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A LEMMA IN COMPLEX FUNCTION THEORY-II

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§1. INTRODUCTION. This is a continuation of [1] and [2]. However the method here is different and is self-contained. In [2] we proved a general result which implied the following

THEOREM 1. Let \( f(z) \) be analytic in \( |z| \leq r \) and there let \( |f(z)| \leq M, (M \geq 3) \). Let \( A \geq 1 \). Then

\[
|f(0)| \leq (24A \log M)\left(\frac{1}{2r} \int_{-r}^{r} |f(iy)| \ dy\right) + M^{-A}. \tag{1}
\]

We also proved a corresponding result with \( |f(z + iy)| \) in place of \( |f(iy)| \), with suitable restrictions on \( z \) and also on the range of integration namely on \( y \). These are statements about \( |f(z)| \) where \( f(z) \) is analytic. We now consider \( |f(z)|^k \) where \( k > 0 \) is any real number independent of \( z \). We prove

THEOREM 2. Let \( k \) be any positive real number. Let \( f(z) \) be analytic in \( |z| \leq 2r \) and there \( |f(z)|^k \leq M(M \geq 9) \). Let \( z = r(\log M)^{-1} \), and let \( x_1 \) be any real number with \( |x_1| \leq x \). Put \( r_0 = \sqrt{4r^2 - x_1^2} \). Then with \( A \geq 1 \) we have

\[
|f(0)|^k \leq 2e^{84A}M^{-A} + \frac{24}{(2\pi)^2}e^{84A}\log M\left(\frac{1}{2r_0} \int_{-r_0}^{r_0} |f(z_1 + iy)|^k \ dy\right). \tag{2}
\]
REMARK 1. It is easy to remember a somewhat crude result namely

\[ |f(0)|^k \leq e^{90A} \{ M^{-A} + (\log M) \left( \frac{1}{2\pi R_0} \int_{-R_0}^{R_0} |f(z_1 + iy)|^k \, dy \} \}. \] (2')

REMARK 2. In Theorem 1 the constants are reasonably small whereas in Theorem 2 they are big. We have not attempted to get optimal constants.

REMARK 3. Let \( k_1, k_2, \ldots, k_m \) be any set of positive real numbers. Let \( f_1(z), f_2(z), \ldots, f_m(z) \) be analytic in \( |z| \leq 2r \), and there

\[ |(f_1(z))^{k_1} \cdots (f_m(z))^{k_m}| \leq M (M \geq 9). \]

Then Theorem 2 holds good with \( |f(z)|^k \) replaced by \( |(f_1(z))^{k_1} \cdots (f_m(z))^{k_m}| \).

REMARK 4. A corollary to our result mentioned in Remark 3 was pointed out to us by Professor J.P. Demailly. It is this: Theorem 2 holds good with \( |f(z)|^k \) replaced by \( \text{Exp}(u) \) where \( u \) is any subharmonic function. To prove this it suffices to note that the set of functions of the form \( \sum_{j=1}^{m} k_j \log |f_j(z)| \) is dense in \( L_{loc}^1 \) in the set of subharmonic functions. (This follows by using Green-Riesz representation formula for \( u \) and approximating the measure \( \Delta_u \) by finite sums of Dirac measures).

REMARK 5. Consider \( k = 1 \) in Theorem 2. Put \( \varphi(z) = f^{(\ell)}(z) \) the \( \ell \)th derivative of \( f(z) \). Then our method of proof gives

\[ |\varphi(0)| \leq CM^{-A} + C(\log M)^{\ell+1} \left( \frac{1}{4r} \int_{-4r}^{4r} |f(iy)| \, dy \right) \]

where \( C \) depends only on \( A \) and \( \ell \).

REMARK 6. (Due to J.-P. Demailly). In view of the example \( f(z) = \left( \frac{\pi z}{n \pi} \right)^2 \), where \( n \) is a large positive integer and \( r = 1 \), the result of Remark 5 is best possible.

