Hardy-Ramanujan Journal 45 (2022), 130-139 submitted 07/11/2022, accepted 28/11/2022, revised 28/11/2022

A Generalization of Iseki's Formula and The Jacobi Theta Function

Maher Me'meh and Ali Saraeb

Abstract. In this paper we give a generalization of Iseki's formula and use it to prove the transformation law of $\theta_1(z,\tau)$.

Keywords. Jacobi Theta Function, Dedekind Eta function, Iseki's Formula, Dedekind Sums.

2010 Mathematics Subject Classification. 11F20, 11F50, 11F55.

1. Introduction

The Dedekind Eta function, defined as

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}),$$

plays an important role in the study of Modular and Jacobi forms. Its transformation over a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$, where with no loss of generality $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c > 0$, is given by

$$\eta(A\tau) = \varepsilon(A)(-i(c\tau+d))^{1/2}\eta(\tau).$$

One obtains the Dedekind eta-character "$\varepsilon(A)$" which is defined as, (check, [Rad12],III)

$$\varepsilon(A) = \begin{cases} \left(\frac{d}{c}\right) i^{(1-c)/2} e^{(\pi i/12)(bd(1-c^2)+c(a+d))} & \text{if } c \textit{ odd} \\ \\ \left(\frac{c}{d}\right) e^{\pi di/4} e^{(\pi i/12)(ac(1-d^2)+d(b-c))} & \text{if } d \textit{ odd}, \end{cases}$$

where $(d, c) = 1$, and $\left(\frac{c}{d}\right)$ is the Legendre-Jacobi symbol given by

$$\left(\frac{c}{d}\right) = \prod_{i=1}^{n}\left(\frac{c}{p_i}\right)^{\alpha_i},$$

if $d = p_1^{\alpha_1}.p_2^{\alpha_2}...p_n^{\alpha_n}$, where each p_i is a prime and each $\left(\frac{c}{p_i}\right)$ is the regular Legendre symbol defined as

$$\left(\frac{c}{p_i}\right) = \begin{cases} 1 & \text{if c is a quadratic residue mod } p_i \\ -1 & \text{if c is a quadratic non-residue mod } p_i \\ 0 & \text{if c is a multiple of } p_i \end{cases}$$

However, it turns out that computations done using this definition of the eta-character can get really messy. On the other hand, Sho Iseki proved in 1952 [Apo89, p.53, Theorem 3.5] the transformation law of the eta function using a functional equation which will be addressed below. Using his proof he was able to write the eta-character using Dedekind sums which proved to be much easier in terms of computations. The eta-character turns out to be

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$$\varepsilon(A) = \exp\left(\pi i \left(\frac{a+d}{12c} + s(-d,c)\right)\right),$$

where

$$s(d,c) = \sum_{r=1}^{c-1} \frac{r}{c}\left(\frac{dr}{c} - \left[\frac{dr}{c}\right] - \frac{1}{2}\right)$$

is the Dedekind sum and $(d,c) = 1$.
Since the eta-character appears significantly in the transformation laws of Jacobi theta functions and Jacobi forms ([Rad12],X), we generalize Iseki's proof of the eta function and apply the generalization to Jacobi theta function $\vartheta_1(z,\tau)$, which is defined as

$$\vartheta_1(z,\tau) = -i\zeta^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^n)(1-\zeta^{-1}q^{n-1}), \tag{1.1}$$

where $\zeta = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$, $z \in \mathbb{C}$ *and* $\tau \in \mathcal{H}$, where $\mathcal{H}$ is the upper half plane. We first generalize Iseki's functional equation of his Λ function defined by

$$\Lambda(\alpha,\beta,z) = \sum_{n=0}^{\infty}\left(-\log(1-e^{-2\pi((n+\alpha)z-i\beta)}) - \log(1-e^{-2\pi((n+1-\alpha)z+i\beta)})\right)$$

and

$$\Lambda(\alpha,\beta,z) = \Lambda(1-\beta,\alpha,z^{-1}) + g_0(\alpha,\beta,z)$$

to four variables using methods from Fourier analysis then we employ this tool to prove the transformation law of ϑ_1.

