Hardy-Ramanujan Journal 45 (2022), 57-73 submitted 14/12/2020, accepted 28/02/2022, revised 06/04/2022

On fixed divisors of the values of the minimal polynomials over $\mathbb{Z}$ of algebraic numbers

M. Ayad, A. Bayad and O. Kihel

Abstract. Let K be a number field of degree n, A be its ring of integers, and A_n (resp. K_n) be the set of elements of A (resp. K) which are primitive over $\mathbb{Q}$. For any $\gamma \in K_n$, let $F_\gamma(x)$ be the unique irreducible polynomial in $\mathbb{Z}[x]$, such that its leading coefficient is positive and $F_\gamma(\gamma) = 0$. Let $i(\gamma) = \text{gcd}_{x\in\mathbb{Z}} F_\gamma(x)$, $i(K) = \text{lcm}_{\theta\in A_n} i(\theta)$ and $\hat{i}(K) = \text{lcm}_{\gamma\in K_n} i(\gamma)$. For any $\gamma \in K_n$, there exists a unique pair (θ, d), where $\theta \in A_n$ and d is a positive integer such that $\gamma = \theta/d$ and $\theta \not\equiv 0 \pmod p$ for any prime divisor p of d. In this paper, we study the possible values of $\nu_p(d)$ when $p|i(\gamma)$. We introduce and study a new invariant of K defined using $\nu_p(d)$, when γ describes K_n. In the last theorem of this paper, we establish a generalisation of a theorem of MacCluer.

Keywords. Values of polynomials, Denominators of algebraic numbers, Splitting of prime numbers.

2010 Mathematics Subject Classification. 11R04, 12Y05.

1. Introduction

Let K be a number field of degree $n \geq 2$ and A be its ring of integers. Denote by A_n (resp. K_n) be the set of elements of A (resp. K) which are primitive over $\mathbb{Q}$, that is those elements which generate K over $\mathbb{Q}$. For any primitive polynomial $g(x) \in \mathbb{Z}[x]$, we define the integer $i(g)$ by

$$i(g) = \text{gcd}_{x\in\mathbb{Z}}\, g(x).$$

Let γ be an algebraic number. When we refer to the minimal polynomial of γ over $\mathbb{Z}$, we mean the unique polynomial $F_\gamma(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$, irreducible such that $a_n > 0$ and $F_\gamma(\gamma) = 0$. The leading coefficient a_n will be denoted by $c(\gamma)$. We set $i(\gamma) = i(F_\gamma)$. In [GuMc70], Gunji and McQuillan defined the integer $i(K)$ by

$$i(K) = \text{lcm}_{\theta\in A_n}\, i(\theta).$$

MacCluer [Mac71] proved that a given prime p divides $i(K)$ if and only if the number of prime ideals of A lying over p is at least equal to p. In [GuMc70] or [AyKi11], it is proved that there exists $\theta \in A_n$ such that

$$i(K) = i(\theta).$$

The smallest positive integer d such that $d\gamma$ is an algebraic integer is called the denominator of γ and will be denoted by $d(\gamma)$.

Arno et al proved in [ARW96] that the density of the set of the algebraic numbers γ such that $c(\gamma) = d(\gamma)$ is equal to $1/\zeta(3) = 0.8319\cdots$. In [ABK15], for a fixed number field K, the set

$$T_p(k) = \{t \geq 1,\ \text{there exists } \gamma \in K_n,\ \nu_p(d(\gamma)) = k \text{ and } \nu_p(c(\gamma)) = t\}$$

We thank episciences.org for providing open access hosting of the electronic journal *Hardy-Ramanujan Journal*

is connected to the splitting of the prime p in K.

In this paper, among other results, we study the possible values of $\nu_p(d)$ when $p \mid i(\gamma)$. After recalling some lemmas in section 2, it is proved in Theorem 3.1 that if $\theta \not\equiv 0 \pmod{pA}$ and $p|i(\theta/p^k)$, for some positive integer k, then the p-adic valuation of the leading coefficient of the minimal polynomial of θ/p^k belongs to the set $\{k, k+1, \dots, (n-p)k\}$. Furthermore any element of this set may occur. Section 4 shows that given $\theta \in A_n$ such that $\theta \not\equiv 0 \pmod{pA}$, it is possible to find explicitly the values of $k \in \mathbb{N}$, if any, such that $p|i(\theta/p^k)$. Fix $\theta \in A_n$ such that $\theta \not\equiv 0 \pmod{pA}$ and define the set $V_p(\theta) = \{k \in \mathbb{N} \; ; \; p|i(\theta/p^k)\}$. On the one hand it is shown that $|V_p(\theta)|$ for $\theta \in A_n$ is bounded by some constant depending on n and p. On the other hand the values of k are bounded by a constant depending on p and $\nu_p(N_{K/\mathbb{Q}}(\theta))$, where $\nu_p(a)$ denotes the p-adic valuation of a. Section 5 deals with this last bound. It is shown that, even if we fix the field K, the values of k may be greater than any given positive constant. In this section, we give examples of Galois number fields K of degree 4 (resp. 3) for which the set of the values of k, when θ runs in A_n is $\{0\}$ (resp. $\mathbb{N}$). Throughout this paper we denote by $\mathbb{N}$ the set of nonnegative integers. The paper ends with remarks and open questions.

2. Indices and denominators of algebraic numbers

Let K be a number field of degree n, A its ring of integers, $\gamma \in K_n$ and let $g(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ be the unique primitive polynomial of degree n such that $a_n > 0$ and $g(\gamma) = 0$. We denote this polynomial $F_r(x)$. This polynomial will be called the minimal polynomial of γ over $\mathbb{Z}$. The leading coefficient a_n will be denoted $c(\gamma)$.

Let

$$\mathfrak{I}(\gamma) = \{m \in \mathbb{Z}, m\gamma \in A\},$$

then $\mathfrak{I}(\gamma)$ is a nonzero ideal of $\mathbb{Z}$, hence a principal ideal generated by some positive integer denoted by $d(\gamma)$. The integer $d(\gamma)$ is called the denominator of γ. Since $c(\gamma) \in \mathfrak{I}(\gamma)$, then $d(\gamma) \mid c(\gamma)$. Write $\gamma = \frac{\theta}{d(\gamma)}$, where $\theta \in A_n$, then θ is unique and we call it the numerator of γ. Let $f(x)$ be the minimal polynomial of θ over $\mathbb{Q}$, then $g(x) = f(d(\gamma)x)/cont(f(d(\gamma)x)$, where the abbreviation $cont(h(x))$ denotes the content of the polynomial $h(x)$. From [ARW96] we have the following result:

Lemma 2.1. *For any prime p, we have*

$$\nu_p(d(\gamma)) = \max\left(0, \max_{j=0}^{n-1} \left\lceil \frac{\nu_p(a_n) - \nu_p(a_j)}{n-j} \right\rceil\right). \tag{2.1}$$

From Lemma 2.1 we see that any prime factor p of $c(\gamma)$ divides $d(\gamma)$.
Summarizing the relations between $c(\gamma)$ and $d(\gamma)$, we have:

Remark 2.1. *Let K be a number field of degree n and $\gamma \in K_n$, then $d(\gamma)$ and $c(\gamma)$ have the same prime factors and for any prime p, we have $\nu_p(d(\gamma)) \le \nu_p(c(\gamma)) \le n\nu_p(d(\gamma))$.*

For any $\theta \in A_n$, let $F_\theta(x)$ its minimal polynomial over $\mathbb{Q}$. Following [GuMc70], we define the integers

$$i(\theta) = \gcd_{x\in\mathbb{Z}} F_\theta(x) \text{ and } i(K) = \mathrm{lcm}_{\theta\in A_n} i(\theta). \tag{2.2}$$

For the integers $i(\theta)$ and $i(K)$ we have the following results:

i) C. R. MacCluer proved in [Mac71] that for a given prime number p, p divides $i(K)$ if and only if the number of prime ideals of A lying over p is at least equal to p.

ii) In [GuMc70] and [AyKi11], it is proved that there exists $\theta \in A_n$ such that $i(K) = i(\theta)$, and that $i(K) = \mathrm{lcm}_{\theta \in A}\, i(\theta)$.

