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Godement–Jacquet L-function, some conjectures and some consequences

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Abstract. In this paper, we investigate the mean square estimate for the logarithmic derivative of the Godement–Jacquet L-function $L_f(s)$ assuming the Riemann hypothesis for $L_f(s)$ and Rudnick–Sarnak conjecture.

Keywords. Godement–Jacquet L-function, Rudnick–Sarnak conjecture, Hecke–Maass form, Riemann Hypothesis.

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1. Introduction

Let $n \geq 2$, and let $v = (v_1, v_2, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$. A Maass form [Gol06] for $SL(n, \mathbb{Z})$ of type v is a smooth function $f \in \mathcal{L}^2(SL(n,\mathbb{Z})\backslash\mathcal{H}^n)$ which satisfies

1. $f(\gamma z) = f(z)$, for all $\gamma \in SL(n,\mathbb{Z}), z \in \mathcal{H}^n$,
2. $Df(z) = \lambda_D f(z)$, for all $D \in \mathfrak{D}^n$ where $\mathfrak{D}^n$ is the center of the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{gl}(n,\mathbb{R})$ is the Lie algebra of $GL(n,\mathbb{R})$,
3. $\int\limits_{(SL(n,\mathbb{Z})\cap U)\backslash U} f(uz)\, du = 0,$

   for all upper triangular groups U of the form

$$U = \left\{ \begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & & * \\ & & \ddots & \\ & & & I_{r_b} \end{pmatrix} \right\},$$

   with $r_1 + r_2 + \cdots + r_b = n$. Here, I_r denotes the $r \times r$ identity matrix, and $*$ denotes arbitrary real entries.

A Hecke–Maass form is a Maass form which is an eigenvector for the Hecke operators algebra.
Let $f(z)$ be a Hecke–Maass form of type $v = (v_1, v_2, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$ for $SL(n,\mathbb{Z})$. Then it has the Fourier expansion

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A(m_1,\dots,m_{n-1})}{\prod_{j=1}^{n-1} |m_j|^{\frac{j(n-j)}{2}}}$$
$$\times W_J\left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v, \psi_{1,\dots,1,\frac{m_{n-1}}{|m_{n-1}|}} \right),$$

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where

$$M = \begin{pmatrix} m_1 \dots m_{n-2} \cdot |m_{n-1}| & & & & \\ & \ddots & & & \\ & & m_1 m_2 & & \\ & & & m_1 & \\ & & & & 1 \end{pmatrix},$$

$$A(m_1, \dots, m_{n-1}) \in \mathbb{C}, \qquad A(1, \dots, 1) = 1,$$

$$\psi_{1,\dots,1,\epsilon}\left(\begin{pmatrix} 1 & u_{n-1} & & & \\ & 1 & u_{n-2} & & * \\ & & \ddots & \ddots & \\ & & & 1 & u_1 \\ & & & & 1 \end{pmatrix}\right) = e^{2\pi i(u_1 + \dots + u_{n-2} + \epsilon u_{n-1})},$$

$U_{n-1}(\mathbb{Z})$ denotes the group of $(n-1) \times (n-1)$ upper triangular matrices with $1s$ on the diagonal and an integer entry above the diagonal and W_J is the Jacquet Whittaker function.

If $f(z)$ is a Maass form of type $(v_1, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$, then

$$\tilde{f}(z) := f(w \cdot (z^{-1})^T \cdot w),$$

$$w = \begin{pmatrix} & & & (-1)^{\left[\frac{n}{2}\right]} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

is a Maass form of type $(v_{n-1}, \dots, v_1)$ for $SL(n, \mathbb{Z})$ called the *dual Maass form.* If $A(m_1, \dots, m_{n-1})$ is the $(m_1, \dots, m_{n-1})$–Fourier coefficient of f, then $A(m_{n-1}, \dots, m_1)$ is the corresponding Fourier coefficient of $\tilde{f}$.