2. Generalization of Iseki's formula

Theorem 2.1. *If $Re(w) > 0$, $0 < \alpha < 1$, θ is real, and $0 < \beta + \theta < 1$, then*

$$\Lambda(\alpha,\beta,w,\theta) = \Lambda(1-\beta,\alpha,w^{-1},-i\theta/w) + g_0(\alpha,\beta,w,\theta), \tag{2.2}$$

where

$$\Lambda(\alpha,\beta,w,\theta) = -\sum_{n=0}^{\infty}\log(1-e^{2\pi i\theta}e^{-2\pi((n+\alpha)w-i\beta)}) + \log(1-e^{-2\pi i\theta}e^{-2\pi((n+1-\alpha)w+i\beta)}), \tag{2.3}$$

with

$$g_0(\alpha,\beta,w,\theta) = \frac{\pi}{w}B_2(\beta+\theta) - \pi w B_2(\alpha) + 2\pi i B_1(\alpha)B_1(\beta+\theta), \tag{2.4}$$

and B_n is the n^{th} Bernoulli polynomial coming from the Taylor expansion of

$$\frac{te^{xt}}{e^t-1} = \sum_{n=1}^{\infty} B_n(x)\frac{t^n}{n!},$$

with an explicit formula

$$B_n(x) = \sum_{m=0}^{n}\frac{1}{m+1}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(x+k)^n.$$

Proof: The proof utilizes the following well-known identities from Fourier analysis (check [Bro59, 5.2, p. 370])

$$\frac{e^{2\pi m\alpha w}}{1-e^{2\pi mw}}+\frac{1}{2\pi wm}=\frac{1}{2\pi i}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i\alpha n}}{wmi+n}. \tag{2.5}$$

Replacing w by w^{-1} and m and $-m$, we get

$$\frac{e^{-2\pi m\alpha w^{-1}}}{1-e^{-2\pi mw^{-1}}}-\frac{1}{2\pi w^{-1}m}=-\frac{1}{2\pi i}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i\alpha n}}{w^{-1}mi-n}. \tag{2.6}$$

We also have

$$\frac{1}{m(wmi-n)}=-\frac{1}{mn}+\frac{w}{n(ni+wm)}. \tag{2.7}$$

We first observe that (2.3) can be rewritten as follows

$$\begin{aligned}\Lambda(\alpha,\beta,w,\theta)&=\sum_{m=1}^{\infty}\frac{e^{2\pi im\beta}}{m}\frac{e^{-2\pi m\alpha w}}{1-e^{-2\pi mw}}e^{2\pi im\theta}-\sum_{m=1}^{\infty}\frac{e^{-2\pi im\beta}}{m}\frac{e^{2\pi m\alpha w}}{1-e^{2\pi mw}}e^{-2\pi im\theta}\\&=-\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi im\beta}}{m}\frac{e^{2\pi m\alpha w}}{1-e^{2\pi mw}}e^{-2\pi im\theta}.\end{aligned} \tag{2.8}$$

Multiplying both sides of equation (2.5) by $\dfrac{-1}{2\pi mi}e^{-2\pi im\beta}e^{-2\pi im\theta}$ and then summing from $m=-\infty$ to $+\infty$, we rewrite (2.8) as follows

$$\begin{aligned}\Lambda(\alpha,\beta,w,\theta)&=-\frac{1}{2\pi i}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{-2\pi im\beta}}{m}\frac{e^{2\pi in\alpha}}{wmi+n}e^{-2\pi im\theta}+\frac{1}{2\pi w}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi im(\beta+\theta)}}{m^2}\\&=-\frac{1}{2\pi i}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{-2\pi im\beta}}{m}\frac{e^{-2\pi in\alpha}}{wmi-n}e^{-2\pi im\theta}+\frac{1}{2\pi w}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{2\pi im(\beta+\theta)}}{m^2}\\&=-\frac{1}{2\pi i}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\left(A_m(\alpha,\beta,w,\theta)\right)+\frac{1}{2\pi w}F_2(\beta+\theta),\end{aligned} \tag{2.9}$$

where

$$A_m(\alpha,\beta,w,\theta)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{-2\pi im\beta}}{m}\frac{e^{-2\pi in\alpha}}{wmi-n}e^{-2\pi im\theta}, \tag{2.10}$$

and

$$F_n(x)=\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{2\pi imx}}{m^n}=\begin{cases}\displaystyle\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi imx}}{m^n} & \text{if n is even.}\\[2ex] \displaystyle-\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi imx}}{m^n} & \text{if n is odd.}\end{cases} \tag{2.11}$$

Using (2.7), (2.10) becomes as follows

$$A_m(\alpha,\beta,w,\theta) = -\frac{e^{-2\pi im(\beta+\theta)}}{m}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{-2\pi in\alpha}}{n} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{-2\pi im\beta}\frac{e^{-2\pi in\alpha}}{n(\frac{ni}{w}+m)}e^{-2\pi im\theta}.$$

