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A LEMMA IN COMPLEX FUNCTION THEORY-II

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§1. **INTRODUCTION.** This is a continuation of [1] and [2]. However the method here is different and is self-contained. In [2] we proved a general result which implied the following

THEOREM 1. *Let $f(z)$ be analytic in $|z| \leq r$ and there let $|f(z)| \leq M, (M \geq 3)$. Let $A \geq 1$. Then*

$$|f(0)| \leq (24A \log M)\left(\frac{1}{2r}\int_{-r}^{r} |f(iy)|\, dy\right) + M^{-A}. \qquad (1)$$

We also proved a corresponding result with $|f(x+iy)|$ in place of $|f(iy)|$, with suitable restrictions on x and also on the range of integration namely on y. These are statements about $|f(z)|$ where $f(z)$ is analytic. We now consider $|f(z)|^k$ where $k > 0$ is any real number independent of z. We prove

THEOREM 2. *Let k be any positive real number. Let $f(z)$ be analytic in $|z| \leq 2r$ and there $|f(z)|^k \leq M (M \geq 9)$. Let $x = r(\log M)^{-1}$, and let x_1 be any real number with $|x_1| \leq x$. Put $r_0 = \sqrt{4r^2 - x_1^2}$. Then with $A \geq 1$ we have*

$$|f(0)|^k \leq 2e^{84A}M^{-A} + \frac{24}{(2\pi)^2}e^{84A}\log M\left(\frac{1}{2r_0}\int_{-r_0}^{r_0} |f(x_1+iy)|^k\, dy\right). \qquad (2)$$

REMARK 1. It is easy to remember a somewhat crude result namely

$$| f(0) |^k \leq e^{90A}\{M^{-A} + (log\ M)(\frac{1}{2r_0}\int_{-r_0}^{r_0} | f(x_1 + iy) |^k\ dy)\}. \qquad (2')$$

REMARK 2. In Theorem 1 the constants are reasonably small whereas in Theorem 2 they are big. We have not attempted to get optimal constants.

REMARK 3. Let $k_1, k_2, ..., k_m$ be any set of positive real numbers. Let $f_1(z), f_2(z), ..., f_m(z)$ be analytic in $| z | \leq 2r$, and there

$$| (f_1(z))^{k_1} ...(f_m(z))^{k_m} | \leq M (M \geq 9).$$

Then Theorem 2 holds good with $| f(z) |^k$ replaced by $| (f_1(z))^{k_1} ...(f_m(z))^{k_m} |$.

REMARK 4. A corollary to our result mentioned in Remark 3 was pointed out to us by Professor J.P. Demailly. It is this : Theorem 2 holds good with $| f(z) |^k$ replaced by $Exp(u)$ where u is any subharmonic function. To prove this it suffices to note that the set of functions of the form $\sum_{j=1}^{m} k_j\ log\ | f_j(z) |$ is dense in L^1_{loc} in the set of subharmonic functions. (This follows by using Green-Riesz representation formula for u and approximating the measure Δ_u by finite sums of Dirac measures).

REMARK 5. Consider $k = 1$ in Theorem 2. Put $\varphi(z) = f^{(\ell)}(z)$ the ℓth derivative of $f(z)$. Then our method of proof gives

$$| \varphi(0) | \leq CM^{-A} + C(log\ M)^{\ell+1}(\frac{1}{4r}\int_{-4r}^{4r} | f(iy) |\ dy),$$

where C depends only on A and ℓ.

REMARK 6. (Due to J.-P. Demailly). In view of the example $f(z) = (\frac{e^{nz}-1}{nz})^2$, where n is a large positive integer and $r = 1$, the result of Remark 5 is best possible.

§ 2. **PROOF OF THEOREM 2.** The proof consists of four steps.

STEP 1. First we consider the circle $| z | = r$. Let

$$0 < 2x \leq r \qquad (3)$$

and let PQS denote respectively the points $re^{i\theta}$ where $\theta = -cos^{-1}(\frac{2x}{r}), cos^{-1}(\frac{2x}{r})$ and π. By the consideration of Riemann mapping theorem and the zero cancellation factors we have for a suitable meromorphic function $\phi(z)$ (in PQSP) that (we can assume that $f(z)$ has no zeros on the boundary)

$$F(z) = (\phi(z)f(z))^k \tag{4}$$

is analytic in the region enclosed by the straight line PQ and the circular arc QSP. Further $\phi(z)$ satisfies

$$|\phi(z)| = 1 \tag{5}$$

on the boundary of PQSP and also

$$|\phi(0)| \geq 1. \tag{6}$$

Let

$$X = Exp(u_1 + u_2 + \ldots + u_n) \tag{7}$$

where $u_1, u_2, \ldots, u_n$ vary over the box B defined by

$$0 \leq u_j \leq B (j = 1, 2, \ldots, n),$$

and $B > 0$.

