

A modular relation involving a generalized digamma function and asymptotics of some integrals containing $\Xi(t)$

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Abstract. A modular relation of the form $F(\alpha, w) = F(\beta, iw)$, where $i = \sqrt{-1}$ and $\alpha\beta = 1$, is obtained. It involves the generalized digamma function $\psi_w(a)$ which was recently studied by the authors in their work on developing the theory of the generalized Hurwitz zeta function $\zeta_w(s, a)$. The limiting case $w \rightarrow 0$ of this modular relation is a famous result of Ramanujan on page 220 of the Lost Notebook. We also obtain asymptotic estimate of a general integral involving the Riemann function $\Xi(t)$ as $\alpha \rightarrow \infty$. Not only does it give the asymptotic estimate of the integral occurring in our modular relation as a corollary but also some known results.

Keywords. Ramanujan's Lost notebook, modular relation, Generalized Hurwitz zeta function, Generalized digamma function, asymptotic estimates.

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1. Introduction

Riemann's functions $\xi(s)$ and $\Xi(t)$ are defined by [Tit86, p. 16]

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s), \quad (1.1)$$

and

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right), \quad (1.2)$$

where $\Gamma(s)$ is the Euler Gamma function and $\zeta(s)$ is the Riemann zeta function.

Over the years, integrals comprising the Riemann Ξ -function in their integrands and their corresponding transformation formulas have been found to be very useful in the theory of the Riemann zeta function. Hardy [Ha1914] was one of the first mathematicians who realized the usefulness of such integrals when he proved his remarkable result that infinitely many non-trivial zeros of $\zeta(s)$ lie on the critical line $\text{Re}(s) = 1/2$. The crux of his argument relies on the identity

$$\begin{aligned} \sqrt{\alpha} \left(\frac{1}{2\alpha} - \sum_{n=1}^{\infty} e^{-\pi\alpha^2 n^2} \right) &= \sqrt{\beta} \left(\frac{1}{2\beta} - \sum_{n=1}^{\infty} e^{-\pi\beta^2 n^2} \right) \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\Xi(t/2)}{1+t^2} \cos\left(\frac{1}{2}t \log \alpha\right) dt, \end{aligned} \quad (1.3)$$

where $\alpha\beta = 1$ with $\text{Re}(\alpha^2) > 0$ and $\text{Re}(\beta^2) > 0$. Note that the first equality in the above identity is equivalent to the transformation formula for the Jacobi theta function:

$$\sum_{n=-\infty}^{\infty} e^{-\pi\alpha^2 n^2} = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\alpha^2} \quad (\text{Re}(\alpha^2) > 0).$$

A generalization of (1.3) is given by [Di13, Theorem 1.2]

$$\begin{aligned} \sqrt{\alpha} \left(\frac{e^{-\frac{w^2}{8}}}{2\alpha} - e^{\frac{w^2}{8}} \sum_{n=1}^{\infty} e^{-\pi\alpha^2 n^2} \cos(\sqrt{\pi}\alpha n w) \right) &= \sqrt{\beta} \left(\frac{e^{-\frac{w^2}{8}}}{2\beta} - e^{-\frac{w^2}{8}} \sum_{n=1}^{\infty} e^{-\pi\beta^2 n^2} \cosh(\sqrt{\pi}\beta n w) \right) \\ &= \frac{1}{\pi} \int_0^{\infty} \frac{\Xi(t/2)}{1+t^2} \nabla \left(\alpha, w, \frac{1+it}{2} \right) dt, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} \nabla(x, w, s) &:= \rho(x, w, s) + \rho(x, w, 1-s), \\ \rho(x, w, s) &:= x^{\frac{1}{2}-s} e^{-\frac{w^2}{8}} {}_1F_1 \left(\frac{1-s}{2}; \frac{1}{2}; \frac{w^2}{4} \right), \end{aligned}$$

where ${}_1F_1(a; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}$, $(a)_n = \Gamma(a+n)/\Gamma(a)$, is the confluent hypergeometric function.

The first equality in (1.4) is due to Jacobi; the integral in (1.4) was found by the first author in [Di13, Theorem 1.2]. Applications of the above identity in generalizing Hardy’s result on the infinitude of the zeros of $\zeta(s)$ can be found in [DKMZ18] and [DRZ15].

A year after Hardy’s paper [Ha1914] appeared, Ramanujan also wrote a paper on some integrals containing the function $\Xi(t)$ in their integrands. One of the results that he provided in [Ra1915] is

$$\int_0^{\infty} \left| \Gamma \left(\frac{-1+it}{4} \right) \Xi \left(\frac{t}{2} \right) \right|^2 \frac{\cos(nt)}{1+t^2} dt = \pi^{3/2} \int_0^{\infty} \left(\frac{1}{\exp(xe^n) - 1} - \frac{1}{xe^n} \right) \left(\frac{1}{\exp(xe^{-n}) - 1} - \frac{1}{xe^{-n}} \right) dx, \tag{1.5}$$

where $n \in \mathbb{R}^+$.

Regarding the integral on the left-hand side of (1.5), Hardy [Ha1915] says, “the properties of this integral resemble those of one which Mr. Littlewood and I have used, in a paper to be published shortly in *Acta Mathematica* to prove that¹

$$\int_{-T}^T \left| \zeta \left(\frac{1}{2} + ti \right) \right|^2 dt \sim \frac{2}{\pi} T \log T \quad (T \rightarrow \infty)."$$

Very recently, Darses and Hillion [DaHi22] studied the polynomial moments with a weighted zeta square measure, namely,

$$\int_{-\infty}^{\infty} t^{2N} \left| \Gamma \left(\frac{1}{2} + it \right) \zeta \left(\frac{1}{2} + it \right) \right|^2 dt.$$

The starting point of their study is nothing but the Ramanujan’s identity (1.5)! Thus Hardy was indeed correct in his assessment of (1.5).