By using $F_1(x) = -2\pi iB_1(x)$, $F_2(x) = \frac{-(2\pi i)^2}{2!}B_2(x)$ [Apo89, theorem 3.5, p:56], (2.7), (2.10), and (2.11) and by carefully manipulating the signs of m and n in the summands, we observe that

$$\begin{aligned}
&-\frac{1}{2\pi i}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} A_{m,n}(\alpha,\beta,w,\theta)\\
&= \frac{1}{2\pi i}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi im(\beta+\theta)}}{m}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{-2\pi in\alpha}}{n} - \frac{1}{2\pi i}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{-2\pi im(\beta+\theta)}\frac{e^{-2\pi i\alpha n}}{n(\frac{ni}{w}+m)}\\
&= \frac{1}{2\pi i}F_1(\beta+\theta)F_1(\alpha) - \frac{1}{2\pi i}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{-2\pi in(\beta+\theta)}\frac{e^{-2\pi im\alpha}}{m(\frac{mi}{z}+n)}\\
&= 2\pi iB_1(\beta+\theta)B_1(\alpha) - \frac{1}{2\pi i}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{2\pi in(\beta+\theta)}\frac{e^{-2\pi im\alpha}}{m(\frac{mi}{w}-n)}\\
&= 2\pi iB_1(\beta+\theta)B_1(\alpha) + \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi m(1-\beta-\theta)w^{-1}}}{1-e^{-2\pi mw^{-1}}}\frac{e^{-2\pi im\alpha}}{m} - \frac{w}{2\pi}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi im\alpha}}{m^2}\\
&= 2\pi iB_1(\beta+\theta)B_1(\alpha) - \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi im\alpha}}{m}\frac{e^{-2\pi m(1-\beta)w^{-1}}}{1-e^{2\pi mw^{-1}}}e^{2\pi im(-i\theta w^{-1})} - \frac{w}{2\pi}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi im\alpha}}{m^2}\\
&= 2\pi iB_1(\beta+\theta)B_1(\alpha) + \Lambda(1-\beta,\alpha,w^{-1},-i\theta/w) - \pi wB_2(\alpha).
\end{aligned}\tag{2.12}$$

Plugging (2.12) in (2.9), we get that

$$\Lambda(\alpha,\beta,w,\theta) = \Lambda(1-\beta,\alpha,w^{-1},-i\theta/w) + \frac{\pi}{w}B_2(\beta+\theta) - \pi wB_2(\alpha) + 2\pi iB_1(\beta+\theta)B_1(\alpha).$$

This completes the proof of Theorem 2.1.

We now use Theorem 2.1. to prove the transformation law for ϑ_1 under the elements of the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 2.2. *For $\tau\in\mathcal{H}$ and $z\in\mathbb{C}$ and $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ with $c>0$, we have*

$$\vartheta_1\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_1(A)\left(-i(c\tau+d)\right)^{1/2}e^{\frac{\pi icz^2}{c\tau+d}}\vartheta_1(z,\tau).\tag{2.13}$$

Here ε appears in the transformation law of the Dedekind eta function as mentioned in the introduction, where

$$\varepsilon_1(A) = -i\varepsilon^3 = -i.\exp\left(3\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right).$$

and

$$s(h,k)=\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\left[\frac{hr}{k}\right]-\frac{1}{2}\right)$$

is the Dedekind sum for $k>0$ and $(k,h)=1$.

Proof. We will start by deriving equivalent functional equations to (2.13). Firstly note that (2.13) is equivalent to

$$\log\left(\vartheta_1\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\right)=\log(\varepsilon_1(A))+\frac{1}{2}\log(-i(c\tau+d))+\frac{\pi icz^2}{c\tau+d}+\log\left(\vartheta_1\left(z,\tau\right)\right)\quad(2.14)$$

where the logarithm is taken over the principle branch.

Note that using the definition of Dedekind eta function, we have

$$\prod_{n=1}^{\infty}(1-q^n)=\eta(\tau).e^{\frac{-\pi i\tau}{12}}.$$

Hence (1.1) becomes

$$\vartheta_1(z,\tau)=-i\zeta^{1/2}q^{1/8}\left(\eta(\tau)e^{\frac{-\pi i\tau}{12}}\right)\prod_{n=1}^{\infty}(1-\zeta q^n)(1-\zeta^{-1}q^{n-1}),\qquad(2.15)$$

where again $\zeta=e^{2\pi iz}$ and $q=e^{2\pi i\tau}$.