We extend the definition of $i(\theta)$ and $i(K)$ to algebraic numbers as follows. Given any $\gamma \in K_n$, we define

$$i(\gamma) := \gcd_{x\in\mathbb{Z}} F_\gamma(x) \text{ and } \hat{\imath}(K) = \mathrm{lcm}_{\gamma\in K_n}\, i(\gamma).$$

We quote from [AyKi11] the following result.

Lemma 2.2. *Let $g(x) \in \mathbb{Z}[x]$. Write $g(x)$ in the form*

$$g(x) = b_n(x)_n + \cdots + b_1(x)_1 + b_0,$$

where $b_0, \ldots, b_n \in \mathbb{Z}$,

$$(x)_0 = 1 \text{ and } (x)_j := x(x-1)\cdots(x-(j-1)), \text{ for } j \geq 1.$$

Then, we have the identity

$$i(g) = \gcd_{j=0}^{n}\left(j!b_j\right).$$

Corollary 2.1. *The integers $i(\gamma)$, $i(K)$ and $\hat{\imath}(K)$ divide $n!$.*

Remark 2.2. *Clearly, from the definitions of $i(K)$ and $\hat{\imath}(K)$, we see that $i(K)$ divides $\hat{\imath}(K)$.*

3. Study of the denominators of some algebraic numbers

We prove the following lemma, which is useful for the rest of this paper:

Lemma 3.1. *Let p be a prime number, γ be an algebraic number and $d(\gamma)$ be its denominator. Write $d(\gamma)$ in the form $d(\gamma) = p^k \cdot q$, where $\gcd(p, q) = 1$ and let $\mu = q\gamma$. Then we have*

$$d(\mu) = p^k,\ \nu_p(c(\mu)) = \nu_p(c(\gamma)) \text{ and } \nu_p(i(\gamma)) = \nu_p(i(\mu)).$$

Proof: Let $\gamma = \theta/_p k_q$ where θ is an algebraic integer such that $\theta \not\equiv 0 \,(\mathrm{mod}\, p)$ and $\theta \not\equiv 0 \,(\mathrm{mod}\ l)$ for any prime factor l of q. Then $\mu = q\gamma = \theta/_p k$, hence $d(\mu) = p_k$. Let n be the degree of γ over $\mathbb{Q}$ and let $g(x) = a_n x_n + \cdots + a_1 x + a_0$ be its minimal polynomial over $\mathbb{Z}$. Since $a_n = c(\gamma)$ and $d(\gamma)|c(\gamma)$, then $q|a_n$. This implies that the polynomial $h(x) = q^{n-1}g(x/q) = (a_n/q)x^n + a_{n-1}x^{n-1} + \cdots + a_1 q^{n-2}x + a_0 q^{n-1}$ has integral coefficients and vanishes for μ. Therefore, the minimal polynomial of μ over $\mathbb{Z}$ is given by $f(x) = h(x)/cont(h)$. Write $cont(h)$ in the form $cont(h) = gcd(q, cont(h)) \cdot \lambda$, where λ is a positive integer. We show that $\lambda = 1$. Suppose that there exists a prime number $l|\lambda$, then $l|cont(h)$ and $l \nmid q$. Since $l|a_0 q^{n-1}$, $a_1 q^{n-2}$,..., a_{n-1}, $a_n|q$, then $l|a_0, \ldots, a_n$ which is a contradiction, hence $\lambda = 1$. Therefore $cont(h)|q$. Since

$$c(\mu) = (a_n/q)/cont(h) = a_n/qcont(h) = c(\gamma)/qcont(h),$$

then $V_p(c(\mu)) = V_p(c(\gamma))$. We now prove the last statement. Suppose that $p^u|i(\gamma)$ for some positive integer u and $x_0 \in \mathbb{Z}$. Since $gcd(p,q) = 1$, there exists $y_0 \in \mathbb{Z}$ such that the congruence $qy_0 \equiv x_0 \,(\mathrm{mod}\, p^u)$ holds. In particular $g(y_0) \equiv 0 \,(\mathrm{mod}\, p^u)$, so that we have, $f(x_0) = h(x_0)/cont(h) = q^{n-1}g(y_0)/cont(h) \equiv 0 \,(\mathrm{mod}\, p^u)$. Since x_0 was arbitrary, then $p^u|i(\mu)$. Conversely, suppose that $p^u|i(\mu)$ and let x_0 and y_0 as above. Then $f(x_0) = 0 \,(\mathrm{mod}\, p^u)$, hence $q^{n-1}g(y_0)/cont(h) \equiv 0 \,(\mathrm{mod}\, p^u)$, thus $g(y_0) \equiv 0 \,(\mathrm{mod}\, p^u)$. Therefore $p^u|i(\gamma)$.
We state the main result of this section.

Theorem 3.1. *Let K be a number field of degree n over $\mathbb{Q}$ and p be a prime number, Let $\gamma \in K_n$ such that $p|i(\gamma)$, $c = c(\gamma)$, $d = d(\gamma)$ and $k = v_p(d) \geqslant 1$. Then $p < n$ and $k \le \nu_p(c) \le (n-p)k$.*

Proof: Set $d = p^k q$ with $gcd(p,q) = 1$. Let $\mu = q\gamma$, then $d(\mu) = p^k$ and by Lemma 3.1, we have $v_p(c(\mu)) = v_p(c(\gamma))$ and $v_p i(\mu) = v_p i(\gamma) \ge 1$. Therefore, $p|i(\mu)$. The minimal polynomial of μ over $\mathbb{Z}$ has the form:

$$g(x) = p^t(x)_n + b_{n-1}(x)_{n-1} + \cdots + b_1(x)_1 + b_0, \text{ with } k \le t \le nk.$$

By corollary 2.1, $p|n!$, hence $p \le n$. If $p = n$, since $p|j!b_j$ for all $j = 0, \dots, n-1$, then we conclude that $p|b_0, b_1, \dots, b_{n-1}$. Therefore $g(x)$ is reducible in $\mathbb{Z}[x]$, which is a contradiction, hence $p < n$. In this case, since $p|b_0, b_1, \dots, b_{p-1}$, then we may write $g(x)$ in the form

$$\begin{aligned} g(x) = x(x-1)\cdots(x-(p-1))\Big(p^t x^{n-p} + \tilde{a}_{n-p-1}x^{n-p-1} + \cdots + \tilde{a}_1 x + \tilde{a}_0\Big) \\ +p\Big(c_{p-1}x^{p-1} + \cdots + c_1 x + c_0\Big) \end{aligned}$$

where all the coefficients $\tilde{a}_i, c_j$ are integral. Since $g(x)$ is irreducible in $\mathbb{Z}[x]$, there exists $j \in \{0, \dots, n-p-1\}$ such that $p \not| \tilde{a}_j$. Denote by j_o the greatest of these integers. Let $\theta \in A_n$ be the unique element such that $\gamma = \frac{\theta}{p^k}$. Then we have

$$\begin{aligned} \theta(\theta - p^k)\cdots(\theta-(p-1)p^k)\Big(p^t\theta^{n-p} + p^k\tilde{a}_{n-p-1}\theta^{n-p-1} \\ + \cdots + p^{k(n-p-j_o)}\tilde{a}_{j_0}\theta^{j_0} + \cdots + p^{k(n-p-1)}\tilde{a}_1\theta + p^{k(n-p)}\tilde{a}_0\Big) \\ +p\cdot p^{k(n-(p-1))}\left(c_{p-1}\theta^{p-1} + c_{p-2}p^k\theta^{p-2} + \cdots + c_0 p^{k(p-1)}\right) = 0. \end{aligned}$$

Write this equation in the form:

$$p^t\theta^n + u_{n-1}\theta^{n-1} + \cdots + u_p\theta^p + \cdots + u_1\theta + u_0 = 0.$$

