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A note on Fourier eigenfunctions in four dimensions

Daniel Lautzenheiser

Abstract. In this note, we exhibit a weakly holomorphic modular form for use in constructing a Fourier eigenfunction in four dimensions. Such auxiliary functions may be of use to the D4 checkerboard lattice and the four dimensional sphere packing problem.

Keywords. sphere packing, Fourier analysis, modular forms.

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1. Introduction

The sphere packing density is the proportion of $\mathbb{R}^d$ occupied by non-overlapping unit balls. In recent years, the sphere packing problem of finding the most dense arrangement of spheres in $\mathbb{R}^d$ has regained interest. This problem has been recently solved in 8 and 24 dimensions [Via17, CKM17] by means of constructing a specialized radial Schwartz function using Fourier and complex analytic methods. The sphere packing problem has otherwise only been solved in dimensions 1,2, and 3 [Tot43, Hal05]. A recent work [CLS] conjectures conditions for which the sphere packing problem in dimension 4 may be solved. In this paper, we present a function for possible use toward this conjecture.

If Λ is a lattice in $\mathbb{R}^d$ with minimal nonzero vector length ρ, then a sphere packing associated to Λ may be defined by placing spheres of radius $\rho/2$ at each lattice point. In this case, there is one sphere for each copy of the lattice fundamental cell $\mathbb{R}^d/\Lambda$ and the sphere packing density is the ratio

$$\frac{\mathrm{Vol}(B^d_{\rho/2})}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}.$$

Let $f : \mathbb{R}^d \to \mathbb{C}$ be integrable with Fourier transform

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} dx, \quad \xi \in \mathbb{R}^d.$$

If $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function, Poisson summation enforces

$$\sum_{x\in\Lambda} f(x) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \sum_{y\in\Lambda^*} \widehat{f}(y)$$

where Λ^* is the dual lattice. We can modify Poisson summation to form an inequality

$$f(0) \geq \sum_{x\in\Lambda} f(x) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \sum_{y\in\Lambda^*} \widehat{f}(y) \geq \frac{\widehat{f}(0)}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}$$

if f is positive definite ($\widehat{f} \geq 0$) and $f(x) \leq 0$ for all $||x|| \geq \rho$. This yields $\mathrm{Vol}(\mathbb{R}^d/\Lambda) \geq 1$ if $\widehat{f}(0) = f(0)$. In light of the lattice sphere packing density, the existence of an auxiliary function f turning the above inequalities into equalities shows that the Λ lattice sphere packing density is $\leq \mathrm{Vol}(B^d_{\rho/2})$. This argument, first put forth in [CoEl03], converts the sphere packing problem into an

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analysis problem. To construct a *magic function* f in which the above inequalities are tight (f and $\widehat{f}$ also vanish on nonzero lattice points), it is typical to write $f = f_+ + f_-$ where $\widehat{f_+} = f_+$ and $\widehat{f_-} = -f_-$ and f_+, f_- have sign radius

$$r_f = \inf\{M \geq 0 : f(\{||x|| \geq M\}) \subseteq [0, \infty)\}$$

equal to ρ. Minimizing the quantity $(r_f r_{\widehat{f}})^{1/2}$ is a type of uncertainty principle which is currently only solved in dimension $d = 12$ when $\widehat{f} = f$ and dimensions $d \in \{1, 8, 24\}$ when $\widehat{f} = -f$, with the cases $d \in \{8, 24\}$ corresponding to the -1 eigenfunctions constructed in the solving of the sphere packing problems in dimensions 8 and 24 [Lo83, BCK10, GSS17, CoGo19, GoRa20].

We construct a radial Schwartz function $f_+ : \mathbb{R}^4 \to \mathbb{R}$ such that $\widehat{f_+} = f_+$ with sign radius $r_{f_+} = \sqrt{2}$, which is also the minimal vector length in the D_4 checkerboard lattice [CoSl13]. f_+ is not sharp in the sense of minimizing the sign radius since numerical results in [CoGo19] demonstrate auxiliary f with $r_f < 0.97$. It is possible, in light of the slackness conditions imposed in conjecture 6.1 of [CLS] that f_+ may be of use toward the sphere packing problem in dimension 4.

To make this note more self-contained, we will include some of the details and arguments from [Via17, CKM17, CoGo19, RoWa20, RaVi19].