As a result of (2.15), (2.14) is equivalent to

$$\begin{aligned}&\log(-ie^{\frac{\pi iz}{c\tau+d}}e^{\frac{\pi i}{4}(\frac{a\tau+b}{c\tau+d})})+\log(\eta(\frac{a\tau+b}{c\tau+d}))-\frac{\pi i}{12}\left(\frac{a\tau+b}{c\tau+d}\right)\\&+\sum_{n=1}^{\infty}\log(1-e^{\frac{2\pi iz}{c\tau+d}}e^{2n\pi i(\frac{a\tau+b}{c\tau+d})})+\sum_{n=1}^{\infty}\log(1-e^{\frac{-2\pi iz}{c\tau+d}}e^{2(n-1)\pi i(\frac{a\tau+b}{c\tau+d})})\\=&\log\left(-i.\exp\left(3\pi i(\frac{a+d}{12c}+s(-d,c))\right)\right)+\frac{1}{2}\log(-i(c\tau+d))+\frac{\pi icz^2}{c\tau+d}\\&+\log(-ie^{\pi iz}e^{\frac{\pi i\tau}{4}})+\log(\eta(\tau))-\frac{\pi i\tau}{12}+\sum_{n=1}^{\infty}\log(1-e^{2\pi iz}e^{2n\pi i\tau})+\sum_{n=1}^{\infty}\log(1-e^{-2\pi iz}e^{2(n-1)\pi i\tau}).\end{aligned}$$

Using the fact that, (check [Apo89, 3.4, p.52])

$$\log\left(\eta\left(\frac{a\tau+b}{c\tau+d}\right)\right)=\pi i\left(\frac{a+d}{12c}\right)+\pi is(-d,c)+\frac{1}{2}\log(-i(c\tau+d))+\log(\eta(\tau)),$$

we obtain

$$\begin{aligned}&\frac{\pi iz}{c\tau+d}+\frac{\pi i}{6}\left(\frac{a\tau+b}{c\tau+d}\right)+\sum_{n=1}^{\infty}\log(1-e^{\frac{2\pi iz}{c\tau+d}}e^{2n\pi i(\frac{a\tau+b}{c\tau+d})})+\sum_{n=1}^{\infty}\log(1-e^{\frac{-2\pi iz}{c\tau+d}}e^{2(n-1)\pi i(\frac{a\tau+b}{c\tau+d})})\\&=\log(-i)+2\pi i(\frac{a+d}{12c})+2\pi is(-d,c))+\frac{\pi icz^2}{c\tau+d}+\pi iz+\frac{\pi i\tau}{6}+\sum_{n=1}^{\infty}\log(1-e^{2\pi iz}e^{2n\pi i\tau})\\&+\sum_{n=1}^{\infty}\log(1-e^{-2\pi iz}e^{2(n-1)\pi i\tau}).\end{aligned}$$

Relocating the terms we get

$$\sum_{n=1}^{\infty}\log(1-e^{\frac{2\pi iz}{c\tau+d}}e^{2n\pi i(\frac{a\tau+b}{c\tau+d})})+\sum_{n=1}^{\infty}\log(1-e^{\frac{-2\pi iz}{c\tau+d}}e^{2(n-1)\pi i(\frac{a\tau+b}{c\tau+d})})$$
$$=\sum_{n=1}^{\infty}\log(1-e^{2\pi iz}e^{2n\pi i\tau})+\sum_{n=1}^{\infty}\log(1-e^{-2\pi iz}e^{2(n-1)\pi i\tau})+\log(-i)+2\pi i(\frac{a+d}{12c})+2\pi is(-d,c)$$
$$+\frac{\pi icz^2}{c\tau+d}-\frac{\pi iz}{c\tau+d}+\pi iz+\frac{\pi i}{6}\left(\tau-\frac{a\tau+b}{c\tau+d}\right). \tag{2.16}$$

Now we prove an equivalence of (2.16) by introducing a classical change of variable, we set

$$-i(c\tau+d)=v \qquad a=H,\ c=k\ \textit{and}\ \ h=-d,$$

where it is clear that $Hh\equiv -1 \pmod{k}$.

