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Some Eichler-Selberg Trace Formulas

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In celebration of Ramanujan's 135th birthday

Abstract. The Eichler-Selberg trace formulas express the traces of Hecke operators on a spaces of cusp forms in terms of weighted sums of Hurwitz-Kronecker class numbers. For cusp forms on $\mathrm{SL}_2(\mathbb{Z})$, Zagier proved these formulas by cleverly making use of the weight 3/2 nonholomorphic Eisenstein series he discovered in the 1970s. The holomorphic part of this form, its so-called *mock modular form*, is the generating function for these class numbers. In this expository note we revisit Zagier's method, and we show how to obtain such formulas for congruence subgroups, working out the details for $\Gamma_0(2)$ and $\Gamma_0(4)$. The trace formulas fall out naturally from the computation of the Rankin-Cohen brackets of Zagier's mock modular form with specific theta functions.

Keywords. Eichler-Selberg trace formula, Harmonic Maass forms, mock modular forms

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1. Introduction and Statement of Results

The celebrated Eichler-Selberg trace formula [Eic55, Sel56] expresses the trace of the action of a Hecke operator on a fixed space of cusp forms in terms of weighted sums of Hurwitz-Kronecker class numbers. These formulas play many important roles in the theory of modular forms. These formulas play a central role in the study of the Shimura correspondence between spaces of half-integral weight and even integer weight cusp forms (for example, see [Shi74]). Notably, Niwa employed these formulas in work [Niw77] that established the first instances of isomorphisms between such spaces as Hecke modules. These formulas also have important implications for the arithmetic statistics of elliptic curves over finite fields. Indeed, Birch [Bir68] and Kaplan and Petrow [KaPe17] used these formulas to determine the asymptotic properties of moments of "traces of Frobenius" for various families of elliptic curves.

In unpublished notes, Zagier (see Chapter 6 of [Zag]) gave a new proof of the Eichler-Selberg trace formula for cusp forms on $\mathrm{SL}_2(\mathbb{Z})$. His ingenious method made use of the weight 3/2 nonholomorphic Eisenstein series he discovered in the 1970s [Zag75]. The key feature of this Eisenstein series is that its holomorphic part, its so-called mock modular form, is the generating function for Hurwitz-Kronecker class numbers. This proof does not seem to be well-known. Therefore, in view of the recent interest in the theory of mock modular forms and harmonic Maass forms (for example, see [BFOR17, Zag10, Zwe02]), here we revisit Zagier's work and we illustrate how to modify the proof to obtain the Eichler-Selberg trace formula for congruence subgroups.

To make this precise, we first fix some notation. If $-D < 0$ such that $-D \equiv 0, 1 \pmod 4$, then denote by $\mathcal{O}(-D)$ the unique imaginary quadratic order with discriminant $-D$. Let $h(D)$ denote the order of the class group of $\mathcal{O}(-D)$, and let $\omega(D)$ denote half the number of roots of unity in $\mathcal{O}(-D)$. In this notation, define[1] the discriminant $-D$ *Hurwitz-Kronecker class number* by

$$H^*(D) := \sum_{f^2 | D} \frac{h(D/f^2)}{\omega(D/f^2)}. \tag{1.1}$$

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[1] We define $H^*(0) := -\frac{1}{12}$ and $h(D) = H^*(D) = 0$ whenever $-D$ is neither zero nor a negative discriminant.

To conveniently state the Eichler-Selberg trace formulas, we require the *Chebyshev polynomials*, that are defined by the recurrence relation

$$U_0(x) := 1, \quad U_1(x) := 2x$$
$$U_m(x) := 2xU_{m-1}(x) - U_{m-2}(x) \text{ for } m \geq 2.$$

Finally, we define

$$\delta(q) := \begin{cases} 1 & \text{if } q \text{ is a perfect square,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

and for positive integers k and N we define

$$\lambda_k(N) := \frac{1}{2}\sum_{d|N} \min(d, N/d)^k. \tag{1.3}$$

We reprove these trace formulas (see [Eic55, Hij74, KaPe17, KnLi06, Sel56]) for cusp forms on $\Gamma_0(2)$ and $\Gamma_0(4)$ for the Hecke operators $T(q)$, where $q = p^r$ is a power of an odd prime. The purpose of this expository note is to illustrate how these formulas arise naturally from the Rankin-Cohen brackets of Zagier's mock modular form with specific theta functions that are chosen to capture the corresponding level structures.

Theorem 1.1. *If r is a positive integer, p is an odd prime, $q = p^r$, and $k \geq 4$ is a positive even integer, then the following are true.*
(1) For cusp forms on $\mathrm{SL}_2(\mathbb{Z})$*, we have*

$$\mathrm{Tr}_k(\mathrm{SL}_2(\mathbb{Z}), q) = \frac{k-1}{12} \cdot \delta(q) q^{\frac{k}{2}-1} - \frac{1}{2} q^{\frac{k}{2}-1} \sum_{s^2<4q} U_{k-2}\left(\frac{s}{2\sqrt{q}}\right) H^\star(4q - s^2) - \lambda_{k-1}(q).$$

(2) For cusp forms on $\Gamma_0(2)$*, we have*

$$\begin{aligned}\mathrm{Tr}_k(\Gamma_0(2), q) = \frac{k-1}{4} \cdot \delta(q) q^{\frac{k}{2}-1} - \frac{1}{2} q^{\frac{k}{2}-1} \sum_{s^2<q} U_{k-2}\left(\frac{s}{\sqrt{q}}\right) H^\star(4q - 4s^2) \\ - q^{\frac{k}{2}-1} \sum_{s^2<q} U_{k-2}\left(\frac{s}{\sqrt{q}}\right) H^\star(q - s^2) - 2\lambda_{k-1}(q).\end{aligned}$$

(3) For cusp forms on $\Gamma_0(4)$*, we have*

$$\mathrm{Tr}_k(\Gamma_0(4), q) = \frac{k-1}{2} \cdot \delta(q) q^{\frac{k}{2}-1} - 3 q^{\frac{k}{2}-1} \sum_{s^2<q} U_{k-2}\left(\frac{s}{\sqrt{q}}\right) H^\star(q - s^2) - 3\lambda_{k-1}(q).$$

Although Theorem 1.1 (1) is proved by Zagier [Zag], we include it in the theorem to juxtapose the formulas, which highlights the modifications that arise when introducing level structure.