2. A +1 eigenfunction in four dimensions

We let $q = e^{2\pi i z}$ with $\mathrm{Im}(z) > 0$. Define the Jacobi theta series

$$\begin{aligned}
\Theta_2(z) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{\frac{1}{2}n^2}, \\
\Theta_3(z) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2}, \\
\Theta_4(z) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2}.
\end{aligned}$$

Under the action of $z \mapsto -1/z$, these theta functions satisfy

$$\begin{aligned}
\Theta_2(-1/z) &= (-iz)^{1/2}\Theta_4(z), \\
\Theta_3(-1/z) &= (-iz)^{1/2}\Theta_3(z), \\
\Theta_4(-1/z) &= (-iz)^{1/2}\Theta_2(z).
\end{aligned}$$

Under the action of $z \mapsto z + 1$ they satisfy

$$\begin{aligned}
\Theta_2(z+1) &= e^{i\pi/4}\Theta_2(z), \\
\Theta_3(z+1) &= \Theta_4(z), \\
\Theta_4(z+1) &= \Theta_3(z).
\end{aligned}$$

The fourth powers Θ_2^4, Θ_3^4, Θ_4^4 have the following leading terms at the cusp $i\infty$:

$$\begin{aligned}
\Theta_2^4 &= 16q^{1/2} + 64q^{3/2} + O(q^{5/2}), \\
\Theta_3^4 &= 1 + 8q^{1/2} + 24q + 32q^{3/2} + 24q^2 + O(q^{5/2}), \\
\Theta_4^4 &= 1 - 8q^{1/2} + 24q - 32q^{3/2} + 24q^2 + O(q^{5/2}).
\end{aligned}$$

From these q-expansions one can observe the *Jacobi identity*: $\Theta_3^4 = \Theta_2^4 + \Theta_4^4$. Additionally, Θ_2^4, Θ_3^4, and Θ_4^4 generate the ring of holomorphic modular forms of weight $k = 2$ for the congruence subgroup $\Gamma(2)$ [Bru08]. We normalize the *modular discriminant* Δ, which can be written in several ways:

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \eta(z)^{24} = \sum_{n\geq 1}\tau(n)q^n = \frac{E_4^3 - E_6^2}{1728}.$$

The function Δ is a weight 12 cusp form for $SL_2(\mathbb{Z})$ [MDG]. A salient point for our purposes is that $\Delta(it) > 0$ for $t > 0$. This can be seen upon setting $z = it$ in the above product expansion.

For $x \in \mathbb{R}^d$, the Fourier transform of a Gaussian function is

$$e^{\pi i||x||^2 z} \to (-iz)^{-d/2} e^{\pi i||x||^2(-1/z)}.$$

Thus, a natural way to construct Fourier eigenfuctions is to work with linear combinations of Gaussians - the continuous version being the Laplace transform. For $r > \sqrt{2}$, let

$$a(r) = -4i\sin(\pi r^2/2)^2 \int_0^\infty \varphi(it)e^{-\pi r^2 t}dt$$

which may be interpreted as the Laplace transform of a weighting function $g(t) = \varphi(it)$ evaluated at πr^2, multiplied by the root forcing function $-4i\sin(\pi r^2/2)^2$. We derive conditions on φ that make $\widehat{a} = a$. We may interpret $a(r) = a(x)$ for $x \in \mathbb{R}^4$ with $||x|| = r$ since we are interested in radial functions [BCK10, CoEl03]. Letting $z = it$ rotates the integration to the positive imaginary axis giving

$$\begin{aligned} a(r) &= \int_0^{i\infty} \varphi(z)e^{\pi ir^2(z+1)}dz + \int_0^{i\infty} \varphi(z)e^{\pi ir^2(z-1)}dz - 2\int_0^{i\infty} \varphi(z)e^{\pi ir^2 z}dz \\ &= \int_1^{1+i\infty} \varphi(z-1)e^{\pi ir^2 z}dz + \int_{-1}^{-1+i\infty} \varphi(z+1)e^{\pi ir^2 z}dz - 2\int_0^{i\infty} \varphi(z)e^{\pi ir^2 z}dz. \end{aligned}$$

