

A q -analog of Jacobi's two squares formula and its applications

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Abstract. We consider a q -analog $r_2(n, q)$ of the number of representations of an integer as a sum of two squares $r_2(n)$. This q -analog is generated by the expansion of a product that was studied by Kronecker and Jordan. We generalize Jacobi's two squares formula from $r_2(n)$ to $r_2(n, q)$. We characterize the signs in the coefficients of $r_2(n, q)$ using the prime factors of n . We use $r_2(n, q)$ to characterize the integers which are the length of the hypotenuse of a primitive Pythagorean triangle.

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1. Introduction

Using the formal identity

$$\prod_{m=1}^{\infty} \frac{1 - (-t)^m}{1 + (-t)^m} = \sum_{n=-\infty}^{\infty} t^{n^2}, \quad (1.1)$$

due to C. F. Gauss [[Gau1866](#)], the generating function for the number of representations of an integer n as the sum of the squares of two integers, denoted $r_2(n)$, immediately follows,

$$\prod_{m=1}^{\infty} \frac{(1 - (-t)^m)^2}{(1 + (-t)^m)^2} = 1 + \sum_{n=1}^{\infty} r_2(n) t^n. \quad (1.2)$$

C. G. J. Jacobi [[Jac1829](#)] expressed $r_2(n)$ as a function of the divisors of $n \geq 1$,

$$r_2(n) = 4d_{1,4}(n) - 4d_{3,4}(n), \quad (1.3)$$

where $d_{k,m}(n)$ is the number of divisors of n which are congruent to k modulo m .

The explicit formula for $r_2(n, q)$ defined by the expansion of the product

$$\prod_{m=1}^{\infty} \frac{(1 - (-qt)^m)^2}{(1 + q(-qt)^m)(1 + q^{-1}(-qt)^m)} = 1 + \sum_{n=1}^{\infty} r_2(n, q) t^n, \quad (1.4)$$

which is a q -deformation¹ of identity (1.2), can be attributed to L. Kronecker [[Kro1890](#)], who proved a more general version and C. Jordan [[Jor1894](#)], who described a method to derive this particular case. The polynomial $r_2(n, q)$ was first introduced in the article [[Cab19c](#)], using the notation $\Gamma_n(q)$, and it was called *Kassel-Reutenauer q -analog of the number of representations as a sum of two squares*. Some versions of the polynomial $r_2(n, q)$, e.g., changing the sign of q and sometimes dividing it by $q - 1$ or by $(q - 1)^2$, have been studied by several authors, in connection with different branches of mathematics: finite fields [[KR18a](#), [KR18b](#), [Cab18](#)], algebraic topology [[HLR13](#)], modular functions [[KR17](#)] and elementary number theory [[Cab19b](#), [Cab20](#), [Cab19a](#)].

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¹A q -deformation of an expression $A(t)$ is another expression $A(t, q)$ satisfying $A(t, 1) = A(t)$.

The aim of the present note is to prove that there are polynomials $4d_{1,4}(n, q)$ and $4d_{3,4}(n, q)$ satisfying the following properties²:

- (i) all the coefficients of $4d_{1,4}(n, q)$ and $4d_{3,4}(n, q)$ are non-negative integers;
- (ii) for every k , q^k cannot appear with non-zero coefficient in both $4d_{1,4}(n, q)$ and $4d_{3,4}(n, q)$;
- (iii) the decomposition of $r_2(n, q)$ into positive and negative parts,

$$r_2(n, q) = 4d_{1,4}(n, q) - 4d_{3,4}(n, q) \tag{1.5}$$

holds;

(iv) $4d_{1,4}(n, q)$ and $4d_{3,4}(n, q)$ are q -analogs of $4d_{1,4}(n)$ and $4d_{3,4}(n)$ respectively, i.e., $4d_{1,4}(n, 1) = 4d_{1,4}(n)$ and $4d_{3,4}(n, 1) = 4d_{3,4}(n)$.

Therefore, the identity (1.5) is a generalization of Jacobi's two squares formula from integers to polynomials. Considering that properties (i), (ii) and (iii) uniquely define the polynomials $4d_{1,4}(n, q)$ and $4d_{3,4}(n, q)$, it is non-trivial that they should also satisfy property (iv). Furthermore, as applications of this formula, we will determine when $r_2(n, q)$ has a negative coefficient by analyzing the prime factors of n . Also, we will use $r_2(n, q)$ to characterize the integers which are the length of the hypotenuse of a primitive Pythagorean triangle.

2. Generalization of Jacobi's formula

In this section we will prove our main result.