Since θ is integral, it follows in particular that $p^t|u_j$ for $j = p, \dots, n-1$. We can set

$$\theta(\theta - p^k)\cdots(\theta - (p-1)p^k) = \theta^p + \sigma_1\theta^{p-1} + \cdots + \sigma_{p-1}\theta.$$

Then

$$\left\{\begin{array}{rcl} \sigma_1 & = & -(p^k + \cdots + p^k(p-1)) = p^k\frac{p(p-1)}{2} \\ \sigma_2 & = & p^{2k} \displaystyle\sum_{\substack{i\neq j \\ i,j\in\{1,\dots,p-1\}}} ij \\ & \cdots & \\ \sigma_{p-1} & = & (-1)^{p-1}p^{k(p-1)}(p-1)!. \end{array}\right.$$

We have

$$(x^p+\sigma_1x^{p-1}+\cdots+\sigma_{p-1}x)\Big(p^tx^{n-p}+\cdots+p^{k(n-p-j_0)}\tilde{a}_{j_0}x^{j_0}+\cdots+\tilde{a}_0p^{k(n-p)}\Big)+$$
$$pp^{k(n-(p-1))}\Big(c_{p-1}x^{p-1}+\cdots+c_1x\Big)=p^tx^n+u_{n-1}x^{n-1}+\cdots+u_px^p+\cdots+u_1x+u_0,$$

hence

$$\begin{cases} u_{n-1} &= p^k\tilde{a}_{n-p-1}+p^t\sigma_1 \\ u_{n-2} &= p^{2k}\tilde{a}_{n-p-2}+p^k\tilde{a}_{n-p-1}\sigma_1+p^t\sigma_2 \\ & \cdots \\ u_{j_0+p} &= \tilde{a}_{j_0}p^{k(n-p-j_0)}+\tilde{a}_{j_0+1}p^{k(n-p-(j_0+1)}\sigma_1+\cdots+\tilde{a}_{j_0+m}p^{k(n-p-(j_0+m)}\sigma_m \\ & + \quad \cdots+p^t\sigma_{n-(j_0+p)} \end{cases}$$

The first equation implies that $p^t|p^k\tilde{a}_{n-p-1}$. Then the second implies that $p^t|p^{2k}\tilde{a}_{n-p-2}$. Iterating the process, the last equation gives $p^t|\tilde{a}_{j_0}p^{k(n-p-j_0)}$. Since $p\not|\tilde{a}_{j_0}$, then

$$t\le k(n-p-j_0)\le k(n-p).$$

Theorem 3.2. *Let p be a prime number, n and k be positive integers, $p<n$. Then for any integer t, such that $k\le t\le (n-p)k$, there exist infinitely many algebraic numbers $\gamma\in\mathbb{C}$ of degree n such that $p|i(\gamma)$, $\nu_p(c(\gamma))=t$ and $\nu_p(d(\gamma))=k$.*

Proof: Dividing t by k, we have two possibilities:

$$t = (n-i)k+\alpha \text{ with } 0<\alpha<k \text{ and } p<i\le n-1 \tag{3.3}$$
$$t = (n-i)k \text{ with } p\le i\le n-1 \tag{3.4}$$

- First case: $t=(n-i)k+\alpha$ with $0<\alpha<k$ and $p<i\le n-1$. Then, we have

$$t>(n-i)\alpha+\alpha=\alpha(n-i+1),\ \text{hence } \alpha<\frac{t}{n-(i+1)}.$$

On the other hand, choose integers $a_0,\dots,a_n$ such that

$$\begin{cases} \gcd(a_0,\dots,a_n)=1, \\ \nu_p(a_j)=t,\ \text{for } j>i \\ \nu_p(a_i)=\alpha, (\text{ note that } \alpha\neq 0) \\ \nu_p(a_{i-1})=0 \\ \nu_p(a_j)=1,\ \text{for } j<i-1. \end{cases} \tag{3.5}$$

Consider the polynomials

$$f(x)=\sum_{j=0}^n a_j(x)_j=\sum_{j=0}^n \tilde{a}_jx^j$$

and

$$g(x)=\tilde{a}_nx^n+q^{e_{n-1}}\tilde{a}_{n-1}x^{n-1}+\cdots+q^{e_1}\tilde{a}_1x+q\tilde{a}_0:=\sum_{j=0}^n b_jx^j,$$

where q is a prime number such that $q \equiv 1 \pmod p$, $q \not| \tilde{a}_0$ and the exponents e_j are arbitrary fixed positive integers. Clearly $g(x)$ is irreducible in $\mathbb{Z}[x]$ by Eisenstein's Theorem. Let γ be a root of g(x). Since $p \leq i-1$, then by Lemma 2.2, we conclude that $p|i(f)$ and since $g(x) \equiv f(x) \pmod p$ for any $x \in \mathbb{Z}$, then $p|i(\gamma)$. We look at the p-adic valuations of the $\tilde{a}_j$. Recall that $\tilde{a}_n = c(\gamma)$ and $\tilde{a}_0 = a_0$, hence $\nu_p(\tilde{a}_n) = t$ and $\nu_p(\tilde{a}_0) = 1$. We claim that

$$\begin{cases} \nu_p(\tilde{a}_j) \geq t, \text{ for } j > i \\ \nu_p(\tilde{a}_i) = \alpha, \\ \nu_p(\tilde{a}_{i-1}) = 0. \end{cases} \tag{3.6}$$

For any $j \geq 1$ we have $\tilde{a}_j = a_j + \sum_{l=j+1}^{n} a_l c_l$, where $c_l \in \mathbb{Z}$ for any l.
If $j > i$ then $\nu_p(a_j) = \nu_p(a_{j+1}) = \cdots = \nu_p(a_l) = t$, hence $\nu_p(\tilde{a}_j) \geq t$.
For $j = i$ we have $\nu_p(a_j) = \alpha < t$ and $\nu_p(a_l) = t$ for $l = i+1, \ldots, n$, hence $\nu_p(\tilde{a}_i) = \alpha$.
For $j = i-1$ we have $\nu_p(a_{i-1}) = 0$ and $\nu_p(a_l) \geq \alpha$ for $l = i, \ldots, n,$, hence $\nu_p(\tilde{a}_{i-1}) = 0$. Thus we obtain the desired claim.

To compute the p-adic valuation of the denominator of γ, we use Lemma 2.1. For $j > i$, we have

$$\frac{\nu_p(b_n) - \nu_p(b_j)}{n-j} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_j)}{n-j} = \frac{t - t_j}{n-j} \leq 0, \text{ because } t_j \geq t.$$

For $j = i$, we have

$$\frac{\nu_p(b_n) - \nu_p(b_i)}{n-i} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_i)}{n-i} = \frac{t-\alpha}{n-i} = k.$$

For $j < i$ we have

$$\frac{\nu_p(b_n) - \nu_p(b_j)}{n-j} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_j)}{n-j} = \frac{t - t_j}{n-j} \leq \frac{t}{n-(i-1)} \leq \frac{(n-i)k+\alpha}{n-(i-1)} < \frac{(n-i)k+k}{n-(i-1)} = k,$$

hence $\nu_p(d(\gamma)) = k$.