We note that the Fourier coefficients $A(m_1, \dots, m_{n-1})$ satisfy the multiplicative relations

$$A(m_1 m_1', \dots, m_{n-1} m_{n-1}') = A(m_1, \dots, m_{n-1}) \cdot A(m_1', \dots, m_{n-1}'),$$

if

$$(m_1 \dots m_{n-1}, m_1' \dots m_{n-1}') = 1,$$

$$A(m, 1, \dots, 1) A(m_1, \dots, m_{n-1}) = \sum_{\substack{\prod_{l=1}^{n} c_l = m \\ c_1 | m_1, c_2 | m_2, \dots, c_{n-1} | m_{n-1}}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \dots, \frac{m_{n-1} c_{n-2}}{c_{n-1}}\right),$$

and

$$A(m_{n-1}, \dots, m_1) = \overline{A(m_1, \dots, m_{n-1})}.$$

Definition 1.1. *[Gol06] The Godement–Jacquet L-function $L_f(s)$ attached to f is defined for $\Re(s) > 1$ by*

$$L_f(s) = \sum_{m=1}^{\infty} \frac{A(m,1,\dots,1)}{m^s} = \prod_p \prod_{i=1}^{n} (1-\alpha_{p,i}p^{-s})^{-1},$$

where $\{\alpha_{p,i}\}, 1 \leq i \leq n$ are the complex roots of the monic polynomial

$$X^n + \sum_{j=1}^{n-1} (-1)^j A(\overbrace{1,\dots,1}^{j-1 \text{ terms}}, p, 1, \dots, 1) X^{n-j} + (-1)^n \in \mathbb{C}[X], \quad \textit{and}$$

$$A(\overbrace{1,\dots,1}^{j-1}, p, 1, \dots, 1) = \sum_{1 \leq i_1 < \dots < i_j \leq n} \alpha_{p,i_1} \dots \alpha_{p,i_j}, \qquad \textit{for } 1 \leq j \leq n-1.$$

$L_f(s)$ satisfies the functional equation:

$$\begin{aligned} \Lambda_f(s) &:= \prod_{i=1}^{n} \pi^{\frac{-s+\lambda_i(v_f)}{2}} \Gamma\left(\frac{s-\lambda_i(v_f)}{2}\right) L_f(s) \\ &= \Lambda_{\tilde{f}}(1-s), \end{aligned}$$

where $\tilde{f}$ is the Dual Maass form.

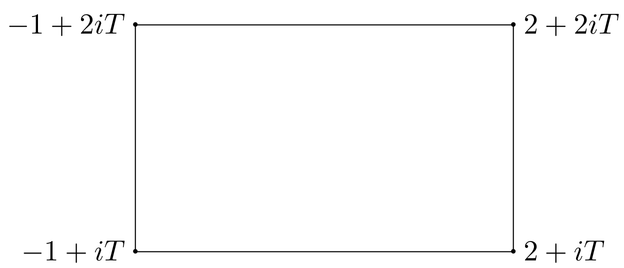
In the case of Godement–Jacquet L-function, Yujiao Jiang and Guangshi Lü [JiLu17] have studied cancellation on the exponential sum $\sum_{m \leq N} \mu(m) A(m,1) e^{2\pi i m\theta}$ related to $SL(3,\mathbb{Z})$ where $\theta \in \mathbb{R}$.

Throughout the paper, we assume that f is self dual i.e., $\tilde{f} = f$.
ϵ, ϵ_1 and η always denote any small positive constants.

If $N_f(T)$ denotes the number of zeros of $L_f(s)$ in the rectangle mentioned below, then from the functional equation and the argument principle of complex function theory we have,

$$N_f(T) \sim c(n) T \log T,$$

where $c(n)$ is a non zero constant depending only on the degree n of $L_f(s)$.



(i) The generalized Ramanujan conjecture:
It asserts that

$$\left|A(m,1,\dots,1)\right| \leq d_n(m)$$

where $d_n(m)$ is the number of representations of m as the product of n natural numbers. The current best estimates are due to Kim and Sarnak [Kim03] for $2 \le n \le 4$ and Luo, Rudnick and Sarnak for $n \ge 5$

$$\begin{aligned} \left|A(m)\right| &\le m^{\frac{7}{64}} d(m), \\ \left|A(m,1)\right| &\le m^{\frac{5}{14}} d_3(m), \\ \left|A(m,1,1)\right| &\le m^{\frac{9}{22}} d_4(m), \\ \left|A(m,1,\ldots,1)\right| &\le m^{\frac{1}{2}-\frac{1}{n^2+1}} d_n(m). \end{aligned}$$

We note that the generalized Ramanujan conjecture is equivalent to

$$\left|\alpha_{p,i}\right| = 1 \qquad \forall \text{ primes } p \text{ and } i = 1, 2, \ldots, n.$$