Addressing the convergence of the integral at $i\infty$ and near the real line, if $|\varphi(it)| = O(e^{2\pi t})$ as $t \to \infty$ and $|\varphi(it)| = O(e^{-\pi/t})$ as $t \to 0^+$, we may apply Cauchy's theorem and write $\int_1^{1+i\infty} + \int_{i\infty}^i + \int_i^1 = 0$ and $\int_{-1}^{-1+i\infty} + \int_{i\infty}^i + \int_i^{-1} = 0$. The path of integration is chosen to be perpendicular to the real line at the endpoints 1 and -1. If also $\varphi(z+2) = \varphi(z)$, then

$$\begin{aligned} a(r) = 2\int_i^{i\infty} &(\varphi(z+1) - \varphi(z))e^{\pi ir^2 z}dz + \int_1^i \varphi(z-1)e^{\pi ir^2 z}dz \\ &+ \int_{-1}^i \varphi(z+1)e^{\pi ir^2 z}dz - 2\int_0^i \varphi(z)e^{\pi ir^2 z}dz. \end{aligned}$$

Taking the Fourier transform and maintaining the variable r (since the Fourier transform of a radial function is radial), substitute $\omega = \frac{-1}{z}$:

$$\begin{aligned} \widehat{a}(r) = 2\int_i^0 &(\varphi(\tfrac{-1}{\omega}+1) - \varphi(\tfrac{-1}{\omega}))i^{-d/2}\omega^{d/2-2}e^{\pi ir^2\omega}d\omega \\ &+ \int_{-1}^i \varphi(\tfrac{-1}{\omega}-1)i^{-d/2}\omega^{d/2-2}e^{\pi ir^2\omega}d\omega \\ &+ \int_1^i \varphi(\tfrac{-1}{\omega}+1)i^{-d/2}\omega^{d/2-2}e^{\pi ir^2\omega}d\omega \\ &-2\int_{i\infty}^i \varphi(\tfrac{-1}{\omega})i^{-d/2}\omega^{d/2-2}e^{\pi ir^2\omega}d\omega. \end{aligned}$$

Note that we are exchanging the Fourier transform with the contour integrals. The equality $\widehat{a} = a$ can be achieved if equality happens at the level of the integrand functions. Comparing the terms in the integral $\int_i^{i\infty}$ results in

$$\varphi(z+1) - \varphi(z) = \varphi(\tfrac{-1}{z}) i^{-d/2} z^{d/2-2}. \tag{2.1}$$

Comparing terms in $\int_0^i$ results in $\varphi(z) = (\varphi(\frac{-1}{z}+1) - \varphi(\frac{-1}{z})) i^{-d/2} z^{d/2-2}$, which is the same as (2.1) after inverting back $z \mapsto \frac{-1}{z}$. Comparing terms in $\int_1^i$, we get

$$\varphi(z+1) = \varphi(\tfrac{-1}{z}+1) i^{-d/2} z^{d/2-2}. \tag{2.2}$$

From (2.1),

$$\begin{aligned} \varphi(z+1) &= \varphi(z) + \varphi(\tfrac{-1}{z}) i^{-d/2} z^{d/2-2} \\ &= (\varphi(\tfrac{-1}{z}+1) - \varphi(\tfrac{-1}{z})) i^{-d/2} z^{d/2-2} + \varphi(\tfrac{-1}{z}) i^{-d/2} z^{d/2-2} \\ &= \varphi(\tfrac{-1}{z}+1) i^{-d/2} z^{d/2-2}. \end{aligned}$$

Thus (2.2) follows from (2.1). The comparison in $\int_{-1}^i$ is similar.

Proposition 2.1. *Let $d = 4$ and let*

$$\varphi = \frac{\Theta_4^{12}(\Theta_3^{12} + \Theta_2^{12})}{\Delta}. \tag{2.3}$$

Then, $f_+(r) = ia(r)$ is a radial Schwartz function invariant under the Fourier transform and f_+ has sign radius $r_{f_+} = \sqrt{2}$.

Proof. Based on the functional equations for $\Theta_2, \Theta_3, \Theta_4$ and Δ,

$$\varphi(z+1) = \frac{\Theta_3^{12}(\Theta_4^{12} - \Theta_2^{12})}{\Delta}$$

and

$$\varphi(-1/z) = \frac{-z^6\Theta_2(z)^{12}(-z^6\Theta_3(z)^{12} - z^6\Theta_4(z)^{12})}{z^{12}\Delta(z)} = \frac{\Theta_2^{12}(\Theta_3^{12} + \Theta_4^{12})}{\Delta}.$$