§ 2. PROOF OF THEOREM 2. The proof consists of four steps.

STEP 1. First we consider the circle \( |z| = r \). Let

\[ 0 < 2z \leq r \] (3)

and let PQS denote respectively the points \( re^{i\theta} \) where \( \theta = -\cos^{-1}(\frac{2\pi}{r}), \cos^{-1}(\frac{2\pi}{r}) \) and \( r \). By the consideration of Riemann mapping theorem and the zero cancellation factors we have for a suitable meromorphic function \( \phi(z) \) (in PQSP)
that (we can assume that \( f(z) \) has no zeros on the boundary)

\[
F(z) = (\phi(z)f(z))^k
\]  

(4)

is analytic in the region enclosed by the straight line PQ and the circular arc QSP. Further \( \phi(z) \) satisfies

\[
| \phi(z) | = 1
\]  

(5)

on the boundary of PQSP and also

\[
| \phi(0) | \geq 1.
\]  

(6)

Let

\[
X = \text{Exp}(u_1 + u_2 + \ldots + u_n)
\]  

(7)

where \( u_1, u_2, \ldots, u_n \) vary over the box \( B \) defined by

\[
0 \leq u_j \leq B(j = 1, 2, \ldots, n),
\]

and \( B > 0 \).

We begin with

**LEMMA 1.** The function \( F(z) \) defined above satisfies

\[
F(0) = I_1 + I_2
\]  

(8)

where

\[
I_1 = \frac{1}{2\pi i} \int_{PQ} F(z)X^z \frac{dz}{z}
\]  

(9)

and

\[
I_2 = \frac{1}{2\pi i} \int_{QSP} F(z)X^z \frac{dz}{z}
\]  

(10)

where the lines of integration are the straight line PQ and the circular arc QSP.

**PROOF.** Follows by Cauchy's theorem.

**LEMMA 2.** We have

\[
| I_1 | \leq \frac{e^{2Bnx}}{2\pi} \int_{PQ} |(f(z))^k \frac{dz}{z} |
\]  

(11)

**PROOF.** Follows since \( |X^z| \leq e^{2Bnx} \) and also \( |\phi(z)| = 1 \) on PQ.
**LEMMA 3.** We have,

\[ |B^{-n} \int_B I_2 du_1 \ldots du_n| \leq e^{2Bn\pi}(\frac{2}{Br})^n M. \]  

**PROOF.** Follows since on QSP we have \( |\phi(z)| = 1 \) (and so \( |F(z)| \leq M \)) and also

\[ |B^{-n} \int_B \left( \int_{QSP} X^z \frac{dz}{2\pi i z} \right) du_1 \ldots du_n| \leq \left( \frac{2}{Br} \right)^n. \]

**LEMMA 4.** We have,

\[ |f(0)|^k \leq e^{2Bn\pi}(\frac{2}{Br})^n M + \frac{e^{2Bn\pi}}{2\pi} \int_{PQ} \left| (f(z))^{k} \frac{dz}{z} \right|. \]  

**PROOF.** Follows by Lemmas 1, 2, and 3.

**STEP 2.** Next in (13), we replace \( |f(z)|^k \) by an integral over a chord \( P_1Q_1 \) (parallel to PQ) of \( |w| = 2r \), of slightly bigger length with a similar error. Let \( z_1 \) be any real number with

\[ |z_1| \leq z. \]  

Let \( P_1Q_1R_1 \) be the points \( 2re^{i\theta} \)

\[ \begin{align*} &where \, \theta = -\cos^{-1}(\frac{2r}{\sqrt{z}}), 0 \, and \, \cos^{-1}(\frac{2r}{z}). \\ & (\text{If } z_1 \text{ is negative we have to consider the points} \\ & \theta = -\frac{\pi}{2} - \sin^{-1}(\frac{2r}{\sqrt{z}}), 0 \, and \frac{\pi}{2} + \sin^{-1}(\frac{2r}{\sqrt{z}})). 
\end{align*} \]  