- Second case: $t = (n-i)k$ with $p \leq i \leq n-1$. Choose integers $a_0, \ldots, a_n$ such that $\gcd(a_0, \ldots, a_n) = 1$ and

$$\begin{cases} \nu_p(a_j) \geq t, \text{ for } j > i, \\ \nu_p(a_i) = 0, \\ \nu_p(a_j) = 1, \text{ for } j < i, \\ \nu_p(a_n) = t. \end{cases} \tag{3.7}$$

Consider the polynomials

$$f(x) = \sum_{j=0}^{n} a_j (x)_j = \sum_{j=0}^{n} \tilde{a}_j x^j$$

and

$$g(x) = \tilde{a}_n x^n + q^{e_{n-1}} \tilde{a}_{n-1} x^{n-1} + \cdots + q^{e_1} \tilde{a}_1 x + q\tilde{a}_0 := \sum_{j=0}^{n} b_j x^j,$$

where q and the e_j have the same meaning as in the preceding case. Clearly $g(x)$ is irreducible in $\mathbb{Z}[x]$ by Eisenstein's Theorem. Let γ be a root of g(x). Since $p \leq i$, then by Lemma 2.2, we conclude that $p|i(f)$ and since $g(x) \equiv f(x) \pmod p$ for any $x \in \mathbb{Z}$, then $p|i(\gamma)$. We look at the p-adic valuations of the $\tilde{a}_j$. We have $\tilde{a}_n = a_n$ hence $\nu_p(\tilde{a}_n) = t$. For $j > i$, we have $\tilde{a}_j = a_j + \sum_{l=j+1}^{n} a_l c_l$ where $c_l \in \mathbb{Z}$ and we have $\nu_p(a_j) \geq t$ and $\nu_p(a_l) \geq t$ for $l \geq j+1$, hence $\nu_p(\tilde{a}_i) \geq t$.
For $j = i$, we have $\nu_p(a_i) = 0$ and $\nu_p(a_l) = t$, $l \geq j+1$, hence $\nu_p(\tilde{a}_i) = 0$.
For $j < i$, we have $\nu_p(\tilde{a}_j) \geq 0$.
For $j < i$, we have $\nu_p(\tilde{a}_j) \geq 0$.
We compute the p-adic valuation of the denominator of γ by using Lemma 2.1.
For $j > i$, we have

$$\frac{\nu_p(b_n) - \nu_p(b_j)}{n-j} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_j)}{n-j} = \frac{t - t_j}{n-j} \leq 0, \text{ because } t_j \geq t.$$

For $j = i$, we have

$$\frac{\nu_p(b_n) - \nu_p(b_i)}{n-i} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_i)}{n-i} = \frac{t}{n-i} = k.$$

For $j < i$ we have

$$\frac{\nu_p(b_n) - \nu_p(b_j)}{n-j} = \frac{\nu_p(\tilde{a}_n) - \nu_p(\tilde{a}_j)}{n-j} < \frac{t - t_j}{n-i} \leq \frac{t}{n-i} = k,$$

hence $\nu_p(d(\gamma)) = k$.

Since we can choose q and the e_j in an infinite number of ways, then the number of γ's is infinite. □

4. Upper bounds for the enumeration of the denominators of some algebraic numbers

Proposition 4.1. *Let $\theta \in A_n$, and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ its minimal polynomial over $\mathbb{Q}$. Suppose that $\theta \not\equiv 0 \,(\mathrm{mod}\, pA)$. Construct the Newton polygon of $f(x)$ by plotting in the (x, y) plan the points A_i whose coordinates are $(i, \nu_p(a_i))$ for all $i \in \{0, \ldots, n\}$ such that $a_i \neq 0$. Suppose that there exists $k \geq 1$ such that $p|i(\theta/p^k)$. Then there exists two integers m, M such that $1 \leq m < M \leq n-1$ and the line joining the points A_m and A_M has the following equation*

$$y + kx - u = 0, \text{ where } kM \leq u < \nu_p(a_0) \text{ and } u = \nu_p(cont(f(p^k x))).$$

Moreover all the points A_i such that $i < m$ or $i > M$ belong to the domain of all points (x, y) such that $y + kx - u > 0$. If $m < i < M$, then we have $\nu_p(a_i) + ki - u \geq 0$.

Proof: The minimal polynomial over $\mathbb{Z}$ of θ/p^k is given by

$$f(p^k x)/p^u = p^{nk-u}x^n + p^{(n-1)k-u}a_{n-1}x^{n-1} + \cdots + p^{k-u}a_1 x + p^{-u}a_0 := g(x),$$

where $u = \nu_p(cont(f(p^k x)))$. Let

$$I = \{i : 1 \leq i \leq n-1, a_i \neq 0 \text{ and } ik + \nu_p(a_i) - u = 0\}.$$

Since θ/p^k is not integral then $nk - u > 0$. Since $g(0) \equiv 0\,(\bmod\ p)$, then $\nu_p(a_0) - u > 0$. Adding these two facts to the property that $g(x)$ is primitive implies that $I \neq \emptyset$. Furthermore $g(1) \equiv 0\,(\mathrm{mod} p)$, hence $|I| \geq 2$.
Let $m = \inf(I)$ and $M = \max(I)$. Clearly the equation of the line joining the points A_m and A_M is given by: $y + kx - u = 0$. Moreover a point $(i, \nu_p(a_i))$ of the Newton polygon belongs to this line if and only if $i \in I$. The definition of m and M implies the properties of the points A_i and of u. $\square$

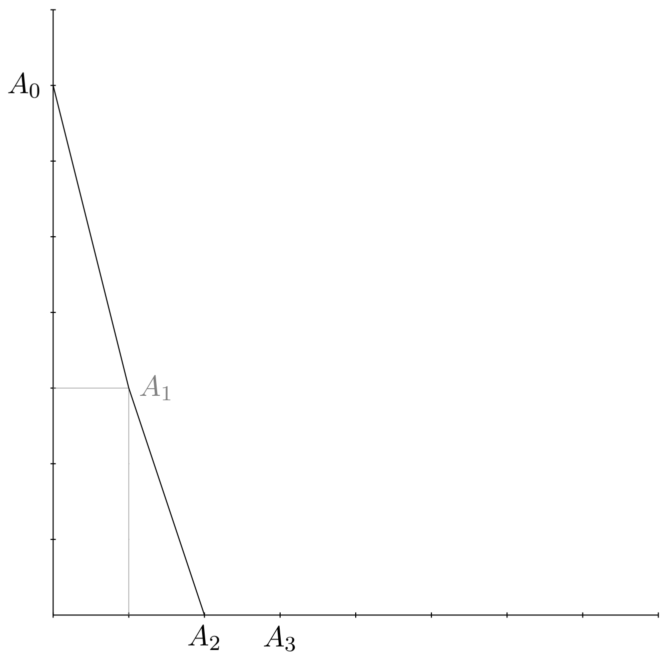
Remark 4.1. *Proposition 4.1 shows that $-k$ is the slope of some line joining two points A_m and A_M. Moreover all the others points belong to the same side of the line (or on the line). Therefore, if we fix a prime p and an algebraic integer θ such that $\theta \neq 0 \pmod{pA}$, it is possible to find explicitly all the values of k such that $p|i(\theta/p^k)$. This proposition shows also that the set of such nonnegative integers k is finite (may be empty).*

Example 4.1. *Let $t \geq 2$ be an integer, $f(x) = x^3 + x^2 + 2^t x + 2^{t+1}$ and θ_t be a root of $f(x)$. It is seen that $f(x)$ is irreducible over $\mathbb{Q}$: if not, it has a root a/b in $\mathbb{Q}$ with $a, b \in \mathbb{Z}$ and $\gcd(a,b) = 1$. Substitution then yields*

$$a^3 + a^2 b + 2^t a b^2 + 2^{t+1} b^3 = 0\,,$$

implying $b \mid a^3$. Thus, $b = \pm 1$, and we then obtain $a \mid 2^{t+1}$. Letting $a = 2^i$, we obtain $2^{3i} + 2^{t+i} = 2^{2i} + 2^{t+1}$, implying that $t + i \leq t + 1$ so $i = 1$ which is impossible.

For any nonnegative integer, let $\gamma_{t,k} = \theta_t / 2^k$. We show that $V_2(\theta_t) = \{0, t\}$. Clearly, $2 \mid i(\theta_t)$, hence $0 \in V_2(\theta_t)$. The Newton diagram for $p = 2$ has the following shape:



The possible edges of the convex hull which may give rise to values of $k \in V_2(\theta_t)$, $k \geq 1$, are $[A_0A_1]$ and $[A_1A_2]$. Their slopes are equal to $-t-1$ and t respectively. Thus, $k = t+1$ or $k = t$.