We begin with

LEMMA 1. *The function $F(z)$ defined above satisfies*

$$F(0) = I_1 + I_2 \tag{8}$$

where

$$I_1 = \frac{1}{2\pi i}\int_{PQ} F(z)X^z \frac{dz}{z} \tag{9}$$

and

$$I_2 = \frac{1}{2\pi i}\int_{QSP} F(z)X^z \frac{dz}{z} \tag{10}$$

where the lines of integration are the straight line PQ *and the circular arc* QSP.

PROOF. Follows by Cauchy's theorem.

LEMMA 2. *We have*

$$|I_1| \leq \frac{e^{2Bnx}}{2\pi}\int_{PQ} \left|(f(z))^k \frac{dz}{z}\right| \tag{11}$$

PROOF. Follows since $|X^z| \leq e^{2Bnx}$ and also $|\phi(z)| = 1$ on PQ.

LEMMA 3. *We have,*

$$| B^{-n} \int_{B} I_2 du_1 ... du_n | \leq e^{2Bnx} (\frac{2}{Br})^n M. \tag{12}$$

PROOF. Follows since on QSP we have $| \phi(z) | = 1$ (and so $| F(z) | \leq M$) and also

$$| B^{-n} \int_{B} (\int_{QSP} X^z \frac{dz}{2\pi i z}) du_1 ... du_n | \leq (\frac{2}{Br})^n.$$

LEMMA 4. *We have,*

$$| f(0) |^k \leq e^{2Bnx} (\frac{2}{Br})^n M + \frac{e^{2Bnx}}{2\pi} \int_{PQ} | (f(z))^k \frac{dz}{z} | . \tag{13}$$

PROOF. Follows by Lemmas 1,2 and 3.

STEP 2. Next in (13), we replace $| f(z) |^k$ by an integral over a chord P_1Q_1 (parallel to PQ) of $| w | = 2r$, of slightly bigger length with a similar error. Let x_1 be any real number with

$$| x_1 | \leq x. \tag{14}$$

$$\left\{ \begin{array}{l} \text{Let } P_1 Q_1 R_1 \text{ be the points } 2re^{i\theta} \\ \text{where } \theta = -cos^{-1}(\frac{x_1}{2r}), 0 \text{ and } cos^{-1}(\frac{x_1}{2r}). \\ \text{(If } x_1 \text{ is negative we have to consider the points} \\ \theta = -\frac{\pi}{2} - sin^{-1}(\frac{x_1}{2r}), 0 \text{ and} \frac{\pi}{2} + sin^{-1}(\frac{x_1}{2r})). \end{array} \right. \tag{15}$$

Let X be as in (7). As before let

$$G(w) = (\psi(w) f(w))^k \tag{16}$$

be analytic in the region enclosed by the circular arc $P_1R_1Q_1$ and the straight line Q_1P_1 (we can assume that $f(z)$ has no zeros on the boundary $P_1R_1Q_1P_1$). By the consideration of Riemann mapping theorem and the zero cancelling factors there exists such a meromorphic function $\psi(w)$ (in $P_1R_1Q_1P_1$) with the extra properties,

$$| \psi(w) | = 1 \text{ on the boundary of } P_1R_1Q_1P_1 \text{ and } | \psi(z) | \geq 1. \tag{17}$$

LEMMA 5. *we have with z on PQ,*

$$G(z) = I_3 + I_4$$

where

$$I_3 = \frac{1}{2\pi i}\int_{Q_1P_1} G(w)X^{-(w-z)}\frac{dw}{w-z} \tag{18}$$

and

$$I_4 = \frac{1}{2\pi i}\int_{P_1R_1Q_1} G(w)X^{-(w-z)}\frac{dw}{w-z}. \tag{19}$$

PROOF. Follows by Cauchy's theorem

LEMMA 6. *We have with z on* PQ

$$| I_3 |\leq \frac{e^{3Bnx}}{2\pi}\int_{P_1Q_1} | (f(w))^k\frac{dw}{w-z} | \tag{20}$$

PROOF. Follows since $| X^{-(w-z)} |\leq e^{3Bnx}$ and $| \psi(w) |= 1$ on P_1Q_1.

LEMMA 7. *We have with z on* PQ,

$$| B^{-n}\int_B I_4du_1...du_n |\leq e^{3Bnx}(\frac{2}{Br})^n M. \tag{21}$$

PROOF. Follows since on $P_1R_1Q_1$ we have $| \psi(w) |= 1$ (and so $| G(w) |\leq M$) and also

$$| B^{-n}\int_B\int x^{-(w-z)}\frac{dw}{2\pi i(w-z)}du_1...du_n |\leq (\frac{2}{Br})^n.$$

LEMMA 8. *We have with z on* PQ,

$$| f(z) |^k\leq e^{3Bnx}(\frac{2}{Br})^n M + \frac{e^{3Bnx}}{2\pi}\int_{P_1Q_1} | f(w) |^k | \frac{dw}{w-z} | . \tag{22}$$

PROOF. Follows from Lemmas 5,6 and 7.

STEP 3. We now combine Lemmas 4 and 8.