Under this change of variable, we have

$$\tau=\frac{iv+h}{k}\quad \textit{and}\quad \frac{a\tau+b}{c\tau+d}=\frac{1}{k}\left(H+\frac{i}{v}\right).$$

Moreover,

$$\frac{\pi i}{6}\left(\tau-\frac{a\tau+b}{c\tau+d}\right)=-2\pi i\left(\frac{a+d}{12c}\right)-\frac{\pi}{6k}\left(v-\frac{1}{v}\right).$$

Plugging in (2.16), where $\log(-i)=-\frac{\pi i}{2}$, we obtain that (2.16) and hence our functional equation (2.13) is equivalent to

$$\sum_{n=1}^{\infty}\log(1-e^{\frac{2\pi z}{v}}e^{\frac{2n\pi i}{k}(H+\frac{i}{v})})+\log(1-e^{-\frac{2\pi z}{v}}e^{\frac{2(n-1)\pi i}{k}(H+\frac{i}{v})})=\sum_{n=1}^{\infty}\log(1-e^{2\pi iz}e^{\frac{2n\pi i}{k}(h+iv)})$$
$$+\log(1-e^{-2\pi iz}e^{\frac{2(n-1)\pi i}{k}(h+iv)})+2\pi is(h,k)-\frac{\pi}{6k}(v-\frac{1}{v})-\frac{\pi i}{2}+\frac{\pi kz^2}{v}+\pi iz-\frac{\pi z}{v}, \tag{2.17}$$

which is what we desire to prove.

We now follow Iseki's proof closely and we let

$$\beta=\frac{\phi}{k}\ \ \textit{where}\ \ 0\le\phi\le k-1.$$

In order to use Theorem 2.1, we need $0<\beta+\theta<1$ for which one can easily prove that it is equivalent to having $0<\theta<\dfrac{1}{k}$.

We will use Theorem 2.2 to prove the functional equation (2.17) for θ real and in the open interval $(0,\ \frac{1}{k})$. Also, we will divide the proof into two cases, for $k=1$ and $k>1$, and then use analytic continuation to extend the result to the whole plane.

$$\Lambda(\alpha,\beta,w,\theta)=\Lambda(1-\beta,\alpha,1/w,-i\theta/w)+g_0(\alpha,\beta,w,\theta). \tag{2.18}$$

For $k=1$ we obtain from (2.17)

$$\sum_{n=1}^{\infty}\log(1-e^{\frac{2\pi z}{v}}e^{2n\pi i(H+\frac{i}{v})})+\log(1-e^{-\frac{2\pi z}{v}}e^{2(n-1)\pi i(H+\frac{i}{v})})$$
$$=\sum_{n=1}^{\infty}\log(1-e^{2\pi iz}e^{2n\pi i(h+iv)})+\log(1-e^{-2\pi iz}e^{2(n-1)\pi i(h+iv)})-\frac{\pi}{6}(v-\frac{1}{v})-\frac{\pi i}{2}+\frac{\pi z^2}{v}-\frac{\pi z}{v}+\pi iz.$$

Note that $e^{2n\pi i H} = 1$, so we end up with

$$\sum_{n=1}^{\infty} \log(1 - e^{\frac{2\pi z}{v}} e^{\frac{-2n\pi}{v}}) + \log(1 - e^{-\frac{2\pi z}{v}} e^{\frac{-2(n-1)\pi}{v}})$$
$$= \sum_{n=1}^{\infty} \log(1 - e^{2\pi i z} e^{-2n\pi v}) + \sum_{n=1}^{\infty} \log(1 - e^{-2\pi i z} e^{-2(n-1)\pi v}) - \frac{\pi}{6}\left(v - \frac{1}{v}\right) - \frac{\pi i}{2} + \frac{\pi z^2}{v} + \pi i z - \frac{\pi z}{v}. \tag{2.19}$$

Setting $\beta = 0$ and $\alpha \to 1$, from (2.18), we see that

$$-\sum_{n=0}^{\infty} \log(1 - e^{-2\pi i\theta} e^{-2\pi(n)w)}) + \log(1 - e^{2\pi i\theta} e^{-2\pi(n+1)w)})$$
$$= -\sum_{n=0}^{\infty} \log(1 - e^{-2\pi\frac{\theta}{w}} e^{\frac{-2\pi(n)}{w}}) + \log(1 - e^{2\pi\frac{\theta}{w}} e^{\frac{-2\pi(n+1)}{w}}) - \frac{\pi w}{6} + \frac{\pi}{w}(\theta^2 - \theta + \frac{1}{6}) - \frac{\pi i}{2} + \pi i\theta$$

Relocating the terms,

$$\sum_{n=1}^{\infty} \log(1 - e^{\frac{2\pi\theta}{w}} e^{\frac{-2\pi n}{w}}) + \sum_{n=1}^{\infty} \log(1 - e^{\frac{-2\pi\theta}{w}} e^{\frac{-2\pi(n-1)}{w}})$$
$$= \sum_{n=1}^{\infty} \log(1 - e^{2\pi i\theta} e^{-2\pi n w}) + \sum_{n=1}^{\infty} \log(1 - e^{-2\pi i\theta} e^{-2\pi(n-1)w})$$
$$= -\frac{\pi}{6}\left(w - \frac{1}{w}\right) + \frac{\pi\theta^2}{w} - \frac{\pi\theta}{w} - \frac{\pi i}{2} + \pi i\theta. \tag{2.20}$$

This is exactly (2.19) if we let $\theta = z$ and $w = v$. This proves the transformation law when $k = 1$.