If $k = t+1$, the minimal polynomial of $\gamma_{t,k}$ over $\mathbb{Z}$ is given by $g(x) = 2^{t+2}x^3 + 2x^2 + x + 1$. This shows that $2 \not| i(g(x))$, hence $t+1 \notin V_2(\theta_t)$.

If $k = t$, the minimal polynomial of $\gamma_{t,k}$ over $\mathbb{Z}$ is given by $h(x) = 2x^3 + x^2 + x + 2$. This shows that $2 \mid i(h(x))$, hence $t \in V_2(\theta_t)$. Thus, $V_2(\theta_t) = \{0, t\}$.

We state now our main result on the upper bounds for the enumeration of the denominators of algebraic numbers γ such that $p|i(\gamma)$.

Theorem 4.1. *Let θ be a root of $f(x) \in \mathbb{Z}[x]$, monic irreducible, p a prime number such that $\theta \not\equiv 0(\ modpA)$ and let $a_0 = f(0)$. We set*

$$V_p(\theta) = \left\{k \geq 0;\ p|i(\theta/p^k)\right\}.$$

Suppose that $V_p(\theta) \neq \emptyset$ then we have

$$|V_p(\theta)| \leq \frac{n-1}{p-1}, \tag{4.8}$$

$$\sum_{k \in V_p(\theta)} k < \frac{\nu_p(a_0)}{p}. \tag{4.9}$$

For the proof of this theorem, we need the following lemma.

Lemma 4.1. *Let p be a prime number and $g(x) = a_M x^M + \cdots + a_m x^m$, $M > m > 0$ such that $p \not| a_M$. If $p|i(g)$, then $M - m \geq p - 1$.*

Proof: Suppose that $p \mid i(g)$, then clearly $p|i(xg_1)$, where $g_1(x) = a_M x^{M-m} + \cdots + a_m$. Write xg_1 in the form

$$xg_1(x) = a_M(x)_{M-m+1} + \sum_{j<M-m+1} b_j(x)_j,$$

then by Lemma 2.2 $p|(M-m+1)!a_M$, hence $p|(M-m+1)!$. Therefore $p \leq M - m + 1$. □

Proof of Theorem 4.1. Suppose that the complete list of elements of $V_p(\theta)$ is given by $k_1 < k_2 < \cdots < k_z$. We have $k_1 = 0$ if and only if $p|i(\theta)$. Set $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. For each $j = 1, \ldots, z$, let $g_j(x)$ be the minimal polynomial of θ/p^{k_j} over $\mathbb{Z}$.
Set $g_j(x) = p^{t_j}x^n + b_{n-1}^{(j)}x^{n-1} + \cdots + b_1^{(j)}x + b_0^{(j)}$, we have

$$g_1(x) = \begin{cases} f(x) & \text{if } k_1 = 0 \\ f(p^{k_1}x)p^{-u_1} & \text{if } k_1 \geq 1 \end{cases} \tag{4.10}$$

and $g_{j+1}(x) = g_j(p^{k_{j+1}-k_j}x)p^{-u_{j+1}}$ for $j = 1, \ldots, z-1$ and $u_1, \ldots, u_z$ are positive integers.
For any $j = 1, \ldots, z$, let $I_j = \{i \in \{1, \ldots, n-1\}, \nu_p(b_i^{(j)}) = 0\}$. Since $t_j > 0$, $\nu_p(b_0^{(j)}) > 0$ and $g_j(x)$ is irreducible in $\mathbb{Z}[x]$, then it follows that $I_j \neq \emptyset$. Let $m_j = \inf(I_j)$ and $M_j = \sup(I_j)$. Since $g_j(1) \equiv 0 \,(\mathrm{mod} p)$ then $|I_j| \geq 2$ and $m_j < M_j$. Clearly $m_j \geq 1$ and $M_j \leq n-1$. We claim that:

- $M_j - m_j \geq p - 1$ for $j = 1, \ldots, z$.

- $u_1 \geq k_1 M_1$ and $u_j \geq (k_j - k_{j-1})M_j$ for $j = 2, \ldots, z$.
- $n \geq M_1 > m_1 \geq M_2 > m_2 \geq \cdots \geq M_z > m_z \geq 1$.

The first claim follows from Lemma 4.1. If $k_1 = 0$, then from (4.10) we have $u_1 = 0 = k_1 M_1$. From the definition of M_1 and the definition of the interval I_1, it follows that

$$\nu_p(b_{M_1}^{(1)}) = 0.$$

Therefore, if $k_1 \geq 1$, equation (10) and the above equality imply that

$$0 = \nu_p(b_{M_1}^{(1)}) = \nu_p(a_{M_1}) + k_1 M_1 - u_1,$$

hence $u_1 \geq k_1 M_1$. Similarly for $j = 2, \ldots, z$, we have

$$0 = \nu_p(b_M^{(j)}) = \nu_p(b_{M_j}^{(j-1)}) + (k_j - k_{j-1})M_j - u_j,$$

hence $u_j \geq (k_j - k_{j-1})M_j$, which proves the second part of the claim. For the last part of the claim it is sufficient to prove that $m_j \geq M_{j+1}$ for $j = 1, \ldots, z-1$. For, suppose that $m_j < M_{j+1}$ for some $j \in \{1, \ldots, z-1\}$. We have $0 = \nu_p(b_{m_j}^{(j)})$, hence

$$\nu_p(b_{m_j}^{(j+1)}) = \nu_p(b_{m_j}^{(j)}.p^{(k_{j+1}-k_j)m_j}.p^{-u_{j+1}}) = (k_{j+1} - k_j)m_j - u_{j+1}.$$

We deduce that $(k_{j+1} - k_j)m_j \geq u_{j+1}$ and then $(k_{j+1} - k_j)M_{j+1} > u_{j+1}$. It follows that

$$\nu_p(b_{M_{j+1}}^{(j+1)}) = \nu_p(b_{M_{j+1}}^{(j)}) + (k_{j+1} - k_j)M_{j+1} - u_{j+1} > 0,$$

which contradicts the definition of M_{j+1} and completes the proof of the claim.
We now come back to the proof of Theorem 4.1.
Completion of proof of Theorem 4.1: We use the first and the third points of the claim. We have

$$n \geq M_1 > M_2 > \cdots > M_{z-1} > M_z \geq p > 1.$$

Using the claim, we obtain

$$n - p \geq M_1 - M_z = (M_1 - M_2) + \cdots + (M_{z-1} - M_z) \geq (p-1)(z-1)$$

hence

$$z \leq \frac{n-p}{p-1} + 1 = \frac{n-1}{p-1}.$$

Therefore (4.8) is proved.
We prove the inequality (4.9) of Theorem 4.1. We have $b_0^{(1)} = a_0 p^{-u_1}$, $b_0^{(j+1)} = b_0^{(j)} p^{-u_{j+1}}$ for $j = 1, \ldots, z-1$ and since $g_z(0) \equiv 0 (\mathrm{mod} p)$, then $\nu_p(b_0^{(z)}) > 0$. Hence $u_1 + u_2 + \cdots + u_z < \nu_p(a_0)$. On the other hand, using the first and the second parts of the claim, we obtain

$$\begin{aligned}
u_1 + u_2 + \cdots + u_z &\geq k_1 M_1 + (k_2 - k_1)M_2 + \cdots + (k_z - k_{z-1})M_z \\
&\geq k_1 z p + (k_2 - k_1)(z-1)p + \cdots + (k_z - k_{z-1})p \\
&= p\,(k_1 z + k_2 z - k_1 z - k_2 + k_1 + k_3 z - k_2 z - 2k_3 + 2k_2 + \cdots + k_z - k_{z-1}) \\
&= p((k_1 + k_2 + \cdots + k_z).
\end{aligned}$$