This note is organized as follows. In Section 2., we recall standard facts about holomorphic modular forms, harmonic Maass forms, and mock modular forms. We require Zagier's weight 3/2 Eisenstein series, the Rankin-Cohen bracket operators, and the method of holomorphic projection. In Section 3., we recall some results from Petersson theory and the theory of newforms as adapted to this setting. Finally, in Section 4., we apply these tools to prove Theorem 1.1.

2. Nuts and Bolts

2.A. Holomorphic Modular Forms

If k and N are positive integers, then denote by $\mathcal{M}_k(\Gamma_0(N))$ the vector space of holomorphic modular forms of weight k and level N. The cuspidal subspace of $\mathcal{M}_k(\Gamma_0(N))$ is denoted by $\mathcal{S}_k(\Gamma_0(N))$. We write $S_k^{\mathrm{new}}(\Gamma_0(N))$ for the new subspace of weight k and level N. We assume that the reader is familiar with the theory of newforms of Atkin and Lehner [AtLe70].

The group $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\{f : \mathbb{H} \to \mathbb{C}\}$ through the slash operator. Namely, if $f : \mathbb{H} \to \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$, then the (weight k) slash operator is defined by

$$(f|_k\gamma)(\tau) = (ad - bc)^{k/2}(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right). \tag{2.4}$$

In our setting, two matrices in $\mathrm{GL}_2^+(\mathbb{Q})$ play a significant role, namely, the matrix $V(d) := \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the Fricke involution $W_N := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. The action of these matrices send cusp forms to cusp forms of possibly higher level. More precisely, if d is a positive integer and $f \in \mathcal{S}_k(\Gamma_0(N))$, then $f|V(d) \in \mathcal{S}_k(\Gamma_0(dN))$ and $f|W_N \in \mathcal{S}_k(\Gamma_0(N))$. Furthermore, if f is a cusp form with Fourier series $f(\tau) = \sum_{n\geq 1} a_f(n)q_\tau^n$, where $q_\tau = e^{2\pi i\tau}$, then we write

$$(f|U(d))(\tau) := \sum_{n\geq 1} a_f(nd)q_\tau^n. \tag{2.5}$$

The vector space of cusp forms admits the structure of a finite-dimensional Hilbert space thanks to the Petersson inner product. To make this precise, if $f, g \in \mathcal{S}_k(\Gamma)$ for some subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of finite index, then define[2] the Petersson inner product[3] of f and g by

$$\langle f, g\rangle_\Gamma := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \iint_{\mathbb{H}/\Gamma} f(x + iy)\overline{g(x + iy)}y^k \frac{dxdy}{y^2}. \tag{2.6}$$

2.B. Zagier's Weight $\frac{3}{2}$ Eisenstein Series

We require the generating function

$$\mathcal{H}^+(\tau) = \sum_{D=0}^{\infty} H^*(D)q_\tau^D = -\frac{1}{12} + \frac{1}{3}q_\tau^3 + \frac{1}{2}q_\tau^4 + q_\tau^7 + q_\tau^8 + \ldots, \tag{2.7}$$

(note: $q_\tau := e^{\pi i\tau}$) for the Hurwitz class numbers, the holomorphic part of the harmonic Maass form [BFOR17] $\mathcal{H}(\tau)$ of weight 3/2 constructed [Zag75] by Zagier in the 1970s by the method of Eisenstein series. This holomorphic part is the so-called *mock modular form* for $\mathcal{H}(\tau)$.

We recall Zagier's construction. To this end, we first recall notation from the theory of modular forms of half-integral weight. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ and $\tau \in \mathbb{H} := \{x + iy \in \mathbb{C}, y > 0\}$, then the automorphy factor is defined by $j(\gamma, \tau) := \theta(\gamma\tau)/\theta(\tau)$, where $\gamma\tau := \frac{a\tau+b}{c\tau+d}$ and

$$\theta(\tau) := \sum_{n\in\mathbb{Z}} q_\tau^{n^2}. \tag{2.8}$$

We recall the metaplectic extension of $\mathrm{GL}_2^+(\mathbb{Q})$ defined as

$$\mathrm{Mp}_2(\mathbb{Q}) := \left\{(\gamma, \phi) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}), \phi : \mathbb{H} \to \mathbb{C} \text{ holomorphic, } \phi(\tau)^2 = \frac{c\tau + d}{\sqrt{ad - bc}}\right\}.$$

[2] We drop the Γ when it is understood from context.

[3] There are different normalizations of the Petersson inner product.

The group $\mathrm{Mp}_2(\mathbb{Q})$ acts on $\{f : \mathbb{H} \to \mathbb{C}\}$ through the slash operator. Namely, if $f : \mathbb{H} \to \mathbb{C}, k$ is an integer, and $\widetilde{\gamma} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \phi\right) \in \mathrm{Mp}_2(\mathbb{Q})$, then the (weight $\frac{k}{2}$) slash operator is defined[4] by

$$\left(f|_{\frac{k}{2}}\widetilde{\gamma}\right)(\tau) = \phi(\tau)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

Finally, if N is a positive integer with $4|N$, we define the Fricke involution

$$W_N := \left(\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, \sqrt{N^{\frac{1}{2}}\tau}\right),$$

where $\sqrt{\cdot}$ is the principal branch of the square root, and if f is an eigenform of $|W_N$, we denote the eigenvalue by $\lambda_N(f)$.

In this notation, if $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{4}$ and $\tau \in \mathbb{H}$, then define the Eisenstein series $E_{\frac{3}{2},s}(\tau)$ by

$$E_{\frac{3}{2},s}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \frac{1}{j(\gamma, \tau)^3} \cdot \Im(\gamma\tau)^s, \tag{2.9}$$

where $\Gamma_\infty \leq \mathrm{SL}_2(\mathbb{Z})$ is the stabilizer group of $i\infty$. Finally, we define the Eisenstein series $F_{\frac{3}{2},s}(\tau)$ by

$$F_{\frac{3}{2},s}(\tau) := (E_{\frac{3}{2},s}|_{\frac{3}{2}} W_4)(\tau). \tag{2.10}$$

Both $E_{\frac{3}{2},s}(\tau)$ and $F_{\frac{3}{2},s}(\tau)$ are analytic in s, and they have analytic continuations to $s = 0$, which we denote by $E_{\frac{3}{2}}(\tau)$ and $F_{\frac{3}{2}}(\tau)$ respectively. In this notation, Zagier proves the following theorem.

