Hardy-Ramanujan journal
Vol. 7 (1984) 11—12

ON THE ALGEBRAIC DIFFERENTIAL EQUATIONS SATISFIED BY SOME ELLIPTIC FUNCTIONS (I)

By

P. CHOWLA AND S. CHOWLA

Summary

When a is an odd positive integer it is implicit in the work of Jacobi that the functions

$$Y = \sum_{1}^{\infty} \sigma_a(n) X^n$$

where

$$\sigma_a(n) = \sum_{d/n} d^a$$

(the sum of the a th powers of the divisors of n) satisfy an algebraic differential equation i. e. there is a polynomial T, not identically 0, such that $T(X, Y, Y_1, \ldots, Y_m) = 0$. When $a = 1$ we give a new argument based on Ramanujan that we may take $m = 3$ here.

§ 1. With Ramanujan (Coll. Papers, pp 138-142) we write

$$P = 1 - 24 \sum_{1}^{\infty} \sigma_1 (n) X^n$$

$$Q = 1 + 240 \sum_{1}^{\infty} \sigma_3 (n) X^n$$

$$R = 1 - 504 \sum_{1}^{\infty} \sigma_5 (n) X^n$$

Then we have

$$(\alpha) \quad X \frac{dP}{dX} = \frac{P^2 - Q}{12}$$

$$(\beta) \qquad X \frac{dQ}{dX} = \frac{PQ - R}{3}$$

$$(\gamma) \qquad X \frac{dR}{dX} = \frac{PR - Q^2}{2}$$

Now differentiate (α) twice w.r.t. "X". The first step gives us

(δ) $\quad \frac{d^2P}{dX^2}$ in terms of P, Q, R, X.

Another differentiation gives us

(ε) $\quad \frac{d^3P}{dX^3}$ in terms of P,Q,R, X.

Now eliminate Q,R from the 3 equations (α), (δ), (ε).

This relation (which can be converted into a polynomial relation) gives us

$$T\left(X, P, \frac{dP}{dX}, \frac{d^2P}{dX^2}, \frac{d^3P}{dX^3}\right) = 0$$

Here T is a polynomial (not identically 0) in the 5 quantities

X, P, P_m ($1 \leqslant m \leqslant 3$) where P_m stands for $\frac{d^mP}{dX^m}$.

Thus we have a third order alg. diff. eqn. satisfied by P, and therefore by $\sum_1^\infty \sigma_1(n) X^n$. We also risk a *conjecture*.

The function

$$Y = \sum_1^\infty \sigma_a(n) X^n$$

does *not* satisfy an alg. differential equation when a is an *even* positive integer.

We have been unable to prove anything in this direction.