Other estimates are equivalent to

$$\left|\alpha_{p,i}\right| \le p^{\theta_n} \qquad \forall \text{ primes } p \text{ and } i = 1, 2, \ldots, n \text{ where}$$

$$\theta_2 := \frac{7}{64}, \qquad \theta_3 := \frac{5}{14}, \qquad \theta_4 := \frac{9}{22}, \qquad \theta_n := \frac{1}{2} - \frac{1}{n^2+1} (n \ge 5).$$

(ii) Ramanujan's generalized weak conjecture:
We formulate this conjecture as:
For $n \ge 2$, the inequality

$$\left|\alpha_{p,i}\right| \le p^{\frac{1}{4}-\epsilon_1}$$

holds for some small $\epsilon_1 > 0$, for every prime p and for $i = 1, 2, \ldots, n$. Of course, this weak conjecture holds good for $n = 2$. For $n \ge 3$, this conjecture is still open.

Taking the logarithmic derivative of $L_f(s)$, we have

$$-\frac{L_f'}{L_f}(s) := \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\Lambda(m) a_f(m)}{m^s}$$

where $a_f(m)$ is multiplicative and

$$a_f(p^r) = \sum_{i=1}^{n} \alpha_{p,i}^r$$

for any integer $r \ge 1$.
In particular,

$$a_f(p) = \sum_{i=1}^{n} \alpha_{p,i} = A(p, 1, \ldots, 1).$$

(iii) Rudnick–Sarnak conjecture:
For any fixed integer $r \ge 2$,

$$\sum_{p} \frac{\left|a_f(p^r)\right|^2 (\log p)^2}{p^r} < \infty.$$

We know that this conjecture is true for $n \leq 4$. (See [Ki06, RuSa96].)

(iv) Riemann hypothesis for $L_f(s)$:
It asserts that $L_f(s) \neq 0$ in $\Re(s) > \frac{1}{2}$.

The aim of this paper is to establish:

Theorem 1.1. *Ramanujan's weak conjecture implies Rudnick–Sarnak conjecture.*

Remark 1.2. Theorem 1.1 is indicated in [Ki06].

Theorem 1.3. *Assume $n \geq 5$ be any arbitrary but fixed integer. Let ϵ be any small positive constant and $T \geq T_0$ where T_0 is sufficiently large. Assume the Rudnick–Sarnak conjecture and Riemann hypothesis for $L_f(s)$. Then the estimate:*

$$\int_T^{2T} \left| \frac{L_f'}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon,\eta} T(\log T)^{2\eta}$$

holds for $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ with η being some constant satisfying $0 < \eta < \frac{1}{2}$.

Remark 1.4. Since Rudnick–Sarnak conjecture is true for $2 \leq n \leq 4$, Theorem 1.3 holds just with the assumption of Riemann hypothesis for $L_f(s)$ whenever $2 \leq n \leq 4$.

Remark 1.5. It is not difficult to see from our arguments that only assuming Riemann Hypothesis for $L_f(s)$, Theorem 1.3 can be upheld for any σ_0 satisfying $1 - \frac{1}{n^2+1} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ by using the bound $\theta_n = \frac{1}{2} - \frac{1}{n^2+1}$ of Luo, Rudnick and Sarnak.
It is also not difficult to see from our arguments that the generalized Ramanujan conjecture and the Riemann hypothesis for $L_f(s)$ together imply the bound

$$\int_T^{2T} \left| \frac{L_f'}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon} T \tag{1.1}$$

to hold for any σ_0 satisfying $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$.
Though we expect the bound stated in Equation 1.1 to hold unconditionally for σ_0 in the said range, this seems to be very hard.

2. Some Lemmas

Lemma 2.1. *If $f(s)$ is regular and*

$$\left| \frac{f(s)}{f(s_0)} \right| < e^M \qquad (M > 1)$$

in $|s - s_0| \leq r_1$, then for any constant b with $0 < b < \frac{1}{2}$,

$$\left| \frac{f'}{f}(s) - \sum_{\rho} \frac{1}{s - \rho} \right| \ll_b \frac{M}{r_1}$$

in $|s-s_0| \leq \left(\frac{1}{2}-b\right) r_1$, *where* ρ *runs over all zeros of* $f(s)$ *such that* $|\rho - s_0| \leq \frac{r_1}{2}$.

Proof. See Lemma α in Section 3.9 of [TiHe86] or see [RaSa91].

