Hardy-Ramanujan Journal
Vol.12 (1989) 1-5

A LEMMA IN COMPLEX FUNCTION THEORY-I

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§1. **INTRODUCTION.** This paper (although self-contained) is a continuation of [1]. The object of this paper is to prove the following theorem which has applications in the theory of the Riemann zeta-function and is also of independent interest.

THEOREM. *Let n be any positive integer, $B > 0$ arbitrary, $r > 0$ arbitrary. Let $f(z)$ be analytic in $|z| \leq r$ and let the maximum of $|f(z)|$ in this disc be $\leq M$. Let $0 \leq x < r, C = Bnx, r_0 = \sqrt{r^2 - x^2}$, and $\alpha = 2(C^{-1} Sinh\ C + Cosh\ C)$. Then (for any fixed combination of signs $\pm$) we have*

$$|f(0)| \leq \left(\frac{\alpha B n r_0}{\pi}\right)\left(\frac{1}{2r_0}\int_{-r_0}^{r_0} |f(\pm x + iy)|\, dy\right) + \left(\frac{2}{Br}\right)^n \{1 + (e^C + 1)(\pi^{-1} Sin^{-1}(\frac{x}{r}))\} M. \tag{1}$$

Putting $x = 0$ we obtain the following

COROLLARY. *We have*

$$|f(0)| \leq \left(\frac{4Bnr}{\pi}\right)\left(\frac{1}{2r}\int_{-r}^{r} |f(iy)|\, dy\right) + \left(\frac{2}{Br}\right)^n M. \tag{2}$$

In particular with $B = 6r^{-1}, M \geq 3, A \geq 1$, and $n =$ the integer part of $(A+1)\log M + 1$, we have

$$|f(0)| \leq \frac{24}{\pi}((A+1)\log M + 1)\left(\frac{1}{2r}\int_{-r}^{r} |f(iy)|\, dy\right) + M^{-A}. \tag{3}$$

REMARK 1. The equations (1), (2) and (3) are statements about $|f(z)|$. In [2] we will extend (1) to more general functions than $|f(z)|$ with some other constants in place of $\alpha, \frac{4}{\pi}, \frac{24}{\pi}$. (For example to $|f(z)|^k$ where $k > 0$ is any real number).

REMARK 2. Corollary (1) shows that

$$|f(0)| \leq (24\, A \log M)(\frac{1}{2r}\int_{-r}^{r} |f(iy)|\, dy) + M^{-A}. \tag{4}$$

We ask the question "Can we replace $\log M$ by a term of smaller order say by $\sqrt{\log M}$ (or omit it altogether) at the cost of increasing the constant 24A?". The answer is no. See the Remark 6 in [2].

REMARK 3. The method of proof is nearly explained in [1]. As for the applications we can state for example the following result. Let $3 \leq H \leq T$. Divide the interval $T, T+H$ into intervals I of length r each. We can assume $0 < r \leq 1$ and omit a small bit at one of the ends. Then for any integer constant $k \geq 1$, (the result is also true if k is real by [2]), we have

$$\sum_{I} \max_{t \text{ in } I} |\zeta(\frac{1}{2}+it)|^k \ll \frac{Ak \log T}{r} \int_{T-r}^{T+H+r} |\zeta(\frac{1}{2}+it)|^k\, dt + r^{-1} H T^{-Ak} \tag{5}$$

the implied constant being absolute. We may retain only one term on the LHS of (5) and if we know for example that RHS of (5) is $\ll HT^{\epsilon}$ then it would follow that

$$\mu(\frac{1}{2}) \leq \frac{1}{k}\lim_{T\to\infty}(\frac{\log H}{\log T}). \tag{6}$$

Since what we want holds for $H = T^{1/3}$ and $k = 2$ and any $r (0 < r \leq 1)$ we obtain the known result $\mu(\frac{1}{2}) \leq \frac{1}{6}$ due to H. Weyl, G.H. Hardy and J.E. Littlewood. Similar remarks apply to L-functions and so on.

REMARK 4. The results of the present paper as well as some of the results in [1] are improvements and generalizations of some lemmas in Ivić's book [3] (see page 172 of this book. Here the results concern Dirichlet series with a functional equation and are of a special nature).

REMARK 5. In (4) we have corresponding results with $|f(iy)|$ on the RHS replaced by $|f(x+iy)|$. These follow from Theorem 1.

