$. One can easily obtain (3.4) by using the techniques developed for $\zeta(s)$ in Chapter 11 of [4] and the estimate

$$\int_1^T \mid F(\frac{1}{2}+it) \mid^2 dt \ll T\ log\ T.$$

This bound follows on representing $F(\frac{1}{2}+it)$ as a sum of Dirichlet polynomials of length $\ll t$, and then using the mean value theorem for Dirichlet polynomials (see Chapter 5 of [4]). A sharp asymptotic formula for the integral in question is established by A. Good [3]. In this way we obtain

THEOREM 2. Let $1 - \frac{A}{loglog\ T} \le \sigma \le 1$ for a constant $A > 0$, and let R be the number of points $t_r \in [T, 2T]$ such that $\mid t_r - t_s \mid \ge 1$ for $r \ne s$ and

$$\mid F(\sigma + it_r) \mid \ge exp\{exp(\frac{2-2\sigma}{A}(loglog\ T)^2) \cdot log(C(A)(loglog\ T)^2)\} \tag{3.5}$$

for a suitable constant $C(A) > 0$. Then for $T \ge T_0(A)$

$$R \le exp(\frac{3A\ log\ T}{loglog\ T}). \tag{3.6}$$

Actually (3.6) can be replaced by the same type of bound as (2.12). To obtain this, it is necessary to show how the analogue of (2.11) holds for the zero-counting function of $F(s)$. We hope to return to this question elsewhere.

§ 4. Large values of the Dedekind zeta-function near $\sigma = 1$.

We shall sketch now how the analogue of Theorem 2 may be established for the Dedekind zeta-function

$$\zeta_K(s) = \sum_{n=1}^{\infty} H(n) n^{-s} \quad (\sigma = Re\ s > 1)$$

of an algebraic number field K such that $[K : Q] = N$. Here $H(n)$ denotes the number of non-zero integral ideals of K with norm equal to n. From the theory of algebraic number fields (see e.g. D.A. Marcus [6], Theorem 21 and Theorem 24) it is known that

$$(p) = \prod_i \mathfrak{p}_i^{e_i}, N\mathfrak{p}_i = p^{f_i}, \sum_i e_i f_i = N,$$

and $e_i = 1$ for almost all primes p. Let P_0 be the finite set of primes which have some $e_i > 1$. Factorising the polynomials $X^{f_i} - 1$ we obtain, for $Re\ s > 1$,

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - (N\mathfrak{p})^{-s})^{-1} = \prod_p \prod_{\mathfrak{p}|(p)} (1 - (N\mathfrak{p})^{-s})^{-1}$$

$$= \prod_{p \notin P_0} \prod_{j=1}^{N} (1 - \chi_j(p) p^{-s})^{-1} \prod_{p \in P_0} \prod_{j=1}^{N_p} (1 - \chi_j(p) p^{-s})^{-1}$$

where $| \chi_j(p) | = 1$ and $N_p < N$. If $p \in P_0$, then for $N_p < j \le N$ we define $\chi_j(p) = 0$. With this notation it follows that

$$\zeta_K(s) = \prod_{j=1}^{N} \prod_p (1 - \chi_j(p) p^{-s})^{-1} \quad (| \chi_j(p) | \le 1, Re\ s > 1). \tag{4.1}$$

Hence if we define $\Lambda_K(n)$, the analogue of $\Lambda(n)$ for $\zeta_K(s)$, by

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{n=1}^{\infty} \Lambda_K(n) n^{-s} \quad (Re\ s > 1), \tag{4.2}$$

then taking the logarithmic derivative of (4.1) and comparing with (4.2) we obtain

$$\Lambda_K(n) = \begin{cases} 0, & n = 1, n \neq p^m, \\ \sum_{j=1}^{N} \chi_j^m(p) \log p, & n = p^m. \end{cases}$$

Thus $0 \leq |\ \Lambda_K(n)\ | \leq N\Lambda(n)$, and in several ways $\zeta_K(s)$ is analogous to $\zeta^N(s)$. The analysis made for $F(s)$ in § 3 can be carried over to $\zeta_K(s)$ with obvious modifications in the proof. For example, (3.3) will become

$$\zeta_K(1 + it) \ll_{N,\alpha_0} (\log\log t)^N \tag{4.3}$$

provided that (2.1) holds with $\zeta_K(s)$ in place of $\zeta(s)$. However, using the prime ideal theorem for algebraic number fields in the form

$$\sum_{N\mathfrak{p} \leq x} 1 = \int_2^x \frac{dt}{\log t} + O(x\ exp(-c\sqrt{\log x})) \quad (c > 0),$$

one obtains by partial summation

$$\sum_{N\mathfrak{p} \leq x} (N\mathfrak{p})^{-1} = \log\log x + O(1).$$

This in turn gives a sharpening of (4.3), namely

$$\zeta_K(1 + it) \ll_{\alpha_0} \log\log t.$$

In this case instead of Theorem 2 we shall obtain

THEOREM 3. Let $1 - \frac{A}{\log\log T} \leq \sigma \leq 1$ for a constant $A > 0$, and let R be the number of points $t_r \in [T, 2T]$ such that $|\ t_r - t_s\ | \geq 1$ for $r \neq s$ and

$$|\ \zeta_K(\sigma + it_r)\ | \geq exp\{exp(\frac{2 - 2\sigma}{A}(\log\log T)^2) \cdot \log(B(\log\log T)^2)\}$$

for some constant $B = B(A, N, K) > 0$. Then for $T \geq T_0(A, N, K)$ and a suitable constant $D = D(A, N, K) > 0$ we have

$$R \leq exp(\frac{D \log T}{\log\log T}). \tag{4.4}$$

A similar type of result could be obtained if, instead of the analogue of (3.4) which yields (4.4), we use the zero-density estimate of W. Staś [13] for $\zeta_K(s)$. As remarked in the Introduction, the whole approach is fairly general and can be used to deal with many other zeta-functions. For example, analogous results may be readily obtained for $L(s, \chi)$ and $L_{K,\mathcal{F}}(s, \chi)$, where χ is a character mod q, if we do not insist on uniformity in q etc.

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(RECEIVED BY EDITORS ON 29TH MARCH 1989).