Theorem 2.1. *Let n be a positive integer. The polynomials*

$$4d_{1,4}(n, q) = (q + 1) \sum_{\substack{d|n \\ d \equiv 1 \pmod{4}}} \left(q^{\binom{2n}{d} + 1} q^{(d-1)/2} + q^{\binom{2n}{d} - 1} q^{(d+1)/2} \right), \tag{2.6}$$

$$4d_{3,4}(n, q) = (q + 1) \sum_{\substack{d|n \\ d \equiv 3 \pmod{4}}} \left(q^{\binom{2n}{d} + 1} q^{(d-1)/2} + q^{\binom{2n}{d} - 1} q^{(d+1)/2} \right), \tag{2.7}$$

satisfy properties (i), (ii), (iii) and (iv).

Proof. Property (i) immediately follows from the explicit expressions (2.6) and (2.7). Property (iv) is just the result of the evaluations of these expressions at $q = 1$.

Property (iii) follows from the following formal manipulation. Take formula (0.100) from [Coo17],

$$\prod_{m=1}^{\infty} \frac{(1 - t^m)^2}{(1 - qt^m)(1 - q^{-1}t^m)} = 1 + (q^{1/2} - q^{-1/2}) \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} t^{dk} \left(q^{d-k/2} - q^{-d+k/2} \right). \tag{2.8}$$

Replace t by qt in (2.8),

$$\prod_{m=1}^{\infty} \frac{(1 - (qt)^m)^2}{(1 - q(qt)^m)(1 - q^{-1}(qt)^m)} = 1 + (q - 1) \sum_{d=1}^{\infty} \sum_{\substack{k \geq 1 \\ k \text{ odd}}} t^{dk} \left(q^{dk+d-k/2-1/2} - q^{dk-d+k/2-1/2} \right), \tag{2.9}$$

where the restriction to only odd values of k is because of the identity

$$\sum_{d=1}^{\infty} \sum_{\substack{k \geq 1 \\ k \text{ even}}} t^{dk} \left(q^{dk+d-k/2-1/2} - q^{dk-d+k/2-1/2} \right) = \sum_{d=1}^{\infty} \sum_{e=1}^{\infty} t^{2de} \left(q^{2de+d-e-1/2} - q^{2de-d+e-1/2} \right) \tag{2.10}$$

$$= q^{-1/2} \left(\sum_{d=1}^{\infty} \sum_{e=1}^{\infty} t^{2de} q^{2de+d-e} - \sum_{d=1}^{\infty} \sum_{e=1}^{\infty} t^{2de} q^{2de-d+e} \right) \tag{2.11}$$

$$= 0. \tag{2.12}$$

²We consider that it is more elegant to work with $4d_{1,4}(n, q)$ and $4d_{3,4}(n, q)$ rather than $d_{1,4}(n, q)$ and $d_{3,4}(n, q)$ because of property (i).

Replace q by $-q$ in (2.9),

$$\prod_{m=1}^{\infty} \frac{(1 - (-qt)^m)^2}{(1 + q(-qt)^m)(1 + q^{-1}(-qt)^m)} = 1 + (q+1) \sum_{d=1}^{\infty} \sum_{\substack{k \geq 1 \\ k \text{ odd}}} t^{dk} \left((-q)^{(2d+1)(k-1)/2} - (-q)^{(2d-1)(k+1)/2} \right) \quad (2.13)$$

$$\begin{aligned} &= 1 + (q+1) \sum_{d=1}^{\infty} \sum_{\substack{k \geq 1 \\ k \equiv 1 \pmod{4}}} t^{dk} \left((-q)^{(2d+1)(k-1)/2} - (-q)^{(2d-1)(k+1)/2} \right) \\ &\quad + (q+1) \sum_{d=1}^{\infty} \sum_{\substack{k \geq 1 \\ k \equiv 3 \pmod{4}}} t^{dk} \left((-q)^{(2d+1)(k-1)/2} - (-q)^{(2d-1)(k+1)/2} \right) \end{aligned} \quad (2.14)$$

$$\begin{aligned} &= 1 + (q+1) \sum_{d=1}^{\infty} \sum_{\substack{k \geq 1 \\ k \equiv 1 \pmod{4}}} t^{dk} \left(q^{(2d+1)(k-1)/2} + q^{(2d-1)(k+1)/2} \right) \\ &\quad - (q+1) \sum_{d=1}^{\infty} \sum_{\substack{k \geq 1 \\ k \equiv 3 \pmod{4}}} t^{dk} \left(q^{(2d+1)(k-1)/2} + q^{(2d-1)(k+1)/2} \right) \end{aligned} \quad (2.15)$$

$$= 1 + \sum_{n=1}^{\infty} (4d_{1,4}(n, q) - 4d_{3,4}(n, q)) t^n. \quad (2.16)$$

To prove property (ii), we proceed by *reductio ad absurdum*. Suppose that for some k , the coefficient of q^k is non-zero in both $4d_{1,4}(n, q)$ and $4d_{3,4}(n, q)$. We need to analyze 16 possible cases. We will use the notations d and e for two arbitrary divisors of n satisfying $d \equiv 1 \pmod{4}$ and $e \equiv 3 \pmod{4}$.