§ 2. **PROOF OF THE THEOREM.** Let P, Q, R, S denote the points $-ri, ri, r,$ and $-r$ respectively. Then we begin with

LEMMA 1. *Let* $X = Exp(u_1 + \cdots + u_n)$ *and* $0 \leq u_j \leq B$ *for* $j = 1, 2, ..., n$ *where* $B > 0$ *is arbitrary. Then*

$$f(0) = \frac{1}{2\pi i}\{\int_{PQ} f(w)\frac{X^w - X^{-w}}{w} dw + \int_{QSP} f(w)\frac{X^w}{w} dw + \int_{PRQ} f(w)\frac{X^{-w}}{w} dw\} \tag{7}$$

where the integrations are respectively along the straight line PQ, *along the semi-circular portion* QSP *of the circle* $|w| = r$, *and along the semi-circular portion* PRQ *of the circle* $|w| = r$.

PROOF. With an understanding of the paths of integration similar to the ones explained in the statement of the lemma we have by Cauchy's theorem that the integral of $f(w)\frac{X^w}{w}$ over $PQSP$ is $2\pi i f(0)$ provided we deform the contour to $P'Q'SP'$ where $P'Q'$ is parallel to PQ and is close to PQ (and to the right of it) and the points P' and Q' lie on the circle $|w| = r$. Also with the same modification the integral of $f(w)\frac{X^{-w}}{w}$ over $PRQP$ is zero. These remarks complete the proof of the lemma.

LEMMA 2. *Denote by* I_1, I_2, I_3 *the integrals appearing in* Lemma 1. *Let* $< du >$ *denote the element of volume* $du_1 du_2 \cdots du_n$ *of the box* $\mathcal{B}$ *defined by* $0 \leq u_j \leq B (j = 1, 2, ..., n)$. *Then*

$$B^{-n} \left| \int_{\mathcal{B}} (\frac{1}{2\pi i}(I_2 + I_3)) < du > \right| \leq (\frac{2}{Br})^n M. \tag{8}$$

PROOF. Trivial since $|X^w| \leq 1$ and $|X^{-w}| \leq 1$ on QSP and PRQ respectively.

LEMMA 3. *Let* $w = x + iy$ *where* x *and* y *are any real numbers, and* $0 \leq L = log\ X \leq Bn$. *Then*

$$\left| \frac{e^{wL} - e^{-wL}}{wL} \right| \leq \frac{1}{C}(e^C - e^{-C}) + e^C + e^{-C} \tag{9}$$

where $C = Bn|x|$.

PROOF. LHS in the lemma is

$$| \{e^{xL}(cos(yL) + i\, sin(yL)) - e^{-xL}(cos(yL) - i\, sin(yL))\}(wL)^{-1} |$$

$$\leq | \frac{e^{xL} - e^{-xL}}{xL} || cos(yL) | + (e^{xL} + e^{-xL}) | \frac{sin(yL)}{yL} | \leq \frac{1}{C}(e^{C} - e^{-C}) + e^{C} + e^{-C}.$$

This completes the proof of the lemma.

In order to obtain the theorem we note that on the line PQ we have $x = 0$. We now assume that $x > 0$ and move the line of integration to $Re\ w = \pm x$, (whatever be the sign) namely the intercept made by this line in the disc $| w | \leq r$. On this line we pass to the absolute value and use Lemma 3. We get the first term on the RHS of (1). For the two circular portions connecting this path with the straight line PQ we integrate over the box B and get

$$B^{-n} | \int_{B} (\frac{1}{2\pi i} \int f(w) \frac{X^{w} - X^{-w}}{w} dw) < du > |$$

$$\leq (\frac{2}{Br})^{n}(e^{C} + 1)(\pi^{-1} sin^{-1}(\frac{x}{r}))M.$$

This with lemma 2 completes the proof of the theorem.

REFERENCES

[1] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *Some local convexity theorems for the zeta-function-like analytic functions,* Hardy-Ramanujan J., 11 (1988), 1-12.

[2] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *A lemma in complex function theory-II,* Hardy-Ramanujan J., vol. 12 (1989), 6-13.

[3] A. IVIĆ, The Riemann zeta-function, *John Wiley and Sons, Interscience Publication* (1985).

ADDRESS OF THE AUTHORS

1. PROFESSOR R. BALASUBRAMANIAN,
(ON DEPUTATION FROM T.I.F.R.)
MATSCIENCE, THARAMANI P.O.,
MADRAS - 600 113,
INDIA

2. PROFESSOR K. RAMACHANDRA,
SCHOOL OF MATHEMATICS,
TATA INSTITUTE OF FUNDAMENTAL RESEARCH,
HOMI BHABHA ROAD,
BOMBAY - 400 005,
INDIA

(MANUSCRIPT COMPLETED ON 18-AUGUST-1989).