Theorem 2.1. *[Zag75] The function*

$$\mathcal{H}(\tau) := \sum_{n=0}^{\infty} H^*(n) q_\tau^n + \frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n\Gamma(-\frac{1}{2}; 4\pi n^2 y) q_\tau^{-n^2},$$

where $\tau = x + iy \in \mathbb{H}$, is a weight $3/2$ harmonic Maass form with manageable growth at the cusps of $\Gamma_0(4)$. In fact, we have[5]

$$\mathcal{H}(\tau) = -\frac{1}{12}\left(E_{\frac{3}{2}}(\tau) + (1-i)2^{-3/2} F_{\frac{3}{2}}(\tau)\right).$$

2.C. Rankin-Cohen Brackets

The expressions in Theorem 1.1 arise from the Rankin-Cohen brackets operators of $\mathcal{H}(\tau)$ with appropriate theta functions. To make this precise, let f and g be smooth functions defined on $\mathbb{H}$ and let $k, l \in \frac{1}{2}\mathbb{N}$ and $m \in \mathbb{N}_0$. The mth Rankin-Cohen bracket (of weight (k, l)) of f and g is

$$[f, g]_m := \frac{1}{(2\pi i)^m} \sum_{r+s=m} (-1)^r \binom{k+m-1}{s} \binom{l+m-1}{r} \frac{d^r}{d\tau^r} f \cdot \frac{d^s}{d\tau^s} g. \tag{2.11}$$

These operators preserve modularity.

Proposition 2.2. (Th. 7.1 of [Coh75]) *Let f and g be (not necessarily holomorphic) modular forms of weights k and l, respectively on a subgroup Γ of $\mathrm{Mp}_2(\mathbb{Q})$. Then the following are true.*
(1) We have that $[f, g]_m$ is modular of weight $k + l + 2m$ on Γ.
(2) If $\tilde{\gamma} \in \mathrm{Mp}_2(\mathbb{Q})$, then under the usual modular slash operator we have

$$[f|_k\tilde{\gamma}, g|_l\tilde{\gamma}]_m = ([f, g]_m)|_{k+l+2m}\tilde{\gamma}.$$

[4] The dependence on k is usually dropped from the notation if understood from context.
[5] This explicit form is computed in [HiZa76]. A different normalization of $E_{\frac{3}{2}}(\tau)$ and $F_{\frac{3}{2}}(\tau)$ is used.

In this notation, the weighted class number sums in Theorem 1.1 appear as the coefficients of $[\mathcal{H}(\tau),\theta(\tau)]_{\frac{k-2}{2}}$ and $[\mathcal{H}(\tau),\theta(4\tau)]_{\frac{k-2}{2}}$ after a straightforward calculation.

Lemma 2.3. *If m and t are positive integers, then*

$$[\mathcal{H}^+(\tau),\theta(t\tau)]_m := \sum_{n\geq 1} c_m(n)q_\tau^n,$$

where

$$c_m(n) = \frac{\binom{2m}{m}}{4^m}\cdot n^m \sum_{s\in\mathbb{Z}} U_{2m}\left(\sqrt{\frac{t}{n}}s\right)\cdot H^\star(n-ts^2).$$

Sketch of the proof. By definition, the Fourier coefficient $c_m(n)$ is given by

$$c_m(n) = (-n)^m \sum_{l=0}^{m}\left[\sum_{r=0}^{l}(-1)^l\binom{m+\frac{1}{2}}{r}\binom{m-\frac{1}{2}}{m-r}\binom{m-r}{l-r}\right]\sum_{s\in\mathbb{Z}} H^\star(n-ts^2)\left(\sqrt{\frac{t}{n}}s\right)^{2l}.$$

Reorganizing the sum, the claim clearly reduces to showing that

$$(-1)^m\cdot\frac{4^m}{\binom{2m}{m}}\sum_{r=0}^{l}(-1)^l\binom{m+\frac{1}{2}}{r}\binom{m-\frac{1}{2}}{m-r}\binom{m-r}{l-r}$$

is the coefficient of x^{2l} in $U_{2m}(x)$. The sum on the right-hand side can be rewritten in terms of

$${}_2F_1\left(\begin{matrix}-\frac{1}{2}-m, & -l \\ & \frac{1}{2}\end{matrix}\mid 1\right).$$

A famous identity of Gauss (see (1.3) of [Bai35]), combined with induction in m, completes the proof.

2.D. Holomorphic projection

To relate the sums in Lemma 2.3 to traces of Hecke operators, we must first relate them to coefficients of a cusp form. To this end, we make use of the method of holomorphic projection, which maps harmonic Maass forms to holomorphic modular forms.

To make this precise, suppose $f:\mathbb{H}\to\mathbb{C}$ is a (not necessarily holomorphic) modular form of weight $k>2$ on $\Gamma_0(N)$ with Fourier expansion

$$f(\tau) = \sum_{n\in\mathbb{Z}} c_f(n,y)q_\tau^n.$$

Furthermore, suppose that $f(\tau)$ has moderate growth at cusps, with $c_f(0,y)=c_0+O(y^{-\varepsilon})$ for some $\varepsilon>0$. Then, the *holomorphic projection* of $f(\tau)$ is defined by

$$(\pi_{\text{hol}}f)(\tau) := c_0 + \sum_{n\geq 1} c(n)q_\tau^n,$$

where

$$c(n) := \frac{(4\pi n)^{k-1}}{(k-2)!}\int_0^\infty c_f(n,y)e^{-4\pi n y}y^{k-2}dy, \tag{2.12}$$

for $n\geq 1$. The following proposition explains the importance of the holomorphic projection operator.

Proposition 2.4. [Section 10.1 of [BFOR17]] *Assuming the hypothesis above, the following are true.*
(1) We have that $\pi_{\text{hol}}(f)\in\mathcal{M}_k(\Gamma_0(N))$.
(2) If $g\in\mathcal{S}_k(\Gamma_0(N))$, then we have $\langle f,g\rangle = \langle\pi_{\text{hol}}(f),g\rangle$, whenever the left-hand side converges.