LEMMA 9. *We have*

$$| f(0) |^k\leq e^{2Bnx}(\frac{2}{Br})^n M + J_1 + J_2 \tag{23}$$

where

$$J_1 = \frac{e^{5Bnx}}{2\pi}(\frac{2}{Br})^n M\int_{PQ} | \frac{dz}{z} |, \tag{24}$$

and

$$J_2 = \frac{e^{5Bnx}}{(2\pi)^2}\int_{P_1Q_1} | f(w) |^k (\int_{PQ} | \frac{dz}{z(w-z)} |) | dw \tag{25}$$

LEMMA 10. *We have*

$$\int_{PQ} \left| \frac{dz}{z} \right| \leq 2 + 2\, log(\frac{r}{2x}). \tag{26}$$

PROOF. On PQ we have $z = 2x + iy$ with $| y | \leq r$ and $2x \leq r$. We split the integral into $| y | \leq 2x$ and $2x \leq | y | \leq r$. On these, we use respectively the lower bounds $| z | \geq 2x$ and $| z | \geq y$. The lemma follows by these observations.

LEMMA 11. *We have for w on P_1Q_1 and z on* PQ,

$$\int_{PQ} \left| \frac{dz}{z(w-z)} \right| \leq \frac{6}{x}. \tag{27}$$

PROOF. On PQ we have $Re\ z = 2x$ and on P_1Q_1 we have $| Re\ w | \leq x$ and so $| Re(w - z) | \geq x$. We have

$$\left| \frac{dz}{z(w-z)} \right| \leq \left| \frac{dz}{z^2} \right| + \left| \frac{dz}{(w-z)^2} \right|.$$

Writing $z = 2x + iy$ we have

$$\begin{aligned} \int_{PQ} \left| \tfrac{dz}{z^2} \right| &\leq \tfrac{2}{(2x)^2} 2x + 2\int_{2x}^{\infty} \tfrac{dy}{y^2} \\ &= \tfrac{2}{x}. \end{aligned}$$

Similarly

$$\begin{aligned} \int_{PQ} \left| \tfrac{dz}{(w-z)^2} \right| &\leq 2(\tfrac{1}{x} + \int_{x}^{\infty} \tfrac{dy}{y^2}) \\ &= \tfrac{4}{x}. \end{aligned}$$

This completes the proof of the lemma.

STEP 4. We collect together the results in Steps 3 and 4 and choose the parameters B and n and this will give Theorem 2. Combining Lemmas 9,10 and 11 we state the following lemma.

LEMMA 12. *We have*

$$\begin{aligned} | f(0) |^k \leq e^{2Bnx}(\frac{2}{Br})^n M + \frac{e^{5Bnx}}{\pi}(\frac{2}{Br})^n(1 + log\frac{r}{2x})M \\ + \frac{e^{5Bnx}}{(2\pi)^2} \cdot \frac{6}{x} \int_{P_1Q_1} | (f(w))^k dw |, \end{aligned} \tag{28}$$

where $0 < 2x \leq r, x_1$ *is any real number with* $| x_1 | \leq x, n$ *any natural number and* B *is any positive real number and* P_1Q_1 *is the straight line joining* $-r_0$ *and* r_0 *where* $r_0 = \sqrt{4r^2 - x_1^2}$.

Next we note that $1 + log\frac{r}{2x} \leq \frac{r}{2x}$ and so by putting $x = r(log\ M)^{-1}$ the first two terms on the RHS of (28) together do not exceed

$$(\frac{2}{Br})^n e^{5Bnx}(1 + \frac{1}{2\pi} log\ M)M \leq 2(\frac{2}{Br})^n e^{5Bnx} M\ log\ M.$$

Also,

$$\frac{6}{x} = \frac{6\ log\ M}{r} = 6\ log\ M(\frac{2r_0}{r})\frac{1}{2r_0} \leq (24\ log\ M)(\frac{1}{2r_0}).$$

Thus RHS of (28) does not exceed

$$2(\frac{2}{Br})^n e^{5Bnx} M\ log\ M + (\frac{24}{(2\pi)^2} e^{5Bnx} log\ M)(\frac{1}{2r_0}\int_{P_1Q_1} | (f(w))^k dw |).$$

We have chosen $x = r(log\ M)^{-1}$. We now choose B such that $Br = 2e$ and $n = [C\ log\ M] + 1$, where $C \geq 1$ is any real number. We have $5Bnx \leq \frac{5Bnr}{log\ M} \leq 10e(C+1) \leq 28(C+1)$ and also

$$(\frac{2}{Br})^n \leq e^{-C\ log\ M} = M^{-C}.$$

With these choices of x, B, n we see that RHS of (28) does not exceed

$$2M^{-C} e^{28(C+1)} M\ log\ M + (\frac{24}{(2\pi)^2} e^{28(C+1)} log\ M)(\frac{1}{2r_0}\int_{P_1Q_1} | f(w))^k dw |).$$

Putting $C = A + 2$ we obtain Theorem 2 since $C + 1 \leq 3A$. This completes the proof of Theorem 2.

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