For $k > 1$ we let

$$\alpha = \frac{\mu}{k} \quad where \quad 1 \le \mu \le k - 1,$$

and writing $h\mu = qk + \phi$ we choose

$$\beta = \frac{\phi}{k} \quad where \quad 1 \le \phi \le k - 1.$$

Note that $\phi \equiv h\mu \pmod{k}$ so $-H\phi \equiv -Hh\mu \equiv \mu \pmod{k}$, and hence $-H\phi/k \equiv \mu/k \pmod{1}$. Therefore

$$\alpha = \mu/k \equiv -H\phi/k \pmod{1}$$
$$\beta = \phi/k \equiv h\mu/k \pmod{1}.$$

Plugging in (2.18) where again $w = v$ and $\theta = z$ we obtain

$$\sum_{n=0}^{\infty} \log(1 - e^{\frac{-2\pi z}{v}} e^{-2\pi((n+\beta)v^{-1} + i\alpha)}) + \sum_{n=0}^{\infty} \log(1 - e^{\frac{2\pi z}{v}} e^{-2\pi((n+1-\beta)v^{-1} - i\alpha)})$$
$$= \sum_{n=0}^{\infty} \log(1 - e^{2\pi i z} e^{-2\pi((n+\alpha)v - i\beta)}) + \sum_{n=0}^{\infty} \log(1 - e^{-2\pi i z} e^{-2\pi((n+1-\alpha)v + i\beta)})$$
$$-\pi v\left(\alpha^2 - \alpha + \frac{1}{6}\right) + \frac{\pi}{v}\left((\beta + z)^2 - (\beta + z) + \frac{1}{6}\right) + 2\pi i\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right) + 2\pi z i\left(\alpha - \frac{1}{2}\right).$$

Using $\alpha \equiv -H\phi/k \pmod{1}$ and $\beta \equiv h\mu/k \pmod{1}$, we obtain

$$\sum_{n=0}^{\infty} \log(1 - e^{\frac{-2\pi z}{v}} e^{-2\pi((n+\phi/k)v^{-1} - i\frac{H\phi}{k})}) + \sum_{n=0}^{\infty} \log(1 - e^{\frac{2\pi z}{v}} e^{-2\pi((n+1-\phi/k)v^{-1} + i\frac{H\phi}{k})})$$
$$= \sum_{n=0}^{\infty} \log(1 - e^{2\pi i z} e^{-2\pi((n+\mu/k)v - i\frac{h\mu}{k})}) + \sum_{n=0}^{\infty} \log(1 - e^{-2\pi i z} e^{-2\pi((n+1-\mu/k)v + i\frac{h\mu}{k})})$$

$$-\pi v\left(\left(\frac{\mu}{k}\right)^2-\frac{\mu}{k}+\frac{1}{6}\right)+\frac{\pi}{v}\left(\left(\frac{\phi}{k}+z\right)^2-(\frac{\phi}{k}+z)+\frac{1}{6}\right)+2\pi i\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{\phi}{k}-\frac{1}{2}\right)+2\pi zi\left(\frac{\mu}{k}-\frac{1}{2}\right). \quad (2.21)$$

Note that $\log(1-e^{-2\pi(x+mi)})=\log(1-e^{-2\pi x})$, i.e it's periodic of period i so the above can be written as

$$\sum_{n=0}^{\infty}\log(1-e^{\frac{-2\pi z}{v}}e^{-2\pi(\frac{(nk+\phi)(\frac{1}{v}-iH)}{k})})+\sum_{n=0}^{\infty}\log(1-e^{\frac{2\pi z}{v}}e^{-2\pi(\frac{(nk+k+\phi)(\frac{1}{v}-iH)}{k})})$$
$$=\sum_{n=0}^{\infty}\log(1-e^{2\pi iz}e^{-2\pi\frac{(nk+\mu)(v-ih)}{k}})+\sum_{n=0}^{\infty}\log(1-e^{-2\pi iz}e^{-2\pi\frac{(nk+k+\mu)(v-ih)}{k}})$$
$$-\pi v\left(\left(\frac{\mu}{k}\right)^2-\frac{\mu}{k}+\frac{1}{6}\right)+\frac{\pi}{v}\left(\left(\frac{\phi}{k}\right)^2+2z(\frac{\phi}{k})+z^2-(\frac{\phi}{k}+z)+\frac{1}{6}\right)+2\pi i\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{\phi}{k}-\frac{1}{2}\right)+2\pi zi\left(\frac{\mu}{k}-\frac{1}{2}\right)$$