The following theorem of Mertens [Mer14] describes the holomorphic projection of the Rankin-Cohen brackets of $\mathcal{H}(\tau)$ with certain weight $1/2$ univariate theta functions.

Theorem 2.5. [Th. V.2.1 of [Mer14]] *If m and t are positive integers, then we have*

$$\pi_{\mathrm{hol}}\left([\mathcal{H}(\tau),\theta(t\tau)]_m\right)=[\mathcal{H}^+(\tau),\theta(t\tau)]_m+\frac{1}{2}\cdot\frac{\binom{2m}{m}}{4^m}\Lambda_t(\tau;m),$$

where

$$\Lambda_t(\tau;m)=2\sum_{n\geq 1}\left(\sum_{\substack{tu^2-v^2=n\\ u,v\geq 1}}(\sqrt{t}u-v)^{2m+1}\right)q_\tau^n+\sum_{n\geq 1}(\sqrt{t}n)^{2m+1}q_\tau^{tn^2}.$$

3. Petersson Inner Products and L-functions

To obtain Theorem 1.1, we partially decompose the holomorphic cusp forms $\pi_{\mathrm{hol}}\left([\mathcal{H}(\tau),\theta(\tau)]_m\right)$ and $\pi_{\mathrm{hol}}\left([\mathcal{H}(\tau),\theta(4\tau)]_m\right)$ in terms of newforms. To this end, we take the Petersson inner product of these cusp forms with newforms. To compute these inner products, we make use of the Hecke relations between Fourier coefficients of a given newform. Namely, we use unfolding theorems due to Rankin and Selberg (see Section 11.12 of [CoSt17]). The following theorem of Rankin [Ran39] relates the Petersson inner product of two cusp forms to their respective Fourier coefficients.

Theorem 3.1. *If $f(\tau)=\sum_{n\geq 1}a_f(n)q_\tau^n$ and $g(\tau)=\sum_{n\geq 1}a_g(n)q_\tau^n$ are cusp forms of weight k on a congruence subgroup Γ, then we have*

$$\langle f,g\rangle=\frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\frac{a_f(n)\overline{a_g(n)}}{n^{k-1}}.$$

We study the inner product of cusp forms with the forms $[\mathcal{H}(\tau),\theta(\tau)]_m$ and $[\mathcal{H}(\tau),\theta(4\tau)]_m$. Since $\mathcal{H}(\tau)$ is defined in terms of Eisenstein series, we obtain the following proposition.

Proposition 3.2. *If m is a positive integer and*

$$f(\tau)=\sum_{n\geq 1}a_f(n)q_\tau^n\in\mathcal{S}_{2m+2}(\Gamma_0(4)),$$
$$g(\tau)=\sum_{n\geq 1}a_g(n)q_\tau^n\in\mathcal{S}_{1/2}(\Gamma_0(4)),$$

then we have

$$\langle f,[E_{\frac{3}{2}},g]_m\rangle=\frac{1}{6}\binom{m+\frac{1}{2}}{m}\frac{(2m)!}{(4\pi)^{2m+1}}\sum_{n\geq 1}\frac{a_f(n)\overline{a_g(n)}}{n^{m+1}}.$$

Sketch of Proof. Formally, we have that

$$\begin{aligned}[E_{\frac{3}{2},s}(\tau),g(\tau)]_m&=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}[j(\gamma,\tau)^{-3}\cdot\Im(\gamma\tau)^s,g(\tau)]_m\\&=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}[\Im(\tau)^s|_{\frac{3}{2}}\gamma,\left((g|_{1/2}\gamma^{-1})|_{1/2}\gamma\right)(\tau)]_m.\end{aligned}$$

Since g is modular of weight $\frac{1}{2}$ on $\Gamma_0(4)$, Proposition 2.2 gives us that

$$[E_{\frac{3}{2},s}(\tau), g(\tau)]_m = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} [\Im(\tau)^s, g(\tau)]_m|_{2m+2}\gamma.$$

A simple computation gives

$$[E_{\frac{3}{2},s}(\tau), g(\tau)]_m = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} \sum_{l=0}^{m} \binom{m+\frac{1}{2}}{m-l}\binom{m-\frac{1}{2}}{l}\frac{s!}{(s-l)!}\frac{\Im(\gamma\tau)^{s-l}}{(c\tau+d)^{2m+2}}\frac{1}{(2\pi i)^{m-l}}\frac{\partial^{m-l}g}{\partial\tau^{m-l}}(\gamma\tau).$$

By the definition of the Petersson inner product and the modularity of f, we have

$$\langle [E_{\frac{3}{2},s}, g]_m, f\rangle = \sum_{l=0}^{m}\frac{1}{(2\pi i)^{m-l}}\binom{m+\frac{1}{2}}{m-l}\binom{m-\frac{1}{2}}{l}\frac{s!}{(s-l)!}\iint_{\mathbb{H}/\Gamma_\infty}\overline{f(\tau)}\cdot\frac{\partial^{m-l}g}{\partial\tau^{m-l}}(\tau)\cdot y^{2m+s-l}dxdy,$$

where $\tau = x+iy$. The rest of the proof follows by writing $f(\tau)$ and $g(\tau)$ as Fourier series, computing the resulting elementary integrals, and taking analytic continuation with $s\to 0$.

To obtain the trace formulas for $\Gamma_0(2)$ and $\Gamma_0(4)$, we require Rankin-Cohen brackets of $\mathcal{H}(\tau)$ with $\theta(4\tau)$. Since $\theta(4\tau)$ is a modular form on $\Gamma_0(16)$, we require an analogue of Proposition 3.2 when f and g are of level dividing 16. Since $E_{\frac{3}{2}}(\tau)$ is the Eisenstein series of weight $\frac{3}{2}$ for $\Gamma_0(4)$, the above proposition doesn't hold. However, we have the following similar result.