Let \( X \) be as in (7). As before let

\[ G(w) = (\psi(w)f(w))^k \]  

be analytic in the region enclosed by the circular arc \( P_1R_1Q_1 \) and the straight line \( Q_1P_1 \) (we can assume that \( f(z) \) has no zeros on the boundary \( P_1R_1Q_1P_1 \)). By the consideration of Riemann mapping theorem and the zero cancelling factors there exists such a meromorphic function \( \psi(w) \) (in \( P_1R_1Q_1P_1 \)) with the extra properties,

\[ |\psi(w)| = 1 \text{ on the boundary of } P_1R_1Q_1P_1 \text{ and } |\psi(z)| \geq 1. \]

**LEMMA 5.** We have with \( z \) on PQ,

\[ G(z) = I_3 + I_4 \]
where
\[ I_3 = \frac{1}{2\pi i} \int_{Q_1, P_1} G(w)X^{-(w-z)} \frac{dw}{w-z} \]  \hspace{1cm} (18)
and
\[ I_4 = \frac{1}{2\pi i} \int_{P_1 R_1 Q_1} G(w)X^{-(w-z)} \frac{dw}{w-z}. \]  \hspace{1cm} (19)

**PROOF.** Follows by Cauchy’s theorem

**LEMMA 6.** We have with \( z \) on \( PQ \)
\[ |I_5| \leq \frac{e^{3B_nz}}{2\pi} \int_{P_1 Q_1} |(f(w))^k \frac{dw}{w-z}| \]  \hspace{1cm} (20)

**PROOF.** Follows since \( |X^{-(w-z)}| \leq e^{3B_nz} \) and \( |\psi(w)| = 1 \) on \( P_1 Q_1 \).

**LEMMA 7.** We have with \( z \) on \( PQ \),
\[ |B^{-n} \int_B I_4 u_1...u_n| \leq e^{3B_nz} \left( \frac{2}{Br} \right)^n M. \]  \hspace{1cm} (21)

**PROOF.** Follows since on \( P_1 R_1 Q_1 \) we have \( |\psi(w)| = 1 \) (and so \( |G(w)| \leq M \)) and also
\[ |B^{-n} \int_B \int z^{-(w-z)} \frac{dw}{2\pi i(w-z)} u_1...u_n| \leq \left( \frac{2}{B_r} \right)^n. \]

**LEMMA 8.** We have with \( z \) on \( PQ \),
\[ |f(z)|^k \leq e^{3B_nz} \left( \frac{2}{B_r} \right)^n M + \frac{e^{3B_nz}}{2\pi} \int_{P_1 Q_1} |f(w)|^k \frac{dw}{w-z}|. \]  \hspace{1cm} (22)

**PROOF.** Follows from Lemmas 5, 6 and 7.

**STEP 3.** We now combine Lemmas 4 and 8.

**LEMMA 9.** We have
\[ |f(0)|^k \leq e^{2B_nz} \left( \frac{2}{B_r} \right)^n M + J_1 + J_2 \]  \hspace{1cm} (23)

where
\[ J_1 = \frac{e^{5B_nz}}{2\pi} \left( \frac{2}{B_r} \right)^n M \int_{PQ} \left| \frac{dz}{z} \right|, \]  \hspace{1cm} (24)

and
\[ J_2 = \frac{e^{5B_nz}}{(2\pi)^2} \int_{P_1 Q_1} |f(w)|^k \left( \int_{PQ} \left| \frac{dz}{z(w-z)} \right| \right) dw \]  \hspace{1cm} (25)
LEMMA 10. We have
\[ \int_{PQ} \left| \frac{dz}{z} \right| \leq 2 + 2 \log \left( \frac{r}{2z} \right). \]  
(26)

PROOF. On PQ we have \( z = 2x + iy \) with \( |y| \leq r \) and \( 2x \leq r \). We split the integral into \( |y| \leq 2z \) and \( 2x \leq |y| \leq r \). On these, we use respectively the lower bounds \( |z| \geq 2x \) and \( |z| \geq y \). The lemma follows by these observations.