Lemma 2.2. *Let* $N_f^*(T)$ *denote the number of zeros of* $L_f(s)$ *in the region* $0 \leq \sigma \leq 1$, $0 \leq t \leq T$. *Then,*

$$N_f^*(T+1) - N_f^*(T) \ll_n \log T.$$

Proof. Let $n(r_1)$ denote the number of zeros of $L_f(s)$ in the circle with centre $2+iT$ and radius r_1. By Jensen's theorem,

$$\int_0^3 \frac{n(r_1)}{r_1} dr_1 = \frac{1}{2\pi} \int_0^{2\pi} \log\left| L_f\left(2+iT+3e^{i\theta}\right)\right| d\theta - \log\left|L_f(2+iT)\right|.$$

From the functional equation, we observe that

$$\left|L_f(s)\right| \ll_f t^A \qquad \text{for } -1 \leq \sigma \leq 5 \text{ where } A \text{ is some fixed positive constant,}$$

and hence we have,

$$\log\left| L\left(2+iT+3e^{i\theta}\right)\right| \ll A \log T.$$

Note that

$$\begin{aligned} \left|1 - \frac{\alpha_{p,i}}{p^{2+it}}\right| &\geq 1 - \frac{\left|\alpha_{p,i}\right|}{p^2} \\ &\geq 1 - \frac{p^{\frac{1}{2}}}{p^2} \\ &= 1 - \frac{1}{p^{\frac{3}{2}}}. \end{aligned}$$

Thus we have,

$$\begin{aligned} \left|L_f(2+it)\right| &= \prod_p \prod_{i=1}^n \left|\left(1 - \frac{\alpha_{p,i}}{p^{2+it}}\right)\right|^{-1} \\ &\leq \prod_p \prod_{i=1}^n \left(1 - \frac{1}{p^{\frac{3}{2}}}\right)^{-1} \\ &\leq \left(\zeta\left(\frac{3}{2}\right)\right)^n \\ &\ll_n 1. \end{aligned}$$

Therefore,

$$\int_0^3 \frac{n(r_1)}{r_1} dr_1 \ll A \log T + A \ll \log T,$$
$$\int_0^3 \frac{n(r_1)}{r_1} dr_1 \geq \int_{\sqrt{5}}^3 \frac{n(r_1)}{r_1} dr_1 \geq n(\sqrt{5}) \int_{\sqrt{5}}^3 \frac{dr_1}{r_1} \geq c.n(\sqrt{5}).$$

Hence,

$$N_f^*(T+1) - N_f^*(T) \ll_n \log T.$$

Lemma 2.3. *Let a_m (m=1,2,...,N) be any set of complex numbers. Then*

$$\int_T^{2T} \left| \sum_{m=1}^{N} a_m m^{-it} \right|^2 dt = \sum_{m=1}^{N} |a_m|^2 \left(T + O(m) \right).$$

Lemma 2.4. *Let b_m be any set of complex numbers such that $\sum m \left(|b_m|\right)^2$ is convergent. Then*

$$\int_T^{2T} \left| \sum_{m=1}^{\infty} b_m m^{-it} \right|^2 dt = \sum_{m=1}^{\infty} |b_m|^2 \left(T + O(m) \right).$$

Proof. See [MoVa74] or [Ram79] for Montgomery and Vaughan theorem.

Hereafter, $Y \geq 10$ is an arbitrary parameter depending on T which will be chosen suitably later. Also, σ_0 satisfies the inequality $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ for any small positive constant ϵ.

Lemma 2.5. *For $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$, we have*

$$\sum_{m > \frac{Y}{2}(\log Y)^2} \frac{m \left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \ll 1.$$

Proof. We have,

$$\sum_{m > \frac{Y}{2}(\log Y)^2} \frac{m \left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \ll \sum_{m > \frac{Y}{2}(\log Y)^2} \frac{m \left|\Lambda_f(m)\right|^2 e^{-\frac{m}{Y}} \frac{Y^2}{m^2}}{m^{2\sigma_0}}$$
$$\ll Y^2 \sum_{m > \frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{m}{Y}}}{m^{1+2\sigma_0}}.$$

Since $\frac{m}{Y} \geq \frac{1}{2}(\log Y)^2$ for $m \geq \frac{Y}{2}(\log Y)^2$, we have $e^{\frac{m}{Y}} \gg Y^B$ for any large positive constant B. Therefore,

$$\sum_{m>\frac{Y}{2}(\log Y)^2} \frac{m\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \ll \frac{Y^2}{Y^B} \sum_{m>\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{1+2\sigma_0}}$$
$$\ll 1.$$