Proposition 3.3. *If m is a positive integer and*

$$f = \sum_{n=1}^{\infty} a_f(n)q_\tau^n \in \mathcal{S}_{2m+2}(\Gamma_0(16)),$$
$$g = \sum_{n=0}^{\infty} a_g(n)q_\tau^n \in \mathcal{M}_{1/2}(\Gamma_0(16)),$$

then we have

$$\langle f, [E_{\frac{3}{2}}, g]_m\rangle = \frac{1}{24}\binom{m+\frac{1}{2}}{m}\frac{(2m)!}{(4\pi)^{2m+1}}\sum_{A\in S}\sum_{n\geq 1}\frac{a_{f,A}(n)\overline{a_{g,A}(n)}}{n^{m+1}},$$

where $S := \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\4&1\end{pmatrix}, \begin{pmatrix}3&-1\\4&-1\end{pmatrix}, \begin{pmatrix}1&0\\8&1\end{pmatrix}\right\}$, $(f|A)(\tau) = \sum_{n\geq 1} a_{f,A}(n)q_\tau^n$, *and* $(g|A)(\tau) = \sum_{n\geq 1} a_{g,A}(n)q_\tau^n$.

Sketch of Proof. If $s\in\mathbb{C}$ with $\Re(s) > \frac{1}{4}$ and $\tau\in\mathbb{H}$, then define

$$G_{\frac{3}{2},s}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(16)}\frac{1}{j(\gamma,\tau)^3}\cdot\Im(\gamma\tau)^s, \tag{3.13}$$

where $j(\gamma,\tau)$ is as in Section 2.B.. It is then clear that

$$E_{\frac{3}{2},s}(\tau) = \sum_{A\in S}(G_{\frac{3}{2},s}|_{\frac{3}{2}}A^{-1})(\tau).$$

The proof follows exactly as in Proposition 3.2.

Remark 3.4. *If $G_{\frac{3}{2}}(\tau)$ denotes the analytic continuation of $G_{\frac{3}{2},s}(\tau)$ to $s=0$, then Proposition 3.2 holds with $G_{\frac{3}{2}}$ and $\Gamma_0(16)$ in place of $E_{\frac{3}{2}}$ and $\Gamma_0(4)$.*

To compute inner products of the form $\langle [F_{3/2}, g]_m, f\rangle$ using Propositions 3.2 and 3.3, we consider the action of the Fricke involution on Eisenstein series of half-integral weight and theta functions.

Lemma 3.5. *The following are true.*
(1) We have

$$E_{\frac{3}{2}}|(W_4^2) = iE_{\frac{3}{2}}; \qquad \theta|W_4 = e^{-\frac{i\pi}{4}}\theta; \qquad \theta(4\tau)|W_{16} = \frac{1}{\sqrt{2}}e^{-\frac{i\pi}{4}}\theta(\tau).$$

(2) If $G_{\frac{3}{2}}$ denotes the analytic continuation of $G_{\frac{3}{2},s}$ at $s=0$, then we have

$$F_{\frac{3}{2}} = 2^{3/2}\cdot G_{\frac{3}{2}}|W_{16} \qquad \text{and} \qquad G_{\frac{3}{2}}|(W_{16}^2) = iG_{\frac{3}{2}}.$$

Proof. The proof of (1) is trivial. On the other hand, note that $G_{\frac{3}{2},s}(\tau) = E_{\frac{3}{2},s}(4\tau)$, and therefore, by analytic continuation, $G_{\frac{3}{2}}(\tau) = E_{\frac{3}{2}}(4\tau)$. The proof of (2) reduces to an elementary computation.

4. Proof of Theorem 1.1

The main tool for proving Theorem 1.1 is the next theorem that gives the Hecke traces in terms of the coefficients of specific holomorphic projections of $\mathcal{H}(\tau)$ with theta functions.

Theorem 4.1. *If $p\geq 3$ is a prime,$m\in\mathbb{N}$ and $k=2m+2$, then the following are true.*
(1) If $\pi_{\mathrm{hol}}([\mathcal{H}(\tau),\theta(\tau)]_m) = \sum\limits_{n\geq 1} a_m(n)q_\tau^n$, and $q=p^r$, then

$$a_m(q) = -\frac{1}{3}\cdot\frac{\binom{2m}{m}}{4^m}\cdot \mathrm{Tr}_k\left(\Gamma_0(4),q\right).$$

(2) $\pi_{\mathrm{hol}}([\mathcal{H}(\tau),\theta(\tau)]_m) = \sum\limits_{n\geq 1} a_m(n)q_\tau^n$, and $q=p^r$, then

$$a_m(4q) = -2\binom{2m}{m}\mathrm{Tr}_k\left(\mathrm{SL}_2(\mathbb{Z}),q\right).$$

(3) If $\pi_{\mathrm{hol}}([\mathcal{H}(\tau),\theta(4\tau)]_m) = \sum\limits_{n\geq 1} b_m(n)q_\tau^n$, then

$$b_m(4q) = -\frac{1}{12}\cdot\binom{2m}{m}\cdot\left(-8\mathrm{Tr}_k\left(\Gamma_0(4),q\right) + 24\mathrm{Tr}_k\left(\Gamma_0(2),q\right)\right).$$

4.A. The proof of Theorem 4.1

We require the following convenient expression for the Petersson norm of newforms f.

Lemma 4.2. *If $f(\tau) = \sum\limits_{n\geq 1} a_f(n)q_\tau^n$ is a normalized newform of weight k and level N, then we have*

$$\langle f,f\rangle = \frac{\pi}{3}\cdot\frac{(k-1)!}{(4\pi)^k}\cdot\prod_{p|N}\left(1-\frac{1}{p}\right)\cdot\sum_{n\geq 1}\frac{a_f(n^2)}{n^k}.$$

Sketch of Proof. It is well known (see Corollary 11.12.3 of [Coh75]) that $\sum\limits_{n\geq 1}\frac{|a_f(n)|^2}{n^s}$ converges for $\Re(s)>k$ and has a simple pole at $s=k$, with residue $\frac{3}{\pi}\cdot\frac{(4\pi)^k}{(k-1)!}\langle f,f\rangle$. Since f is a newform on $\Gamma_0(N)$ with trivial nebentypus, $a(n)$ is real for all n and we have (see p. 80 of [Shi75]) that

$$\sum_{n\geq 1}\frac{a_f(n)^2}{n^s} = \zeta_N(s-k+1)\sum_{n\geq 1}\frac{a_f(n^2)}{n^s},$$

where $\zeta_N(s) = \sum_{\substack{n\geq 1\\(n,N)=1}} \frac{1}{n^s}$. The lemma follows by taking residues at $s=k$.

