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ON ALGEBRAIC DIFFERENTIAL EQUATIONS SATISFIED BY SOME ELLIPTIC FUNCTIONS (II)

By

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Summary:

In (I) we obtained the "implicit" algebraic differential equation for the function defined by

\[ Y = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} \]

where \( a \) is an odd positive integer, and conjectured that there are no algebraic differential equations for the case when \( a \) is an even integer. In this note we obtain a simple proof that (this has been known for almost 200 years)

\[ Y = \sum_{n=1}^{\infty} \frac{x^{2n}}{1} \quad (|x| < 1) \]

satisfies an algebraic differential equation, and conjecture that

\[ Y = \sum_{n=1}^{\infty} \frac{x^{kn}}{1} \quad (k \text{ a positive integer } > 2) \]

does not satisfy an algebraic differential equation.

§1 In the standard notation of elliptic functions we write

\[ K = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \]
Here $0 < k < 1$. We also define $k'$ by

$$k^2 + k'^2 = 1$$

and the corresponding integrals

$$K' = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k'^2 \sin^2 \phi}}$$

$$E' = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k'^2 \sin^2 \phi}}$$

We know that

$$KE' + K'E - KK' = \frac{\pi}{2}$$

which is often called Legendre's relation.

Our proof rests on several identities for $K$, $K'$, $E$, $E'$ and their derivatives. Most important:

$$\sqrt{\frac{2k}{\pi}} = \sum_{n=0}^{\infty} \frac{q^n}{\pi n^2}$$

where

$$q = e^{-\frac{\pi K'}{k}}$$

§ 2. Next we need:

$$\frac{dK}{dk} = \frac{E-k'{E}^2 K}{kk'2}$$

$$\frac{dE}{dk} = \frac{E-K}{k}$$

Since

$$\frac{d}{dk} \frac{k'}{k} = -\frac{k}{k'}$$
we easily express
\[
\frac{dE'}{dk}, \quad \frac{dK'}{dk}
\]
as easy functions of \(K, K', E, E'\) and \(k\). From these
\[
\frac{d}{dk} (EK' + KE' - KK') = 0
\]
Hence
\[
EK' + KE' - KK' = c
\]
where \(c\) is a constant independent of \(k\).

We easily see that \(c = \frac{\pi}{2}\).

§ 3. Next
\[
\frac{d q}{d k} = \frac{\pi^2 q}{2k k'^2 K^2}
\]
(Ramanujan, Collected Papers, p. 32)

§ 4. Next find
\[
\frac{dK}{dq} = \frac{dK}{dk} \frac{dk}{dq}
\]
which is, from our previous relations a simple rational function of \(K, E, k, q\).

Form
\[
\frac{d^2 K}{dq^2} \quad \text{and} \quad \frac{d^3 K}{dq^3}
\]
which again are simple rational functions of \(K, E, k, q\).

Eliminate \(E\) and \(k\) from the relations obtained in this section.

We then get a "polynomial" relation between
which is a third order algebraic differential equation

$$T(K, q, \frac{dK}{dq}, \frac{d^2K}{dq^2}, \frac{d^3K}{dq^3}) = 0$$

where \(T\) is a polynomial in the five quantities following \(T\).

§ 5. It would be nice to obtain "explicitly" the polynomial \(T\) of the last section, as well as those mentioned in the beginning of this paper i.e. algebraic differential equations satisfied by

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{1 - x^n}$$

i.e. a relation with an explicit polynomial \(T\) with

$$T(Y, X, Y', Y'', Y''') = 0$$

As far we know this has never been done.

Conclusion.

Professor Dwork (Princeton University) pointed out to us that G. Valerion in a monograph (in French) on automorphic forms in analysis obtained the algebraic differential equation explicitly in the case of the function

$$Y = \prod_{1}^{\infty} \frac{1 - x^n}{1}.$$