Thus, φ satisfies (2.1) for $d = 4$. To show that φ is a weight 0 weakly holomorphic modular form for the congruence subgroup $\Gamma(2)$, written $\varphi \in M_0^!(\Gamma(2))$, it suffices to check that $\varphi(\gamma z) = \varphi(z)$ for a set of generators of $\Gamma(2)$. $\Gamma(2)$ is generated by $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$, and $-I$. Observe that $\varphi\left(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right) z\right) = \varphi(z+2) = \varphi(z)$ directly from the definitions of $\Delta, \Theta_2, \Theta_3$, and Θ_4. Using (2.1),

$$\begin{aligned} \varphi\left(\begin{pmatrix}1&0\\2&1\end{pmatrix} z\right) &= \varphi\left(\frac{z}{2z+1}\right) = \varphi\left(\frac{-1}{-1/z-2}\right) \\ &= \varphi\left(\frac{-1}{z}-2\right) - \varphi\left(\frac{-1}{z}-2+1\right) \\ &= \varphi\left(\frac{-1}{z}\right) - \left(\varphi\left(\frac{-1}{z}\right) - \varphi\left(\frac{-1}{-1/z}\right)\right) = \varphi(z). \end{aligned}$$

The first few Fourier coefficients of φ are

$$\begin{aligned} \varphi(z) &= q^{-1} - 24 + 4096q^{1/2} - 98028q + O(q^{3/2}), \\ \varphi(-1/z) &= 8192q^{1/2} + O(q^{3/2}). \end{aligned} \tag{2.4}$$

The asymptotics and the fact that f_+ is a Schwartz function follow directly from the growth rate of these Fourier coefficients since φ is a weakly holomorphic modular form [Bru02]. f_+ has sign radius $\sqrt{2}$ since we may observe that for $t > 0$, $\Delta(it) > 0$ and each $\Theta_j(it) \in \mathbb{R}$. Thus, $f_+(r) \geq 0$ for $r > \sqrt{2}$.

Based on the q-series expansion (2.4), it is possible to write

$$a(r) = -4i\sin(\pi r^2/2)^2 \int_0^\infty (e^{2\pi t} - 24 + O(e^{-\pi t}))e^{-\pi r^2 t}dt$$
$$= -4i\sin(\pi r^2/2)^2 \left(\frac{1}{\pi(r^2-2)} - \frac{24}{\pi r^2} + \int_0^\infty (\varphi(it) - e^{2\pi t} + 24)e^{-\pi r^2 t}dt\right)$$

which converges for all $r \geq 0$. In particular, $f_+(0) = 0$, so f_+ is not positive definite.

3. A -1 eigenfunction and concluding remarks

To construct a -1 eigenfunction, it is possible to begin with a similar function shape

$$b(r) = -4\sin(\pi r^2/2)^2 \int_0^{i\infty} \phi(-1/z)z^{d/2-2}e^{\pi i r^2 z}dz.$$

Comparing integrand terms as before, we get

$$\phi\left(\frac{-1}{z-1}\right)(z-1)^{d/2-2} + \phi\left(\frac{-1}{z+1}\right)(z+1)^{d/2-2} - 2\phi\left(\frac{-1}{z}\right)z^{d/2-2} = 2\phi(z). \tag{3.5}$$

So, the form given for $b(r)$ enforces (3.5) when $\widehat{b} = -b$. Using the Eisenstein series $E_2(z) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$ (which is usually introduced as a basic example of a quasi-modular form) and the transformation property

$$E_2\left(\frac{-1}{z}\right) = z^2E_2(z) + \frac{6z}{\pi i},$$

a straightforward calculation shows that E_2 satisfies (3.5) for $d = 4$. Thus, setting $\phi = E_2$ above gives a -1 eigenfunction also with sign radius $\sqrt{2}$. So, $b(r)$ provides a simple construction of an eigenfunction with the basic depth 1 Eisenstein series E_2 within the integral. As before, $b(r)$ is not sharp since the numerical bounds from table 4.1 in [CoGo19] give $A_-(4) \leq 1.204 < \sqrt{2}$. Furthermore, using the checkerboard lattice D_4 from [CoSl13] with minimal radius $\sqrt{2}$, the dual lattice D_4^* is homothetic to D_4 but will have minimal radius 1. We can scale D_4 so that D_4 and D_4^* both have minimal radius $2^{1/4}$, however constructing eigenfunctions $f : \mathbb{R}^4 \to \mathbb{R}$ of the above shape with sign radius $2^{1/4}$ appears to remain a difficult problem.

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Daniel Lautzenheiser
Eastern Sierra College Center
4090 W. Line St
Bishop, CA, 93514, USA

e-mail: daniel.lautzenheiser@cerrocoso.edu