To prove Theorem 4.1, we now compute $\pi_{\text{hol}}([\mathcal{H}(\tau),\theta(\tau)]_m)$ and $\pi_{\text{hol}}([\mathcal{H}(\tau),\theta(4\tau)]_m)$ with newforms and their images under $V(d)$ operator. This depends on the level of the newform. The first lemma concerns the level 4.

Lemma 4.3. *If $f(\tau)\in\mathcal{S}^{new}_{2m+2}(\Gamma_0(4))$ is a normalized newform, then the following are true.*
(1) We have that

$$\langle \pi_{\text{hol}}([\mathcal{H}(\tau),\theta(\tau)]_m), f\rangle = -\frac{1}{3}\cdot\frac{\binom{2m}{m}}{4^m}\langle f,f\rangle.$$

(2) We have that

$$\langle \pi_{\text{hol}}([\mathcal{H}(\tau),\theta(4\tau)]_m), f\rangle = -\frac{1}{6}\cdot\frac{\binom{2m}{m}}{4^m}\langle f,f\rangle.$$

(3) We have that

$$\langle \pi_{\text{hol}}([\mathcal{H}(\tau),\theta(4\tau)]_m), f|V(2)\rangle = 0.$$

(4) We have that

$$\langle \pi_{\text{hol}}([\mathcal{H}(\tau),\theta(4\tau)]_m), f|V(4)\rangle = \frac{1}{6}\cdot\frac{\binom{2m}{m}}{4^m}\langle f,f\rangle.$$

Proof. For brevity, we only prove (1). By Proposition 2.4, we have

$$\langle \pi_{\text{hol}}([\mathcal{H},\theta]_m), f\rangle = \langle [\mathcal{H},\theta]_m, f\rangle.$$

We write $\mathcal{H}$ in terms of Eisenstein series to obtain

$$\langle [\mathcal{H},\theta]_m, f\rangle = -\frac{1}{12}\langle [E_{\frac{3}{2}},\theta]_m, f\rangle - \frac{1}{12}\cdot\frac{1-i}{2^{3/2}}\langle [E_{\frac{3}{2}}|W_4,\theta]_m, f\rangle.$$

Since W_4^2 acts trivially on $\mathcal{M}_k(\Gamma_0(4))$ when k is an even integer, Proposition 2.2 (2) shows that

$$\langle [E_{\frac{3}{2}}|W_4,\theta]_m, f\rangle = \langle [E_{\frac{3}{2}}|(W_4)^2,\theta|W_4]_m, f|W_4\rangle.$$

Using Lemma 3.5, we have that

$$\langle [\mathcal{H},\theta]_m, f\rangle = -\frac{1}{12}\langle [E_{\frac{3}{2}},\theta]_m, f\rangle - \frac{1}{24}\langle [E_{\frac{3}{2}},\theta]_m, f|W_4\rangle.$$

Since f is a newform of level 4, $f|W_4=-f$ (see Theorem 7 of [AtLe70]). Therefore, we have

$$\langle \pi_{\text{hol}}([\mathcal{H},\theta]_m), f\rangle = -\frac{1}{24}\langle [E_{\frac{3}{2}},\theta]_m, f\rangle.$$

Proposition 3.2 then gives

$$\langle \pi_{\text{hol}}([\mathcal{H},\theta]_m), f\rangle = -\frac{1}{144}\cdot\binom{m+\frac{1}{2}}{m}\frac{(2m)!}{(4\pi)^{2m+1}}\sum_{n\geq 1}\frac{2\cdot a_f(n^2)}{n^{2m+2}}.$$

We apply Lemma 4.2 to obtain

$$\langle \pi_{\text{hol}}([\mathcal{H},\theta]_m), f\rangle = -\frac{1}{72}\cdot\binom{m+\frac{1}{2}}{m}\frac{(2m)!}{(4\pi)^{2m+1}}\cdot 2\cdot\frac{(4\pi)^{2m+2}}{(2m+1)!}\cdot\frac{3}{\pi}\langle f,f\rangle.$$

Since $\frac{\binom{m+\frac{1}{2}}{m}}{2m+1} = \frac{\binom{2m}{m}}{4^m}$, the claim follows.

The proofs of (2),(3), and (4) are similar, where Proposition 3.3 takes the place of Proposition 3.2. Furthermore, the Fourier coefficients at the cusps given in Proposition 3.3 are the same as the Fourier coefficients at $i\infty$, since the matrices in S are in $\Gamma_0(4)$, and f is modular on $\Gamma_0(4)$.

If the level is 2, we have the following lemma.

Lemma 4.4. *If $f(\tau) \in \mathcal{S}^{new}_{2m+2}(\Gamma_0(2))$ is a normalized newform, then the following are true.*
(1) We have that

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}, \theta]_m), f\rangle = -\frac{1}{2} \cdot \frac{\binom{2m}{m}}{4^m} \langle f, f\rangle.$$

(2) We have that

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}, \theta]_m), f|V(2)\rangle = 0.$$

(3) We have that

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}(\tau), \theta(4\tau)]_m), f\rangle = -\frac{1}{4} \cdot \frac{\binom{2m}{m}}{4^m} \langle f, f\rangle.$$

(4) We have that

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}(\tau), \theta(4\tau)]_m), f|V(2)\rangle = \langle \pi_{\mathrm{hol}}([\mathcal{H}(\tau), \theta(4\tau)]_m), f|V(4)\rangle = 0.$$

(5) We have that

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}(\tau), \theta(4\tau)]_m), f|V(8)\rangle = -\frac{\lambda_2(f)}{4} \cdot \frac{\binom{2m}{m}}{4^m} \langle f, f\rangle.$$

