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ON ALGEBRAIC DIFFERENTIAL EQUATIONS SATISFIED BY SOME ELLIPTIC FUNCTIONS (II)

By

P. CHOWLA and S. CHOWLA

Summary :

In (I) we obtained the "implicit" algebraic differential equation for the function defined by

$$Y = \sum_{1}^{\infty} \frac{n^a x^n}{1 - x^n}$$

where a is an odd positive integer, and conjectured that there are no algebraic differential equations for the case when a is an even integer. In this note we obtain a simple proof that (this has been known for almost 200 years)

$$Y = \sum_{1}^{\infty} x^{n^2} \qquad (|x| < 1)$$

satisfies an algebraic differential equation, and conjecture that

$$Y = \sum_{1}^{\infty} x^{n^k} \qquad (k \text{ a positive integer } > 2)$$

does not satisfy an algebraic differential equation.

§1 In the standard notation of elliptic functions we write

$$K = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}$$

$$E = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2\phi}\, d\phi$$

Here $0 < k < 1$. We also define k' by

$$k^2 + k'^2 = 1$$

and the corresponding integrals

$$K' = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k'^2 \sin^2\phi}}$$

$$E' = \int_0^{\pi/2} \sqrt{1-k'^2 \sin^2\phi}\, d\phi$$

We know that

$$KE' + K'E - KK' = \frac{\pi}{2}$$

which is often called Legendre's relation.

Our proof rests on several identities for K, K', E, E' and their derivatives. Most important:

$$\sqrt{\frac{2k}{\pi}} = \sum_{-\infty}^{\infty} q^{n^2}$$

where

$$q = e^{\frac{-\pi K'}{k}}$$

§ 2. Next we need:

$$\frac{dK}{dk} = \frac{E-k'^2K}{k\,k'^2},$$

$$\frac{dE}{dk} = \frac{E-K}{k}$$

Since

$$\frac{dk'}{dk} = -\frac{k}{k'}$$

we easily express

$$\frac{d E'}{d k}, \frac{d K'}{d k}$$

as easy functions of K, K′, E, E′ and k. From these

$$\frac{d}{dk}(E K' + KE' - K K') = 0$$

Hence $$EK' + KE' - K K' = c$$

where c is a constant independent of k.

We easily see that $c = \frac{\pi}{2}$.

§ 3. Next

$$\frac{d q}{d k} = \frac{\pi^2 q}{2k k'^2 K^2}$$

(Ramanujan, Collected Papers, p. 32)

§ 4. Next find

$$\frac{d K}{d q} = \frac{d K}{d k} \frac{d k}{d q}$$

which is, from our previous relations a simple rational function of K, E, k, q.

Form

$$\frac{d^2 K}{d q^2} \text{ and } \frac{d^3 K}{d q^3}$$

which again are simple rational functions of K, E, k, q. Eliminate E and k from the relations obtained in this section. We then get a "polynomial" relation between

$$K, q, \frac{dK}{dq}, \frac{d^2K}{dq^2}, \frac{d^3K}{dq^3}$$

which is a third order algebraic differential equation

$$T\left(K, q, \frac{dK}{dq}, \frac{d^2K}{dq^2}, \frac{d^3K}{dq^3}\right) = 0$$

where T is a polynomial in the five quantities following T.

§ 5. It would be nice to obtain "explicitly" the polynomial T of the last section, as well as those mentioned in the beginning of this paper i. e. algebraic differential equations satisfied by

$$Y = \sum_{1}^{\infty} \frac{n^a x^n}{1-x^n}$$

i. e. a relation with an explicit polynomial T with

$$T(Y, X, Y', Y'', Y''') = 0$$

As far we know this has never been done.

Conclusion.

Professor Dwork (Princeton University) pointed out to us that G. Valerion in a monograph (in French) on automorphic forms in analysis obtained the algebraic differential equation explicitly in the case of the function

$$Y = \prod_{1}^{\infty} (1 - x^n).$$