Lemma 2.6. *Assuming Rudnick–Sarnak conjecture and taking Y sufficiently large, we have*

$$\sum_{m\leq\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \ll (\log Y)^2.$$

Proof. Note that

$$\sum_{m\leq\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \leq \sum_{p\leq\frac{Y}{2}(\log Y)^2} \frac{(\log p)^2\left|a_f(p)\right|^2}{p^{2\sigma_0}} + \sum_{r=2}^{\left[\frac{\log\frac{Y}{2}}{\log 2}\right]+1} \sum_{p} \frac{(\log p)^2\left|a_f(p^r)\right|^2}{(p^r)^{2\sigma_0}},$$

and

$$\left|a_f(p)\right| = \left|\sum_{i=1}^{n} \alpha_{p,i}\right| = \left|A(p,1,\ldots,1)\right|.$$

We have,

$$\sum_{m\leq Y} \frac{c_m}{m^l} = \int_1^Y \frac{d\left(\sum_{m\leq u} c_m\right)}{u^l}$$
$$= \frac{\sum_{m\leq u} c_m}{u^l}\Bigg|_1^Y - \int_1^Y (-l)\frac{\sum_{m\leq u} c_m}{u^{l+1}} du.$$

From Remark 12.1.8 of [Gol06], we have

$$\sum_{m_1^{n-1}m_2^{n-2}\ldots m_{n-1}\leq Y} \left|A(m_1,m_2,\ldots,m_{n-1})\right|^2 \ll_f Y.$$

Therefore,

$$\sum_{m\leq Y} \left|A(m,1,\ldots,1)\right|^2 \leq \sum_{m_1^{n-1}m_2^{n-2}\ldots m_{n-1}\leq Y} \left|A(m_1,m_2,\ldots,m_{n-1})\right|^2 \ll_f Y.$$

Taking $l = 2\sigma_0$ and $c_m = \left|A(m,1,\ldots,1)\right|^2$,

$$\sum_{m\le \frac{Y}{2}(\log Y)^2} \frac{\left|A(m,1,\ldots,1)\right|^2}{m^{2\sigma_0}} \ll 1.$$

Hence,

$$\sum_{p\le \frac{Y}{2}(\log Y)^2} \frac{(\log p)^2\left|a_f(p)\right|^2}{p^{2\sigma_0}} \ll (\log Y)^2 \sum_{m\le \frac{Y}{2}(\log Y)^2} \frac{\left|A(m,1,\ldots,1)\right|^2}{m^{2\sigma_0}} \ll (\log Y)^2.$$

By Rudnick–Sarnak conjecture and the bound $\left|\alpha_{p,i}\right| \le p^{\theta_n}$ with $\theta_n = \frac{1}{2} - \frac{1}{n^2+1}$,

$$\sum_{r\ge 2}\sum_{p} \frac{(\log p)^2\left|a_f(p^r)\right|^2}{p^r}$$

converges (as in proof of Theorem 1.1) and in particular,

$$\sum_{r=2}^{\left[\frac{\log \frac{Y}{2}}{\log 2}\right]+1}\sum_{p} \frac{(\log p)^2\left|a_f(p^r)\right|^2}{p^r} \ll 1.$$

Therefore,

$$\sum_{m\le \frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} \ll (\log Y)^2.$$

3. Proof of Theorem 1.1

Assuming $\left|\alpha_{p,i}\right| \le p^{\theta_n}$ with $\theta_n \le \frac{1}{4} - \epsilon_1$, we need to prove that for every integer $n \ge 5$ and for every integer $r \ge 2$,

$$\sum_{p} \frac{(\log p)^2\left|a_f(p^r)\right|^2}{p^r} < \infty.$$

It is enough to show that

$$\sum_{r=2}^{\infty}\sum_{p} \frac{(\log p)^2\left|a_f(p^r)\right|^2}{p^r} < \infty.$$