Proof. The proof is analogous to the proof of Lemma 4.3. Case (5) is more involved when computing the Fourier coefficients of $f|V(8)$ at cusps. We note that

$$\begin{aligned}\left(f|V(8)\middle| \begin{pmatrix}1 & 8\\ 0 & 1\end{pmatrix}\right)(\tau) &= \left(f|W_2 \cdot V(4) \begin{pmatrix}1 & 0\\ -8 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{1}{8}\\ 0 & 1\end{pmatrix}\right)(\tau)\\ &= \lambda_2(f)\left(f|V(4)\middle| \begin{pmatrix}1 & \frac{1}{8}\\ 0 & 1\end{pmatrix}\right)(\tau)\\ &= \lambda_2(f) \cdot 4^{m+1} \sum_{n\geq 1} a_f(n)(-1)^n q_\tau^{4n}.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\left(f|V(8)\middle| \begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}\right)(\tau) &= \left(f\middle| \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} W_2 \begin{pmatrix}1 & 0\\ -4 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{1}{4}\\ 0 & 1\end{pmatrix}\right)(\tau)\\ &= \lambda_2(f) \sum_{n\geq 1} a_f(n) \cdot i^n q_\tau^n.\end{aligned}$$

and

$$\left(f|V(8)\middle| \begin{pmatrix}3 & -1\\ 4 & -1\end{pmatrix}\right)(\tau) = \lambda_2(f) \sum_{n\geq 1} a_f(n) \cdot (-i)^n q_\tau^n.$$

For newforms on $\mathrm{SL}_2(\mathbb{Z})$ (i.e. level 1), we have the following lemma.

Lemma 4.5. *If $f(\tau) \in \mathcal{S}_{2m+2}(\mathrm{SL}_2(\mathbb{Z}))$ is a normalized newform, then the following are true.*
(1) We have that

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}, \theta]_m), f\rangle = -\frac{1}{2} \cdot \frac{\binom{2m}{m}}{4^m} \langle f, f\rangle.$$

(2) We have that

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}, \theta]_m), f|V(2)\rangle = -\frac{1}{6} \cdot \frac{a_f(2)}{2^m} \cdot \frac{\binom{2m}{m}}{4^m} \langle f, f\rangle.$$

(3) We have that

$$\langle \pi_{\mathrm{hol}}([\mathcal{H},\theta]_m), f|V(4)\rangle = -\frac{1}{2}\cdot\frac{\binom{2m}{m}}{4^m}\langle f,f\rangle.$$

(4) If $l\in\{0,2,4\}$, *then*

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}(\tau),\theta(4\tau)]_m), f|V(2^l)\rangle = -\frac{1}{4}\cdot\frac{\binom{2m}{m}}{4^m}\langle f,f\rangle.$$

(5) If $l\in\{1,3\}$, *then*

$$\langle \pi_{\mathrm{hol}}([\mathcal{H}(\tau),\theta(4\tau)]_m), f|V(2^l)\rangle = -\frac{1}{6}\cdot\frac{a_f(2)}{2^{m+1}}\frac{\binom{2m}{m}}{4^m}\langle f,f\rangle.$$

Proof. The proof is similar to Lemmas 4.3 and 4.4 with the following modification.

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{2^{m+1}}\frac{a_f(2n^2)}{n^{2m+2}} &= \sum_{n\geq 1}\frac{1}{2^{m+1}}\frac{1}{n^{2m+2}}\cdot\left[a_f(2)a_f(n^2)-2^{2m+1}a_f\left(\frac{n^2}{2}\right)\right]\\ &= \frac{a_f(2)}{2^{m+1}}\sum_{n\geq 1}\frac{a_f(n^2)}{n^{2m+2}}-\frac{1}{2}\sum_{n\geq 1}\frac{1}{2^{m+1}}\frac{a_f(2n^2)}{n^{2m+1}}\\ &= \frac{2}{3}\cdot\frac{a_f(2)}{2^{m+1}}\sum_{n\geq 1}\frac{a_f(n^2)}{n^{2m+2}}.\end{aligned}$$

Since f and $f|V(d)$ are not always orthogonal when $d>1$, we compute $\langle f, f|V(d)\rangle$ in terms of $\langle f,f\rangle$. This depends on the level of f. If f is of level 4, we have the following lemma.

Lemma 4.6. *If* $f(\tau)\in\mathcal{S}_k^{new}(\Gamma_0(4))$ *is a normalized newform and* $l\geq 1$, *then* $\langle f, f|V(2^l)\rangle = 0$.

Proof. Theorem 3.1 implies that

$$\begin{aligned}\langle f, f|V(2^l)\rangle &= \frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\frac{a_f(n)\cdot 2^{\frac{lk}{2}}\overline{a_f(\frac{n}{2^l})}}{n^{k-1}}\\ &= \frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}\lim_{x\to\infty}\frac{2^{-\frac{lk}{2}}}{x/2^l}\sum_{n\leq x/2^l}\frac{a_f(2^l n)\overline{a_f(n)}}{n^{k-1}}.\end{aligned}$$

It is well-known (see Proposition 13.3.14 (a) of [CoSt17]) that the coefficients with even index of a normalized newform of level 4 vanish, proving the claim.

If f is of level 2, we have the following lemma whose proof is analogous to that of Lemma 4.6.

Lemma 4.7. *If* $f(\tau)=\sum_{n\geq 1}a_f(n)q_\tau^n\in\mathcal{S}_k^{new}(\Gamma_0(2))$ *is a normalized newform and* $l\geq 0$, *then*

$$\langle f, f|V(2^l)\rangle = \left(-\frac{\lambda_2(f)}{2}\right)^l\langle f,f\rangle.$$

For newforms on $\mathrm{SL}_2(\mathbb{Z})$, we have the following lemma.

Lemma 4.8. *If $f(\tau) \in \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ is a normalized newform, then the following are true.*
(1) We have that

$$\langle f, f|V(2)\rangle = \frac{1}{3} \cdot \frac{a_f(2)}{2^{k/2-1}}\langle f, f\rangle.$$

(2) We have that

$$\langle f, f|V(4)\rangle = \left(\frac{1}{6} \cdot \frac{a_f(2)^2}{4^{k/2-1}} - \frac{1}{2}\right)\langle f, f\rangle.$$

(3) We have that

$$\langle f, f|V(8)\rangle = \left(\frac{1}{12} \cdot \frac{a_f(2)^3}{8^{k/2-1}} - \frac{5}{12} \cdot \frac{a_f(2)}{2^{k/2-1}}\right)\langle f, f\rangle.$$

(4) We have that

$$\langle f, f|V(16)\rangle = \left(\frac{1}{24} \cdot \frac{a_f(2)^4}{16^{k/2-1}} - \frac{7}{24} \cdot \frac{a_f(2)^2}{4^{k/2-1}} + \frac{1}{4}\right)\langle f, f\rangle.$$

Proof. The same argument in the proofs of Lemmas 4.6 and 4.7 gives us that

$$\langle f, f|V(2^l)\rangle = 2^{-\frac{lk}{2}}\langle f|U(2^l), f\rangle,$$

for all $l \geq 0$. Since f is of level 1, the relation between $a(2^l n)$ and $a(n)$ is more involved. Therefore, we must determine the Hecke operator $T(2^l)$ in terms of U and V operators. We make this precise for (1), and leave the remaining cases to the reader.