Now sum both sides on μ from $\mu=1,2...k-1$ and also notice that

$$\{nk+\mu, n=0,1,2...;\mu=1,2,...k-1\}=\{r:r\not\equiv 0 \pmod{k}\},$$

and the same goes for the set of number $nk+k-\mu$, and since $\phi\equiv h\mu \pmod{k}$ as μ runs over the number $1,2,...k-1$ so does ϕ but in some other order. Hence we get

$$\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{\frac{-2\pi z}{v}}e^{-2\pi r(\frac{(\frac{1}{v}-iH)}{k})})+\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{\frac{2\pi z}{v}}e^{-2\pi r(\frac{(\frac{1}{v}-iH)}{k})})$$
$$=\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{2\pi iz}e^{-2\pi r\frac{(v-ih)}{k}})+\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{-2\pi iz}e^{-2\pi r\frac{(v-hi)}{k}})$$
$$-\pi v\sum_{\mu=1}^{k-1}\left(\left(\frac{\mu}{k}\right)^2-\frac{\mu}{k}+\frac{1}{6}\right)+\frac{\pi}{v}\sum_{\mu=1}^{k-1}\left(\left(\frac{\phi}{k}\right)^2+2z(\frac{\phi}{k})+z^2-(\frac{\phi}{k}+z)+\frac{1}{6}\right)$$
$$+2\pi i\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{\phi}{k}-\frac{1}{2}\right)+2\pi zi\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right). \quad (2.22)$$

Checking [Apo89, 3.7, p: 61], one can see that

$$\sum_{\mu=1}^{\infty}\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{\phi}{k}-\frac{1}{2}\right)=\sum_{\mu=1}^{\infty}\left(\frac{\mu}{k}\right)\left(\frac{\phi}{k}-\frac{1}{2}\right)=s(h,k),$$

so (22) transforms into

$$\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{\frac{-2\pi z}{v}}e^{2\pi ir(\frac{(\frac{i}{v}+H)}{k})})+\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{\frac{2\pi z}{v}}e^{2\pi ir(\frac{\frac{i}{v}+H}{k})})$$
$$=\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{2\pi iz}e^{2\pi ir\frac{(vi+h)}{k}})+\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{-2\pi iz}e^{2\pi ir\frac{(vi+h)}{k}})$$
$$-\pi v\sum_{\mu=1}^{k-1}\left(\left(\frac{\mu}{k}\right)^2-\frac{\mu}{k}+\frac{1}{6}\right)+\frac{\pi}{v}\sum_{\mu=1}^{k-1}\left(\left(\frac{\phi}{k}\right)^2+2z(\frac{\phi}{k})+z^2-(\frac{\phi}{k}+z)+\frac{1}{6}\right)$$
$$+2\pi is(h,k)+2\pi zi\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right). \quad (2.23)$$

Notice that the four sums resemble the desired form, so we intend to look at the residues g_0, where

$$-\pi v\sum_{\mu=1}^{k-1}\left(\left(\frac{\mu}{k}\right)^2-\frac{\mu}{k}+\frac{1}{6}\right)+\frac{\pi}{v}\sum_{\mu=1}^{k-1}\left(\left(\frac{\phi}{k}\right)^2+2z(\frac{\phi}{k})+z^2-(\frac{\phi}{k}+z)+\frac{1}{6}\right)$$
$$+2\pi is(h,k)+2\pi zi\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)$$
$$=-\pi v\left(\frac{1}{k^2}.\frac{k(k-1)(2k-1)}{6}\right)+\pi v\left(\frac{1}{k}.\frac{k(k-1)}{2}\right)-\pi v\left(\frac{k-1}{6}\right)$$
$$+\ 2\pi is(h,k)+\frac{\pi}{v}\left(\frac{1}{k^2}.\frac{k(k-1)(2k-1)}{6}+(2z-1)\left(\frac{k-1}{2}\right)+(z^2-z+\frac{1}{6})(k-1)\right). \tag{2.24}$$