Therefore, we have

$$\nu_p(a_0) > \sum_{j=1}^{z} u_j \geq p \sum_{j=1}^{z} k_j,$$

hence

$$\sum_{j=1}^{z} k_j < \nu_p(a_0)/p.$$

Remark 4.2. *Theorem 4.1, shows that if $\nu_p(a_0) \leq p$, then $V_p(\theta) = \{0\}$ or $V_p(\theta) = \emptyset$.*

The following result shows that the bound (4.8) in Theorem 4.1 is the best possible. More precisely, we have

Proposition 4.2. *Let p and q be distinct prime numbers such that $q \equiv 1 \pmod p$, θ be a root of*

$$f(x) = x^n + \sum_{i=1}^{N} a_i x^{n-i(p-1)} + qp^{\lambda},$$

where $N = \lfloor (n-1)/(p-1) \rfloor$, $a_i = (-1)^i qp^{(p-1)i(i-1)/2}$, for $i = 1, \ldots, N$ and

$$\lambda > \frac{2n(N-1) - (p-1)\big((N-1)^2 + N - 1\big)}{2}.$$

Then

$$|V_p(\theta)| = \Big\lfloor \frac{n-1}{p-1} \Big\rfloor.$$

Proof: Clearly, by Eisenstein's criterion, $f(x)$ is irreducible over $\mathbb{Q}$. The coefficient of $x^{n-(p-1)}$ is coprime to p, hence $\theta \not\equiv 0 \, (\mathrm{mod} pA)$. By Theorem 4.1, we have

$$|V_p(\theta)| \leq \Big\lfloor \frac{n-1}{p-1} \Big\rfloor.$$

We show that the integers $0, 1, \ldots, \Big\lfloor \frac{n-1}{p-1} \Big\rfloor - 1$ belong to $V_p(\theta)$ and this will complete the proof of Proposition 4.2. Since $f(x) \equiv x^n - qx^{n-(p-1)} \pmod{p\mathbb{Z}[x]}$ and $q \equiv 1 \pmod p$, then $f(x) \equiv x^{n-(p-1)}(x^p - 1) \pmod{p\mathbb{Z}[x]}$. Thus, $p \mid i(f)$ and $0 \in V_p(\theta)$. Set $a_0 = 1$ and fix $k \in \Big\{1, \ldots, \Big\lfloor \frac{n-1}{p-1} \Big\rfloor - 1\Big\}$. We have

$$f(p^k x) = p^{nk} x^n + \sum_{i=1}^{N} a_i p^{nk - i(p-1)k} x^{n-i(p-1)} + qp^{\lambda}.$$

We claim, omitting the proofs that

$$\nu_p(a_k p^{nk-k(p-1)k}) = \nu_p(a_{k+1} p^{nk-k(p-1)(k+1)}) = \frac{2nk - (p-1)(k^2+k)}{2}$$

and

$$\nu_p(a_i p^{nk-i(p-1)k}) > \frac{2nk - (p-1)(k^2+k)}{2} \text{ if } i \neq k, k+1.$$

Moreover, since the function $x \mapsto \psi(x) = 2nx - (p-1)(x^2+x)$ is increasing in $[0, N-1]$ and since $\lambda > \frac{2n(N-1)-(p-1)(N-1)^2+N-1}{2}$, then $\lambda > \frac{2nk-(p-1)(k^2+k)}{2}$. It follows that $cont(f(p^k x)) = \frac{2nk-(p-1)(k^2+k)}{2}$ and the minimal polynomial over $\mathbb{Z}$ of $\gamma_k = \frac{\theta}{p^k}$ is given by

$$g_k(x) = f(p^k x)p^{-(2nk-(p-1)k^2+k)/2}.$$

From the above it is seen that

$$g_k(x) \equiv (-1)^k x^{n-k(p-1)} + (-1)^{k+1} x^{n-(k+1)(p-1)} (\mathrm{mod} p) \equiv (-1)^k x^{n-(k+1)(p-1)}(x^{p-1}-1)(\mathrm{mod} p),$$

hence $p|i(\gamma_k)$, thus $k \in V_p(\theta)$. □

Corollary 4.1. *Let p be a prime number, and θ be a root of*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x],$$

irreducible over $\mathbb{Q}$. Suppose that there exists $i \in \{0, \ldots, \min(p,n)-1\}$ such that $\nu_p(a_i) = 0$. Then

$$\theta \not\equiv 0(\mathrm{mod} pA) \textit{ and } V_p(\theta) = \emptyset \textit{ or } V_p(\theta) = \{0\}.$$

Moreover if $p > n$, then $V_p(\theta) = \emptyset$.

Proof: Let α be an algebraic integer and $g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$ be its minimal polynomial over $\mathbb{Q}$. It is easy to prove that $\alpha \equiv 0(\mathrm{mod} p)$ if and only if $\nu_p(b_i) \geq n-i$ for $i = 0, \ldots, n-1$. Our assumption then implies that $\theta \not\equiv 0(\mathrm{mod} pA)$. Suppose that $V_p(\theta) \neq \emptyset$ and let $k \in V_p(\theta)$. Assume that $k \geq 1$ and let $\gamma = \theta/p^k$ and $u = cont(f(p^k x))$. Then the minimal polynomial of γ over $\mathbb{Z}$ is given by

$$g(x) = f(p^k x)p^{-u} = p^{nk-u}x^n + p^{(n-1)k-u}a_{n-1}x^{n-1} + \cdots + p^{-u}a_0.$$

As in Theorem 4.1, let

$$I = \{j \in \{1, \ldots, n-1\}; \nu_p(a_j) + kj - u = 0\}, m = \inf(I), M = \sup(I).$$

Suppose first that $m \leq i$. We have $ik - u = \nu_p(a_i) + ik - u \geq 0$. Since $M - m \geq p-1$, then $M > i$ which implies $\nu_p(a_M) + Mk - u \geq Mk - u > ik - u \geq 0$, a contradiction. We deduce that $m > i$ and then $\nu_p(a_m)) + km - u \geq km - u > ki - u = \nu_p(a_i) + ki - u \geq 0$, a contradiction again. Therefore $V_p(\theta) = \{0\}$. □

5. A new invariant of number fields and a generalisation of MacCluer's Theorem

Let p be a fixed prime integer. We have shown that for any algebraic integer θ such that $\theta \not\equiv 0\,(\mathrm{mod} pA)$, $p|i(\theta/p^k)$ for some $k \geq 1$, then $k < \nu_p(N_{\mathbb{Q}(\theta)/\mathbb{Q}}(\theta))/p$. Does there exist some constant $c > 0$ such that if $\theta \in \overline{\mathbb{Q}}, \theta \not\equiv 0\,(\mathrm{mod} pA)$ and $p|i(\theta/p^k)$ then $k < c$?
Even if we fix the degree n of θ and suppose that the constant c depends on n, the answer is negative as it is shown by the following result.

Proposition 5.1. *Let n, N be positive integers and p be a prime number such that $p < n$. Then there exists an integer $k > N$ and an algebraic integer θ of degree n such that*

$$\theta \not\equiv 0 \,(\mathrm{mod} pA) \text{ and } p|i(\theta/p^k).$$

Proof: Let F be a number field of degree $n-1$ such that $p|i(F)$. In particular, we can take F such that p completely splits in F, so that $p \mid i(F)$ by MacCluer's Theorem. Such a field F exists by Tchebotarev's theorem [Neu99]. Let α be a primitive element of $F/\mathbb{Q}$. Suppose that α is integral and $p|i(\alpha)$. Let $F_\alpha(x)$ be the minimal polynomial of α over $\mathbb{Q}$. Let q be a prime number such that

$$q \neq p \text{ and } q \not| N_{F/\mathbb{Q}}(\alpha).$$

Let t be an integer such that $t > nN$ and let $g(x) = p^t x^n + qF_\alpha(x)$. Then Eisenstein's criterion shows that $g(x)$ is irreducible over $\mathbb{Q}$. Obviously $g(x)$ is primitive, hence it is irreducible in $\mathbb{Z}[x]$. Let γ be a root of $g(x)$, then clearly $p|i(\gamma)$ and $d(\gamma) = p^k$ for some positive integer k such that $k \leq t \leq nk$, hence $k \geq t/n > N$. The algebraic integer $\theta = p^k\gamma$ satisfies all the conditions of the proposition and the proof is complete. $\square$

Let K be a number field of degree n over $\mathbb{Q}$ and A be its ring of integers. We define the integer $\nu_p(K)$ as follows.