Using

$$a_f(p^r) := \sum_{i=1}^{n} \alpha_{p,i}^r \qquad \text{and} \qquad \left|\alpha_{p,i}\right| \le p^{\theta_n}$$

we get,

$$\begin{aligned}
\sum_{r=2}^{\infty}\sum_{p}\frac{(\log p)^2\left|a_f(p^r)\right|^2}{p^r} &\leq \sum_{r=2}^{\infty}\sum_{p}\frac{(\log p)^2\left(\sum\limits_{i=1}^{n}p^{r\theta_n}\right)^2}{p^r}\\
&=\sum_{r=2}^{\infty}\sum_{p}\frac{(\log p)^2 n^2 p^{2r\theta_n}}{p^r}\\
&\leq n^2\sum_{p}(\log p)^2\sum_{r=2}^{\infty}\frac{p^{2r\left(\frac{1}{4}-\epsilon_1\right)}}{p^r}\\
&=n^2\sum_{p}(\log p)^2\sum_{r=2}^{\infty}\frac{1}{p^{\frac{r}{2}+2r\epsilon_1}}\\
&=n^2\sum_{p}(\log p)^2\frac{p^{-(1+4\epsilon_1)}}{1-p^{-\left(\frac{1}{2}+2\epsilon_1\right)}}\\
&=n^2\sum_{p}(\log p)^2\frac{1}{p^{\frac{1}{2}+2\epsilon_1}\left(p^{\frac{1}{2}+2\epsilon_1}-1\right)}\\
&\ll_{n,\epsilon_1} 1.
\end{aligned}$$

This proves Theorem 1.1.

4. Proof of Theorem 1.3

First, we wish to approximate $\frac{L_f'}{L_f}(s)$ uniformly for $\frac{1}{2}<\sigma_0\leq\sigma\leq\sigma_1<1$ when $T\leq t\leq 2T$. We assume throughout below the Riemann hypothesis for $L_f(s)$.

From the work of Godement–Jacquet [GoJa06], it is known that the function $L_f(s)$ is of finite order in any bounded vertical strip. Hence, we can very well assume that

$$L_f(s)\ll T^A=e^{A\log T}$$

for $-1\leq\sigma\leq 2$, $T\leq t\leq 2T$ and A some fixed positive constant.

Taking $s_0=2+it$ with $t\in\mathbb{R}$, we have

$$L_f(2+it)=\prod_{p}\prod_{i=1}^{n}\left(1-\frac{\alpha_{p,i}}{p^{2+it}}\right)^{-1}.$$

Observe that

$$\begin{aligned}\left|1-\frac{\alpha_{p,i}}{p^{2+it}}\right| &\leq 1+\frac{|\alpha_{p,i}|}{p^2}\\ &\leq 1+\frac{p^{\theta_n}}{p^2}\\ &= 1+\frac{1}{p^{2-\theta_n}}\\ &\leq 1+\frac{1}{p^{\frac{3}{2}}}\end{aligned}$$

because $\theta_n \leq \frac{1}{2}$ for $n \geq 2$.

Therefore,

$$\begin{aligned}\left|L_f(2+it)\right| &\geq \prod_p \prod_{i=1}^{n}\left(1+\frac{1}{p^{\frac{3}{2}}}\right)^{-1}\\ &= \prod_p \left(1+\frac{1}{p^{\frac{3}{2}}}\right)^{-n}\\ &= \prod_p \left(\frac{1-\frac{1}{p^{\frac{3}{2}}}}{1-\frac{1}{p^3}}\right)^{n}\\ &= \left(\frac{\zeta(3)}{\zeta\left(\frac{3}{2}\right)}\right)^{n}\end{aligned}$$

which is a constant depending only on n. Therefore, $L_f(2+it) \neq 0 \ \forall \ t \in \mathbb{R}$.

Hence from Lemma 2.1, with $r = 12$, $s_0 = 2+iT$, $f(s) = L_f(s)$, $M = A\log T$, we obtain

$$-\frac{L_f'}{L_f}(s) = \sum_{|s-s_0|\leq 6} \frac{1}{s-\rho} + O(\log T).$$

For $|s-s_0| \leq 3$ and so in particular for $-1 \leq \sigma \leq 2, t = T$, replacing T by t in the particular case, we obtain

$$-\frac{L_f'}{L_f}(s) = \sum_{|\rho-s_0|\leq 6} \frac{1}{s-\rho} + O(\log t).$$

Any term occurring in $\sum\limits_{|t-\gamma|\leq 1} \frac{1}{s-\rho}$ but not in $\sum\limits_{|s-s_0|\leq 6} \frac{1}{s-\rho}$ is bounded and the number of such terms does not exceed

$$N_f^*(t+6) - N_f^*(t-6) \ll \log t,$$

where $N_f^*(t)$ is the number of zeros of $L_f(s)$ in the region $0 \leq \sigma \leq 1$ and $0 \leq t \leq T$. Thus, we get

$$-\frac{L_f'}{L_f}(s) = \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O(\log t).$$

