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ON ALGEBRAIC DIFFERENTIAL EQUATIONS SATISFIED BY SOME ELLIPTIC FUNCTIONS (II)

By

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Summary:

In (I) we obtained the "implicit" algebraic differential equation for the function defined by

$$Y = \sum_{n=1}^{\infty} \frac{a^n x^n}{1 - x^n}$$

where a is an odd positive integer, and conjectured that there are no algebraic differential equations for the case when a is an even integer. In this note we obtain a simple proof that (this has been known for almost 200 years)

$$Y = \sum_{n=1}^{\infty} \frac{2^n}{1}$$

(\(|x| < 1\))

satisfies an algebraic differential equation, and conjecture that

$$Y = \sum_{n=1}^{\infty} \frac{k^n}{1}$$

(k a positive integer > 2)

does not satisfy an algebraic differential equation.

§1 In the standard notation of elliptic functions we write

$$K = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$
Here $0 < k < 1$. We also define $k'$ by

$$k^2 + k'^2 = 1$$

and the corresponding integrals

$$K' = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k'^2 \sin^2 \phi}}$$

$$E' = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k'^2 \sin^2 \phi}}$$

We know that

$$KE' + K'E - KK' = \frac{\pi}{2}$$

which is often called Legendre's relation.

Our proof rests on several identities for $K, K', E, E'$ and their derivatives. Most important:

$$\sqrt{\frac{2k}{\pi}} = \frac{\infty}{-\infty} q^n$$

where

$$q = \frac{-\pi K'}{k}$$

§ 2. Next we need:

$$\frac{dK}{dk} = \frac{E - k'^2 K}{k k'^2}$$

$$\frac{dE}{dk} = \frac{E - K}{k}$$

Since

$$\frac{d}{dk} \frac{k'}{k'} = \frac{k}{k'}$$
we easily express
\[ \frac{dE'}{dk}, \frac{dK'}{dk} \]
as easy functions of K, K', E, E' and k. From these
\[ \frac{d}{dk} (EK' + KE' - KK') = 0 \]
Hence
\[ EK' + KE' - KK' = c \]
where c is a constant independent of k.
We easily see that \( c = \frac{\pi}{2} \).

§ 3. Next
\[ \frac{d q}{d k} = \frac{\pi^2 q}{2k k'2 K^2} \]
(Ramanujan, Collected Papers, p. 32)

§ 4. Next find
\[ \frac{d K}{d q} = \frac{d K}{d k} \frac{d k}{d q} \]
which is, from our previous relations a simple rational function of K, E, k, q.
Form
\[ \frac{d^2 K}{d q^2}, \frac{d^3 K}{d q^3} \]
which again are simple rational functions of K, E, k, q.
Eliminate E and k from the relations obtained in this section.
We then get a "polynomial" relation between
which is a third order algebraic differential equation

\[ T(K, q, \frac{dK}{dq}, \frac{d^2K}{dq^2}, \frac{d^3K}{dq^3}) = 0 \]

where \( T \) is a polynomial in the five quantities following \( T \).

§ 5. It would be nice to obtain "explicitly" the polynomial \( T \) of the last section, as well as those mentioned in the beginning of this paper i.e. algebraic differential equations satisfied by

\[ Y = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} \]

i.e. a relation with an explicit polynomial \( T \) with

\[ T(Y, X, Y', Y'', Y''') = 0 \]

As far we know this has never been done.

Conclusion.

Professor Dwork (Princeton University) pointed out to us that G. Valerion in a monograph (in French) on automorphic forms in analysis obtained the algebraic differential equation explicitly in the case of the function

\[ Y = \prod_{n=1}^{\infty} (1 - x^n). \]