Let $f(x) = \left(\frac{2n}{x} + 1\right) \frac{x-1}{2}$. Notice that $f(x)$, on the domain $x > 0$, is strictly increasing and satisfies the inequality $f(x+2) - f(x) > 1$. Furthermore, $f\left(\frac{2n}{x}\right) = \left(\frac{2n}{x} - 1\right) \frac{x+1}{2}$.

Notice that, $k = f(d) = f(e)$ implies $d = e$. Nevertheless, this is impossible because $d \not\equiv e \pmod{4}$. In the same way, it is easy to prove that $k = f\left(\frac{2n}{d}\right) = f\left(\frac{2n}{e}\right)$ also implies an absurde. Similarly, $k = f(d) + 1 = f(e) + 1$ and $k = f\left(\frac{2n}{d}\right) + 1 = f\left(\frac{2n}{e}\right) + 1$ are impossible.

Notice that $k = f(d) = f\left(\frac{2n}{e}\right)$ implies $d = \frac{2n}{e}$. If this is the case, $de = 2n$, which is absurd, since d and e are odd. In the same vein, $k = f\left(\frac{2n}{d}\right) = f(e)$ is also absurd. Similarly, we exclude the cases $k = f(d) + 1 = f\left(\frac{2n}{e}\right) + 1$ and $k = f\left(\frac{2n}{d}\right) + 1 = f(e) + 1$.

Assume that $k = f(d) + 1 = f(e)$. It follows that $e > d$. Because e and d share the same parity (both are odd), $e \geq d + 2$. Hence, $f(e) - f(d) \geq f(d+2) - f(d) > 1$, which contradicts our assumption. In the same vein, it is easy to prove that $k = f(d) = f(e) + 1$ implies an absurd conclusion. Similarly, we exclude the cases, $k = f\left(\frac{2n}{d}\right) + 1 = f\left(\frac{2n}{e}\right)$ and $k = f\left(\frac{2n}{d}\right) = f\left(\frac{2n}{e}\right) + 1$ by considering that $\frac{2n}{d}$ and $\frac{2n}{e}$ share the same parity (both are even).

Assume that $k = f(d) + 1 = f\left(\frac{2n}{e}\right)$. On the one hand, $f(d) = \left(\frac{2n}{d} + 1\right) \frac{d-1}{2}$ is even, since $\frac{d-1}{2}$ is even. On the other hand $f\left(\frac{2n}{e}\right) = \left(\frac{2n}{e} - 1\right) \frac{e+1}{2}$ is also even, since $\frac{e+1}{2}$ is even. We derive the absurd conclusion that $1 = f\left(\frac{2n}{e}\right) - f(d)$ should be even. In the same vein, we can easily prove that $k = f(d) = f\left(\frac{2n}{e}\right) + 1$ implies an absurde. Similarly, we can exclude the cases $k = f\left(\frac{2n}{d}\right) + 1 = f(e)$ and $k = f\left(\frac{2n}{d}\right) = f(e) + 1$.

3. Applications

In this section we derive some immediate consequences of our generalization of Jacobi's formula.

Corollary 3.1. *Let n be a positive integer. The polynomial $r_2(n, q)$ has a negative coefficient if and only if some of the prime factors of n are congruent to 3 modulo 4.*

Proof. Considering that $d_{3,4}(n, q) \neq 0$ if and only if some of the prime factors of n are congruent to 3 modulo 4, the result immediately follows from Theorem 2.1 and the definition of $d_{1,4}(n, q)$ and $d_{3,4}(n, q)$.

We recall that n is the hypotenuse of a primitive Pythagorean triangle if and only if for some pair of positive integers u and v the equality $u^2 + v^2 = n^2$ holds and u , v and n are relatively prime.

Corollary 3.2. *An odd integer n larger than 1 is the length of the hypotenuse of a primitive Pythagorean triangle if and only if all the coefficients of the polynomial $r_2(n, q)$ are non-negative.*

Proof. E. J. Eckert [Eck84] proved that an integer larger than 1 is the hypotenuse of a primitive Pythagorean triangle if and only if all its prime factors are congruent to 1 modulo 4. Combining this result with Corollary 3.1, the result follows.

4. Final remarks

In the spirit of the work of C. Kassel and C. Reutenauer [KR18b], the value of the polynomial $r_2(n, q)$, when q is a prime power, may have a combinatorial interpretation in the ring $\mathbb{F}_q[X, Y, X^{-1}, Y^{-1}]$.

Let $r_4(n)$ be the number of representations of n as the sum of 4 squares of integers. We suggest to empirically study the q -analog of $r_4(n)$ obtained from the square

$$\left(1 + \sum_{n=1}^{\infty} r_2(n, q) t^n\right)^2 = 1 + \sum_{n=1}^{\infty} r_4(n, q) t^n \quad (4.17)$$

and check whether some of the classical results about $r_4(n)$ can be generalized to $r_4(n, q)$. The expansion of the corresponding product can be found in equation (0.101) of [Coo17].

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