Assume $\frac{1}{2} < \sigma < 1$ and $T \le t \le 2T$, then

$$\sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{Y}} = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L_f'}{L_f}(s+w)\Gamma(w)Y^w dw.$$

Note also that from the above reasoning

$$\frac{L_f'}{L_f}(s) \ll \log t \qquad \text{on any line } \sigma \neq \frac{1}{2}.$$

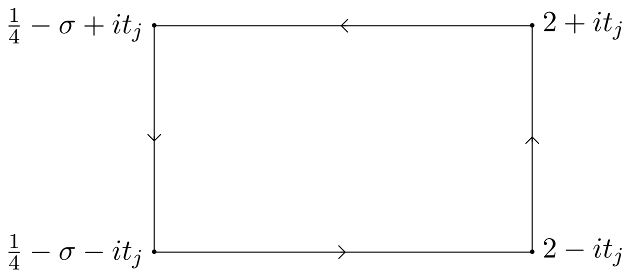
Also,

$$\frac{L_f'}{L_f}(s) \ll \frac{\log t}{\min(|t-\gamma|)} + \log t \qquad \text{uniformly for } -1 \le \sigma \le 2.$$

From Lemma 2.2, we observe that each interval $(j, j+1)$ contains values of t whose distance from the ordinate of any zero exceeds $\frac{A}{\log j}$, there is a t_j in any such interval for which

$$\frac{L_f'}{L_f}(s) \ll (\log t)^2 \qquad \text{where } -1 \le \sigma \le 2 \text{ and } t = t_j.$$

Applying Cauchy's residue theorem to the rectangle, we get



$$\frac{1}{2\pi i}\left(\int_{2-it_j}^{2+it_j} + \int_{2+it_j}^{\frac{1}{4}-\sigma+it_j} + \int_{\frac{1}{4}-\sigma+it_j}^{\frac{1}{4}-\sigma-it_j} + \int_{\frac{1}{4}-\sigma-it_j}^{2-it_j}\right) \frac{L_f'}{L_f}(s+w)\Gamma(w)Y^w dw$$
$$= \frac{L_f'}{L_f}(s) + \sum_{-t_j<\gamma<t_j} \Gamma(\rho - s)Y^{\rho-s}.$$

In the sum appearing on the right hand side above, zeros ρ are counted with its multiplicity if there are any multiple zeros. The integrals along the horizontal lines tend to zero as $j \to \infty$ since gamma function decays exponentially and Y is going to be at most a power of T only, so that

$$\sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{Y}} = \frac{1}{2\pi i} \int_{\frac{1}{4}-\sigma-i\infty}^{\frac{1}{4}-\sigma+i\infty} \frac{L_f'}{L_f}(s+w)\Gamma(w)Y^w dw - \frac{L_f'}{L_f}(s) - \sum_{\rho} \Gamma(\rho-s)Y^{\rho-s}.$$

Note that $\Gamma(w) \ll e^{-A|v|}$ so that the integral on $\Re(w) = \frac{1}{4} - \sigma$ is

$$\begin{aligned}
&\ll \int_{-\infty}^{\infty} e^{-A|v|} \log(|t+v|+2) Y^{\frac{1}{4}-\sigma} dv \\
&\ll \int_{0}^{2t} e^{-A|v|} \log(10|t|+2) Y^{\frac{1}{4}-\sigma} dv + \left(\int_{-\infty}^{0} + \int_{2t}^{\infty} \right) e^{-A|v|} \log(|v|+10) Y^{\frac{1}{4}-\sigma} dv \\
&\ll Y^{\frac{1}{4}-\sigma} \log T + Y^{\frac{1}{4}-\sigma} \\
&\ll Y^{\frac{1}{4}-\sigma} \log T.
\end{aligned}$$

Note that for $\frac{1}{2} < \sigma_0 \le \sigma \le \sigma_1 < 1$,

$$\left|\Gamma(\rho - s)\right| < A_1 e^{-A_2|\gamma - t|}$$

uniformly for σ in the said range. Therefore,

$$\sum_{\rho} \left|\Gamma(\rho - s)\right| < A_1 \sum_{\rho} e^{-A_2|\gamma - t|} = A_1 \sum_{m=1}^{\infty} \sum_{m-1 \le \gamma \le m} e^{-A_2|t-\gamma|}.$$