LEMMA 11. We have for \( w \) on \( P_1Q_1 \) and \( z \) on \( PQ \),
\[ \int_{PQ} \left| \frac{dz}{z(w - z)} \right| \leq \frac{6}{z}. \]  
(27)

PROOF. On PQ we have \( Re \ z = 2x \) and on \( P_1Q_1 \) we have \( Re \ w \leq x \) and so \( Re (w - z) \geq z \). We have
\[ \left| \frac{dz}{z(w - z)} \right| \leq \left| \frac{dz}{z^2} \right| + \left| \frac{dz}{(w - z)^2} \right|. \]

Writing \( z = 2z + iy \) we have
\[ \int_{PQ} \left| \frac{dz}{z} \right| \leq \frac{2}{(2z)^2} 2x + 2 \int_{2z}^{\infty} \frac{dy}{y^2} \]
\[ = \frac{2}{x}. \]

Similarly
\[ \int_{PQ} \left| \frac{dz}{(w - z)^2} \right| \leq 2(\frac{1}{x} + \int_{x}^{\infty} \frac{dy}{y}) \]
\[ = \frac{4}{x}. \]

This completes the proof of the lemma.

STEP 4. We collect together the results in Steps 3 and 4 and choose the parameters \( B \) and \( n \) and this will give Theorem 2. Combining Lemmas 9, 10 and 11 we state the following lemma.

LEMMA 12. We have
\[ |f(0)|^k \leq e^{2Bn\pi}(\frac{2}{Br})^nM + \frac{e^{5Bn\pi}}{r}(\frac{2}{Br})^n(1 + log \frac{r}{2z})M \]
\[ + \frac{e^{5Bn\pi}}{(2\pi)^2} \frac{6}{z} \int_{P_1Q_1} \left| (f(w))^k \right| dw, \]  
(28)
where $0 < 2x \leq r, x_1$ is any real number with $|x_1| \leq x, n$ any natural number and $B$ is any positive real number and $P_1Q_1$ is the straight line joining $-r_0$ and $r_0$ where $r_0 = \sqrt{4r^2 - x_1^2}$.

Next we note that $1 + \log \frac{x}{2x} \leq \frac{x}{2x}$ and so by putting $x = r \log M$ the first two terms on the RHS of (28) together do not exceed

$$\left(\frac{2}{Br}\right)^n e^{5Bnx}(1 + \frac{1}{2x} \log M)M \leq 2\left(\frac{2}{Br}\right)^n e^{5Bnx}M \log M.$$  

Also,

$$\frac{6}{x} = \frac{6 \log M}{r} = 6 \log M \left(\frac{2r_0}{r}\right) \frac{1}{2r_0} \leq (24 \log M)(\frac{1}{2r_0}).$$

Thus RHS of (28) does not exceed

$$2\left(\frac{2}{Br}\right)^n e^{5Bnx}M \log M + \left(\frac{24}{(2\pi)^2}\right) e^{5Bnx} \log M \left(\frac{1}{2r_0} \int_{P_1Q_1} |(f(w))^k dw| \right).$$

We have chosen $x = r \log M$. We now choose $B$ such that $Br = 2e$ and $n = [C \log M] + 1$, where $C \geq 1$ is any real number. We have $5Bnx \leq \frac{5Bnx}{\log M} \leq 10e(C + 1) \leq 28(C + 1)$ and also

$$(\frac{2}{Br})^n \leq e^{-C \log M} = M^{-C}.$$  

With these choices of $z, B, n$ we see that RHS of (28) does not exceed

$$2M^{-C}e^{28(C+1)}\log M + \left(\frac{24}{(2\pi)^2}\right) e^{28(C+1)} \log M \left(\frac{1}{2r_0} \int_{P_1Q_1} |f(w)|^k dw\right).$$

Putting $C = A + 2$ we obtain Theorem 2 since $C + 1 \leq 3A$. This completes the proof of Theorem 2.

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