Hecke's recurrence relations (see Corollary 10.4.4 of [CoSt17]) imply that $T(2)f = f|U(2) + 2^{\frac{k}{2}-1}f|V(2)$. On the other hand, since f is a normalized eigenform of $T(2)$, we have that $T(2)f = a_f(2)f$. Combining these two equations together, we have that

$$a_f(2)f = f|U(2) + 2^{\frac{k}{2}-1}f|V(2).$$

Taking the inner product with f on both sides, we find that

$$\begin{aligned} a_f(2)\langle f, f\rangle &= \langle f, f|U(2)\rangle + 2^{\frac{k}{2}-1}\langle f, f|V(2)\rangle \\ &= 2^{\frac{k}{2}}\langle f, f|V(2)\rangle + 2^{\frac{k}{2}-1}\langle f, f|V(2)\rangle. \end{aligned}$$

This completes the proof of (1).

Lemmas 4.3-4.8 combine to prove Theorem 4.1.

Proof of Theorem 4.1. We prove (1) and (2). By Lemmas 4.3-4.8, some linear algebra gives

$$\begin{aligned} \pi_{\mathrm{hol}}([\mathcal{H}(\tau), \theta(\tau)]_m) = \frac{\binom{2m}{m}}{4^m}\Big[&-\frac{1}{3}\sum f - \frac{2}{3}\sum g - \sum h \\ &- \frac{1}{3}\sum \lambda_2(g) g|V(2) + \frac{1}{2}\sum_h \frac{a_h(2)}{2^m} h|V(2) - \sum h|V(4)\Big], \end{aligned} \tag{4.14}$$

where f, g and h run over newforms of levels $4, 2$ and 1 respectively. Comparing the coefficients of q_τ^q on both sides of 4.14 gives (1).

To prove (2), we write $f(\tau) = \sum_{n\geq 1} a_f(n)q_\tau^n, g(\tau) = \sum_{n\geq 1} a_g(n)q_\tau^n$ and $h(\tau) = \sum_{n\geq 1} a_h(n)q_\tau^n$. Using (4.14), we have that

$$\begin{aligned} a_m(4q) = \frac{\binom{2m}{m}}{4^m}\Big[&-\frac{1}{3}\sum_f a_f(4q) - \frac{2}{3}\sum_g (a_g(4q) + \lambda_2(g)\cdot 2^m a_g(2q)) \\ &- \sum_h (a_h(4q) - a_h(2)a_h(2q) + 4^{m+1}a_h(q))\Big]. \end{aligned} \tag{4.15}$$

Since f denotes a newform of level 4, then $a_f(4q) = 0$. Furthermore, since g denotes a newform of level 2 and weight $2m+2$, we have that $\lambda_2(g) = -2^{-m}a_g(2)$. (for example, see Proposition 13.3.14 (b) of [CoSt17]). Furthermore, $a_g(2n) = a_g(2)a_g(n)$. This implies that $a_g(4q) + \lambda_2(g)\cdot 2^m a_g(2q) = 0$. Finally, the Hecke relations give that $a_h(2)a_h(2q) = a_h(4q) + 2^{2m+1}a_h(q)$, and so we have

$$a_h(4q) - a_h(2)a_h(2q) + 4^{m+1}a_h(q) = 2\cdot 4^m \cdot a_h(q).$$

Combining this with (4.15) gives (2).

4.B. Proof of Theorem 1.1

For brevity, we prove (2), leaving the proofs of (1) and (3) to the reader. Thanks to Theorem 4.1, we have

$$a_m(q) = -\frac{1}{3}\cdot\frac{\binom{2m}{m}}{4^m}\cdot \mathrm{Tr}_k\left(\Gamma_0(4), q\right)$$

and

$$b_m(4q) = -\frac{1}{12}\cdot\binom{2m}{m}\cdot\left(-8\mathrm{Tr}_k\left(\Gamma_0(4), q\right) + 24\mathrm{Tr}_k\left(\Gamma_0(2), q\right)\right).$$

Using Lemma 2.3 and Theorem 2.5, after some simplification we find that

$$q^m\sum_{s\in\mathbb{Z}} U_{2m}\left(\frac{s}{\sqrt{q}}\right)H^\star(q-s^2) + \lambda_{2m+1}(q) = -\frac{1}{3}\cdot \mathrm{Tr}_k\left(\Gamma_0(4), q\right)$$

and

$$q^m\sum_{s\in\mathbb{Z}} U_{2m}\left(\frac{s}{\sqrt{q}}\right)H^\star(4q-4s^2) + 2\lambda_{2m+1}(q) = -\frac{1}{12}\left(-8\mathrm{Tr}_k\left(\Gamma_0(4), q\right) + 24\mathrm{Tr}_k\left(\Gamma_0(2), q\right)\right).$$

Taking an appropriate linear combination, we then have

$$-q^m\sum_{s\in\mathbb{Z}} U_{2m}\left(\frac{s}{\sqrt{q}}\right)H^\star(q-s^2) - \frac{1}{2}q^m\sum_{s\in\mathbb{Z}} U_{2m}\left(\frac{s}{\sqrt{q}}\right)H^\star(4q-4s^2) - 2\lambda_{2m+1}(q) = \mathrm{Tr}_k\left(\Gamma_0(2), q\right).$$

Since the summands are nonzero only if $s^2 < q$ or $s^2 = q$ and q is a square, (2) follows from the fact that $H^\star(0) = -\frac{1}{12}$ and $U_{2m}(1) = 2m+1$. This proves (2).

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