The number of terms in the inner sum is

$$\ll \log(|t|+m) \ll \log|t| + \log(m+1)$$

and hence

$$\sum_{\rho} \left|\Gamma(\rho - s)\right| \ll \sum_{m=1}^{\infty} e^{-A_2 m} (\log|t| + \log(m+1)) \ll \log T,$$

$$\left| \sum_{\rho} \Gamma(\rho - s) Y^{\rho - s} \right| \ll Y^{\frac{1}{2}-\sigma} \log T.$$

Thus for $\frac{1}{2} < \sigma_0 \le \sigma \le \sigma_1 < 1$, we have

$$-\frac{L_f'}{L_f}(s) = \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{Y}} + O_f(Y^{\frac{1}{2}-\sigma} \log T).$$

Thus for $\frac{1}{2} + \epsilon \le \sigma_0 \le 1 - \epsilon$ and $T \le t \le 2T$, we obtain

$$\left| \frac{L_f'}{L_f}(\sigma_0 + it) \right|^2 \ll \left| \sum_{m=1}^{\infty} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0 + it}} \right|^2 + \left(Y^{\frac{1}{2}-\sigma_0} \log T \right)^2.$$

Thus,

$$\int_T^{2T} \left| \frac{L_f'}{L_f}(\sigma_0 + it) \right|^2 dt \ll_f \int_T^{2T} \left| \sum_{m=1}^{\infty} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0 + it}} \right|^2 dt + Y^{1-2\sigma_0} T (\log T)^2.$$

We note that

$$\left|\sum_{m=1}^{\infty}\frac{\Lambda_f(m)e^{-\frac{m}{Y}}}{m^{\sigma_0+it}}\right|^2 \ll \left|\sum_{m\leq\frac{Y}{2}(\log Y)^2}\frac{\Lambda_f(m)e^{-\frac{m}{Y}}}{m^{\sigma_0+it}}\right|^2 + \left|\sum_{m>\frac{Y}{2}(\log Y)^2}\frac{\Lambda_f(m)e^{-\frac{m}{Y}}}{m^{\sigma_0+it}}\right|^2,$$

and hence

$$\int_T^{2T}\left|\frac{L_f'}{L_f}(\sigma_0+it)\right|^2 dt \ll_f \int_T^{2T}\left|\sum_{m\leq\frac{Y}{2}(\log Y)^2}\frac{\Lambda_f(m)e^{-\frac{m}{Y}}}{m^{\sigma_0+it}}\right|^2 + \int_T^{2T}\left|\sum_{m>\frac{Y}{2}(\log Y)^2}\frac{\Lambda_f(m)e^{-\frac{m}{Y}}}{m^{\sigma_0+it}}\right|^2 + Y^{1-2\sigma_0}T(\log T)^2.$$

By Montgomery–Vaughan theorem (Lemmas 2.3 and 2.4) and Lemma 2.5, we get

$$\begin{aligned}\int_T^{2T}\left|\frac{L_f'}{L_f}(\sigma_0+it)\right|^2 dt &\ll_f \sum_{m\leq\frac{Y}{2}(\log Y)^2}\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}\left(T+O(m)\right)\\ &\quad+\sum_{m>\frac{Y}{2}(\log Y)^2}\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}\left(T+O(m)\right)+Y^{1-2\sigma_0}T(\log T)^2\\ &\ll_f T\sum_{m\leq\frac{Y}{2}(\log Y)^2}\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}+\sum_{m\leq\frac{Y}{2}(\log Y)^2}m\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}\\ &\quad+T\sum_{m>\frac{Y}{2}(\log Y)^2}\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}+\sum_{m>\frac{Y}{2}(\log Y)^2}m\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}\\ &\quad+Y^{1-2\sigma_0}T(\log T)^2.\end{aligned}$$

By Lemmas 2.5 and 2.6, we obtain

$$\int_T^{2T}\left|\frac{L_f'}{L_f}\left(\frac{1}{2}+\epsilon+it\right)\right|^2 dt \ll_{f,n,\epsilon} T(\log Y)^2+Y(\log Y)^4+Y^{1-2\sigma_0}T(\log T)^2.$$

We choose $Y=\exp\{(\log T)^\eta\}$ with any η satisfying $0<\eta<\frac{1}{2}$ so that we obtain

$$\int_T^{2T}\left|\frac{L_f'}{L_f}(\sigma_0+it)\right|^2 dt \ll_{f,n,\epsilon,\eta} T(\log T)^{2\eta}.$$

This proves Theorem 1.3.

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