Note that

$$\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)=\frac{1}{k}\left(\frac{k(k-1)}{2}\right)-\frac{(k-1)}{2}=0,$$

so the last term in (2.23) cancels. After some simplification of the terms, (2.24) becomes

$$-\frac{\pi}{6k}(k-1)(2k-1)\left(v-\frac{1}{v}\right)+\frac{\pi}{3}(k-1)\left(v-\frac{1}{v}\right)+\frac{k\pi z^2}{v}-\frac{\pi z^2}{v}$$
$$=\frac{\pi}{6}\left(v-\frac{1}{v}\right)-\frac{\pi}{6k}\left(v-\frac{1}{v}\right)+\frac{k\pi z^2}{v}-\frac{\pi z^2}{v}.$$

Hence (2.23) is equivalent to

$$\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{\frac{-2\pi z}{v}}e^{2\pi ir(\frac{(\frac{i}{v}+H)}{k})})+\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{\frac{2\pi z}{v}}e^{2\pi ir(\frac{\frac{i}{v}+H}{k})})$$
$$=\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{2\pi iz}e^{2\pi ir\frac{(vi+h)}{k}})+\sum_{\substack{r=1\\ r\not\equiv 0(\bmod\ k)}}^{\infty}\log(1-e^{-2\pi iz}e^{2\pi ir\frac{(vi+h)}{k}})$$
$$+2\pi is(h,k)+\frac{\pi}{6}\left(v-\frac{1}{v}\right)-\frac{\pi}{6k}\left(v-\frac{1}{v}\right)+\frac{k\pi z^2}{v}-\frac{\pi z^2}{v}. \tag{2.25}$$

Adding equation (2.25) to equation (2.20), which corresponds to the case when $k=1$:

$$\sum_{n=1}^{\infty}\log(1-e^{\frac{2\pi z}{v}}e^{\frac{-2\pi n}{v}})+\sum_{n=1}^{\infty}\log(1-e^{\frac{-2\pi z}{v}}e^{\frac{-2\pi(n-1)}{v}})$$
$$=\sum_{n=1}^{\infty}\log(1-e^{-2\pi iz}e^{-2\pi(n-1)v})+\sum_{n=1}^{\infty}\log(1-e^{2\pi iz}e^{-2\pi nv})-\frac{\pi}{6}\left(v-\frac{1}{v}\right)+\frac{\pi z^2}{v}-\frac{\pi z}{v}-\frac{\pi i}{2}+\pi iz.$$

Adding equation (2.20) accounts for the missing r where $r\equiv 0\ (\bmod\ k)$ if we write $r=mk$, then the functional equation becomes

$$\sum_{r=1}^{\infty}\log(1-e^{\frac{-2\pi z}{v}}e^{2\pi ir(\frac{(H+\frac{i}{v})}{k})})+\sum_{r=1}^{\infty}\log(1-e^{\frac{2\pi z}{v}}e^{2\pi i(r-1)(\frac{H+\frac{i}{v}}{k})})$$
$$=\sum_{r=1}^{\infty}\log(1-e^{2\pi iz}e^{2\pi ir\frac{(h+iv)}{k}})+\sum_{r=1}^{\infty}\log(1-e^{-2\pi iz}e^{2\pi i(r-1)\frac{(h+iv)}{k}})$$
$$+2\pi is(h,k)-\frac{\pi}{6k}\left(v-\frac{1}{v}\right)+\frac{\pi kz^2}{v}-\frac{\pi z}{v}+\pi iz-\frac{\pi i}{2}.$$

This is exactly (2.17), and this completes the proof of theorem 2 for $0<\theta<\frac{1}{k}$ where $\theta=z$. However by considering equation (2.13), both sides are entire in z, and the set $E=(0,\frac{1}{k})$ clearly has a limit point. Hence by proving that the functional equation holds for $\theta\in E$ then it holds for all $z\in\mathbb{C}$, and hence our proof is done.

Acknowledgements. We would like to express our deepest gratitude to our advisor Professor Wissam Raji. We also like to thank the Department of Mathematics and the Center of Advanced Mathematical Sciences (CAMS) at the American university of Beirut (AUB) for the guidance and support we are receiving from the summer research camp (SRC). And finally we would like to thank the referee for his valuable and fruitful feedback for the paper.

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Maher Me'meh
Department of Mathematics
American University of Beirut
Beirut, Lebanon

e-mail: mmm133@mail.aub.edu

Ali Saraeb
Department of Mathematics
American University of Beirut
Beirut, Lebanon

e-mail: ays11@mail.aub.edu