Definition 5.1. *Let*

$$V_p(K) = \left\{k \geq 0, \text{ there exists } \theta \in A_n, \theta \not\equiv 0 \,(\mathrm{mod} pA), \text{ and } p|i(\theta/p^k)\right\},$$

and we define

$$v_p(K) = \begin{cases} -\infty & \text{if } \ V_p(K) = \emptyset, \\ \infty & \text{if } \ V_p(K) \ \text{ is infinite}, \\ \max(V_p(K)) & \text{if } \ V_p(K) \ \text{ is finite}. \end{cases}$$

Remark 5.1. *By Theorem 3.1, we have $v_p(K) = -\infty$ if and only if $p \not| i(K)$. So there is no need to give examples illustrating this fact. Theorem 3.1 again shows that if the degree of the number field K is a prime p then $v_p(K) = 0$ if $p|i(K)$ and $v_p(K) = -\infty$ if $p \not| i(K)$.*

In the following we compute explicitly $v_2(K)$ for some number fields of degree 3 or 4 over $\mathbb{Q}$.

Proposition 5.2. (Galois field of degree 4**)** *Let $K/\mathbb{Q}$ be a Galois number field of degree 4 in which the prime 2 splits into a product of two prime ideals having their residual degree equal to 2. Then we have $v_2(K) = 0$.*

Proof: By MacCluer's theorem, $2|i(K)$, hence $0 \in V_p(K)$. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be the conjugate prime ideals of A lying over 2 and having their residual degree equal to 2. Suppose that $2|i(\theta/2^k)$ for some $k \geq 1$ and $\theta \in A_n$ such that $\theta \not\equiv 0 \,(\mathrm{mod}\ 2A)$. Since $N_{K/\mathbb{Q}}(\theta) \equiv 0 \pmod 2$, then we may suppose that $\mathfrak{p}^e || \theta$ and $\mathfrak{p}' \not| \theta$ for some $e \geq 1$. We suppose that the conjugates $\theta_1 = \theta, \theta_2, \theta_3, \theta_4$ of θ satisfy the following conditions:

$$\mathfrak{p}^e || \theta_1, \mathfrak{p}^e || \theta_3, \mathfrak{p}' \not| \theta_1\theta_3, \mathfrak{p}'^e || \theta_2, \mathfrak{p}'^e || \theta_4, \mathfrak{p} \not| \theta_2\theta_4.$$

Let $f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of θ over $\mathbb{Q}$. Let $g(x) \in \mathbb{Z}[x]$ be the minimal polynomial of $\gamma = \theta/2^k$ over $\mathbb{Z}$, then

$$g(x) = f(2^k x).2^{-u} = 2^{4k-u}x^4 + 2^{3k-u}a_3x^3 + 2^{2k-u}a_2x^2 + 2^{k-u}a_1x + 2^{-u}a_0,$$

where u is the content of $f(2^k x)$. Using the elementary symmetric functions of the θ_j and our assumption on their $\mathfrak{p}$-adic and $\mathfrak{p}'$-adic valuations, we get

$$\nu_2(a_0) = 2e, \nu_2(a_1) \geq e \text{ and } \nu_2(a_2) = 0.$$

If $k \geq e$, then $\nu_2(2^{4k-u}) \geq 4e - u$, $\nu_2(2^{3k-u}a_3) \geq 3e - u$, $\nu_2(2^{2k-u}a_2) = 2k - u \geq 2e - u$, $\nu_2(2^{k-u}a_1) \geq 2e - u$, $\nu_2(2^{-u}a_0) = 2e - u$. Since these five 2-adic valuations must be nonnegative, then $u \leq 2e$. Furthermore one (at least) of these valuations must be 0, hence $u = 2e$. In this case, $g(0) \not\equiv 0 \pmod 2$ which is a contradiction to $2|i(\gamma)$. If $k < e$, then $\nu_2(2^{4k-u}) = 4k - u$, $\nu_2(2^{3k-u}a_3) \geq 3k-u$, $\nu_2(2^{2k-u}a_2) = 2k-u$, $\nu_2(2^{k-u}a_1) > 2k-u$, $\nu_2(2^{-u}a_0) > 2k-u$. Using similar arguments as in the preceding case, we obtain $u = 2k$. We conclude that all the coefficients of $g(x)$ have their 2-adic valuations positive except the coefficient of x^2 which has a 2-adic valuation equal to 0. In this case also we reach a contradiction since $g(1) \not\equiv 0((\bmod)2)$. It follows that $V_2(K) = \{0\}$ and $v_2(K) = 0$. $\square$

For the proof of the next proposition, we will need the following lemma.

Lemma 5.1. (Engstrom) *Let K be a number field, A be its ring of integers and p be a prime integer. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be distinct prime ideals of A lying over p and let $\Phi_1(x), \ldots, \Phi_s(x)$ be monic irreducible polynomials over $\mathbb{F}_p$ not necessarily distincts of degree $d_1, \ldots, d_s$ respectively, where d_i divides the residual degree of $\mathfrak{p}_i$. Let $h_1, \ldots, h_s$ be positive integers. Then there exists a primitive element $\theta \in A$ such that $\mathfrak{p}_i^{h_i}||\Phi_i(\theta)$ for $i = 1, \ldots, s$*

Proof: see [Eng30]. $\square$

Proposition 5.3. (Cubic Galois) *Let $K/\mathbb{Q}$ be a Galois number field of degree 3 in which the prime 2 splits completely. Then $V_2(K) = \mathbb{N}$.*

Proof: Let k and e be positive integers such that $e > k$. Let $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$ be the prime ideals of A lying over 2. By Lemma 5.1 there exists $\theta \in A_n$ such that

$$\mathfrak{p}_1^e||\theta, \mathfrak{p}_2^k||\theta \text{ and } \mathfrak{p}_3 \not|\theta.$$

Assume that the conjugates of θ, $\theta_1 = \theta, \theta_2, \theta_3$ are labelled in order to satisfy the following conditions:

$$\mathfrak{p}_2^e||\theta_2, \mathfrak{p}_3^k||\theta_2, \mathfrak{p}_1 \not|\theta_2,$$
$$\mathfrak{p}_3^e||\theta_3, \mathfrak{p}_1^k|||\theta_3, \mathfrak{p}_2 \not|\theta_3.$$

Let $f(x) = x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of θ over $\mathbb{Q}$. Expressing a_0, a_1 and a_2 in terms of $\theta_1, \theta_2, \theta_3$, we get

$$\nu_2(a_0) = e + k, \nu_2(a_1) = k \text{ and } \nu_2(a_2) = 0.$$

We have

$$f(2^k x) = 2^{3k}x^3 + 2^{2k}a_2x^2 + 2^k a_1 x + a_0.$$

Set

$$b_3 = 2^{3k}, b_2 = 2^{2k}a_2, b_1 = 2^k a_1, b_0 = a_0.$$

Using the 2-adic valuation of a_0, a_1, a_2 we obtain

$$\nu_2(b_1) = \nu_2(b_2) = 2k, \nu_2(b_0) = e + k > 2k, \nu_2(b_3) = 3k > 2k.$$

Therefore $cont(f(2^{2k}x)) = 2^{2k}$ and the minimal polynomial of $\theta/2^k$ is given by

$$g(x) = f(2^k x) \cdot 2^{-2k}.$$

Clearly we have $g(0) \equiv g(1) \equiv 0 \,(\mathrm{mod} 2)$ hence $2|i(\theta/p^k)$. Since the prime 2 splits completely in K, then $0 \in V_2(K)$. Therefore $V_2(K) = \mathbb{N}$ and $v_2(K) = \infty$. $\square$

Remark 5.2. *Our result in the sequel can be viewed as a generalization of MacCluer's theorem which establishes a relation between the number of prime ideals of A lying over p and the property of p to be a divisor of $i(K)$.*

Fix a prime number p and define, for any primitive element $\theta \in A$ of K, the integer $j_p(\theta)$ as follows.

Definition 5.2. *Let $F_\theta(x)$ be the minimal polynomial of θ over $\mathbb{Q}$. Let $j_p(\theta)$ be the largest integer y, if it exists, $1 \le y \le p$ such that $F_\theta(1) \equiv F_\theta(2) \equiv \cdots \equiv F_\theta(y) \equiv 0 \,(\mathrm{mod}\, p)$. If not set $j_p(\theta) = 0$. We define also $j_p(K) = \max_{\theta \in A_n} j_p(\theta)$.*

Theorem 5.1. *Let r be the number of prime ideals of A lying over p. Then*

$$j_p(K) = \inf(r, p).$$

Moreover

$$p|i(K) \iff j_p(K) = p.$$

Proof: Suppose first that $r \le p$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the distinct prime ideals of A lying over p. By Lemma 5.1 there exists $\theta \in A_n$ such that $\theta \equiv i (\mathrm{mod} p)$ for $i = 1, \ldots, r$. It follows that the minimal polynomial $F_\theta(x)$ of θ satisfies the condition

$$F_\theta(x) \equiv (x-1)(x-2)\cdots(x-r)g(x) (\mathrm{mod}\, p)$$

hence $j_p(\theta) \ge r$ which implies that $j_p(K) \ge r$. On the other hand, let $\theta \in A_n$ such that

$$j_p(K) = j_p(\theta) := \ t \ ,$$

then

$$F_\theta(x) \equiv (x-1)(x-2)\cdots(x-t)g(x) (\mathrm{mod}\, p)$$

hence, by Hensel's Lemma, we deduce that $F_\theta(x)$ has at least t irreducible factors over $\mathbb{Z}_p$, the ring of p-adic integers. Again by Theorem 5.1 of chap. 2 of [Jan96], we have $t \le r$. We conclude that

$j_p(K) = r = \inf(p, r)$.
Suppose now that $r > p$. By Lemma 5.1, let $\theta \in A_n$ such that $\theta \equiv i(\text{mod}p)$ for $i = 1, \ldots, p$. Then

$$F_\theta(x) \equiv (x-1)(x-2)\cdots(x-p)g(x)(\text{mod}\, p).$$

therefore we have $j_p(\theta) \geq p$ which implies $j_p(K) \geq p$. From the definition we have $j_p(K) \leq p$, hence $j_p(K) = p = \inf(r, p)$.
We now prove the last statement of the proposition. We have

$$p|i(K) \iff r \geq p \text{ (by MacCluer's theorem)} \iff \inf(r,p) = p \iff j_p(K) = p. \quad \square$$

6. Concluding remarks

Questions Let K be a number field of degree n. If $[K : \mathbb{Q}] = 2$, then by Corollary 2.1, $i(K)$ and $\hat{\imath}(K)$ are equal to 1 or 2. Theorem 3.1 shows that $2 \not| i(\gamma)$ if $\gamma \notin A_n$, hence $i(K) = \hat{\imath}(K) \in \{1, 2\}$.

If $[K : \mathbb{Q}] = 3$, then $i(K)$ and $\hat{\imath}(K) \in \{1, 2, 3, 6\}$. Moreover, Theorem 3.1 shows that $3 \mid \hat{\imath}(K)$ if and only if $3 \mid i(K)$.

Suppose that there exists $\gamma = \theta/2^k$ with $k \geq 1$, $k \leq t \leq 3k$ and $\theta \not\equiv 0 \pmod p$ such that $2 \mid i(\gamma)$. Let $g(x) = 2^t x^3 + b_2 x^2 + b_1 x + b_0$ be the minimal polynomial of γ over $\mathbb{Z}$. Since $g(0) \equiv 0 \pmod 2$, then $b_0 \equiv 0 \pmod 2$. Since $g(1) \equiv 0 \pmod 2$, then $b_1 + b_2 \equiv 0 \pmod 2$, thus $b_1 \equiv b_2 \pmod 2$. Moreover, since $g(x)$ is primitive, then $b_1 \equiv b_2 \equiv 1 \pmod 2$. By Theorem 3.1, $t \leq k$. Since $k \leq t$, then $k = t$. The minimal polynomial of θ is then given by

$$f(x) = x^3 + b_2 x^2 + b_1 2^t x + b_0 2^{2t}.$$

This shows that $2 \mid i(f)$ and then $2 \mid i(\theta)$, thus $2 \mid i(K)$. We conclude that $i(K) = \hat{\imath}(K)$.

Let K be a number field of degree n and let $\gamma \in K_n \setminus A_n$. Set $\gamma = \theta/d$, where d is an integer at least equal to 2 such that $\theta \not\equiv 0 \pmod p$ for any prime divisor p of d. It is proved in Lemma 3.1 that if $d = p^k q$ with $\gcd(p, q) = 1$ and $k \geq 1$, then $p \mid i(\gamma)$ if and only if $p \mid i(\theta/p^k)$. We ask that following: Is it true that if $p \mid i(\theta/p^k)$ with $k \geq 1$ and $\theta \not\equiv 0 \pmod p$, then $p \mid i(K)$? Do we have $\hat{\imath}(K) = i(K)$?

Recall that $\nu_p(K)$ is the greatest element of the set $V_p(K)$, when this set is finite. Do we have $\{0, 1, \ldots, \nu_p(K)\} = V_p(K)$? The example given in section 4 shows that $V_p(\theta_t) = \{0, t\} \neq \{0, 1, \ldots, t\}$. We may ask a similar question when $V_p(K)$ is infinite. Do we have $V_p(K) = \mathbb{N}$?

Acknowledgements. The authors express their gratitude to the anonymous referee for constructive suggestions which improved the quality of the paper. The third author was supported in part by NSERC.

References

[ARW96] S. Arno, M. L. Robinson, and F. S. Wheeler, *On denominators of algebraic numbers and integer Polynomials*, J. Number theory **57** (1996), 292–302.

[AyKi11] M. Ayad, O. Kihel, *Common Divisors of Values of Polynomials and Common Factors of Indices in a Number Field*, Inter. J. Number Theory **7** (2011), 1173–1194.

[ABK15] M. Ayad, A. Bayad, and O. Kihel, *Denominators of Algebraic Numbers in a Number Field* , J. Number Theory **149** (2015), 1–14.

[Eng30] H. T. Engstrom, *On the common index divisors of an algebraic field*, Trans. A.M.S **32** (1930), 223–237.

[GuMc70] H. Gunji, D. L. McQuillan, *On a class of ideals in an algebraic number field*, J. Number Theory **2** (1970), 207–222.

[Jan96] G. J. Janusz, *Algebraic Number Fields, Graduate Studies in Math*, A.M.S. (1996).

[Mac71] C. R. MacCluer, *Common divisors of values of polynomials*, J. Number Theory **3** (1971), 33–34.

[Neu99] O. Neukirch, *Algebraic Number Theory*, Springer (1999).

Mohamed Ayad
Laboratoire de Mathématiques Pures et Appliquées
Université du Littoral, F-62228 Calais, France
e-mail: ayad@lmpa.univ-littoral.fr

Abdelmejid Bayad
Laboratoire de Mathématiques et Modélisation d'Évry (LAMME)
Université Paris-Saclay, CNRS (UMR 8071)
Bâtiment I.B.G.B.I., 23 Boulevard de France,
CEDEX, 91037 Evry, France
e-mail: abdelmejid.bayad@univ-evry.fr and
e-mail: abayad@maths.univ-evry.fr

Omar Kihel
Department of Mathematics
Brock University, Ontario, Canada L2S 3A1
e-mail: okihel@brocku.ca