In his Lost Notebook [Ra88], Ramanujan provided a beautiful modular relation containing the integral on the left-hand side of (1.5):

$$\begin{aligned} \sqrt{\alpha} \left\{ \frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty} \phi(n\alpha) \right\} &= \sqrt{\beta} \left\{ \frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty} \phi(n\beta) \right\} \\ &= -\frac{1}{\pi^{3/2}} \int_0^1 \left| \Xi \left(\frac{1}{2}t \right) \Gamma \left(\frac{-1+it}{4} \right) \right|^2 \frac{\cos \left(\frac{1}{2}t \log \alpha \right)}{1+t^2} dt, \end{aligned} \tag{1.6}$$

where $\alpha, \beta > 0$ such that $\alpha\beta = 1$, and $\phi(x) := \psi(x) + \frac{1}{2x} - \log(x)$, with $\psi(x) = \Gamma'(x)/\Gamma(x)$ being the digamma function. By a modular relation, we mean a transformation of the form $F(-1/z) = F(z)$,

¹Note that there is a typo in this formula in that π should not be present.

where z is in the upper-half plane. Unlike modular forms, these transformations may not be governed by the translation $z \rightarrow z + 1$. Note that $z \rightarrow -1/z$ can be equivalently rewritten in the form $\alpha \rightarrow \beta$ with $\alpha\beta = 1$, where $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$. For more details, see [Di18, p. 49].

The first ever proof of this identity appeared in [BeDi10]. Guinand [Gui47] rediscovered the first equality and remarked that “*This formula also seems to have been overlooked*”. Very recently, Gupta and the second author [GuKu23] showed that this formula of Ramanujan fits in the theory of the Herglotz function.

One of the goals of this paper is to provide a new generalization of (1.6). Before we do this though, we first record an existing generalization of (1.6) given by the first author [Di11, Theorem 1.4].

Theorem 1.1. *Let $-1 < \operatorname{Re} z < 1$. Define $\varphi(z, x)$ by*

$$\varphi(z, x) := \zeta(z + 1, x) - \frac{1}{z}x^{-z} - \frac{1}{2}x^{-z-1}, \quad (1.7)$$

where $\zeta(z, x)$ denotes the Hurwitz zeta function. Then if α and β are any positive numbers such that $\alpha\beta = 1$,

$$\begin{aligned} \alpha^{\frac{z+1}{2}} \left(\sum_{n=1}^{\infty} \varphi(z, n\alpha) - \frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{\zeta(z)}{\alpha z} \right) &= \beta^{\frac{z+1}{2}} \left(\sum_{n=1}^{\infty} \varphi(z, n\beta) - \frac{\zeta(z+1)}{2\beta^{z+1}} - \frac{\zeta(z)}{\beta z} \right) \\ &= \frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)} \int_0^{\infty} \Gamma\left(\frac{z-1+it}{4}\right) \Gamma\left(\frac{z-1-it}{4}\right) \Xi\left(\frac{t+iz}{2}\right) \Xi\left(\frac{t-iz}{2}\right) \frac{\cos\left(\frac{1}{2}t \log \alpha\right)}{(z+1)^2 + t^2} dt. \end{aligned} \quad (1.8)$$

Note that if we let $z \rightarrow 0$ in the theorem above, we get (1.6). For more details, see [Di11, pp. 1163–1165].

Recently, in their quest for putting the theta structure on the modular relation (1.8), the current authors [DK21] introduced a new generalization of the Hurwitz zeta function, which is stated next. Let $\mathfrak{B} := \{\xi : \operatorname{Re}(\xi) = 1, \operatorname{Im}(\xi) \neq 0\}$. The generalized Hurwitz zeta function for $w \in \mathbb{C} \setminus \{0\}$, $\operatorname{Re}(s) > 1$ and $a \in \mathbb{C} \setminus \mathfrak{B}$ is defined by [DK21, Equation (1.1.14)]

$$\zeta_w(s, a) := \frac{4}{w^2 \sqrt{\pi} \Gamma\left(\frac{s+1}{2}\right)} \sum_{n=1}^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{(uv)^{s-1} e^{-(u^2+v^2)} \sin(wv) \sinh(wu)}{(n^2 u^2 + (a-1)^2 v^2)^{s/2}} dudv. \quad (1.9)$$

The function $\zeta_w(s, a)$ satisfies several interesting properties, one of them being $\zeta_w(s, a) = \zeta_w(s, 2-a)$ for $\operatorname{Re}(s) > 1$. Observe that the symmetry in the variable a here along the line $\operatorname{Re}(a) = 1$ does not hold for the Hurwitz zeta function $\zeta(s, a)$. The reader is referred to [DK21] for more properties of $\zeta_w(s, a)$.

Note that for $\operatorname{Re}(s) > 1$ [DK21, Theorem 1.1.2], $\zeta_w(s, a)$ reduces to the Hurwitz zeta function $\zeta(s, a)$ as follows:

$$\lim_{w \rightarrow 0} \zeta_w(s, a) = \begin{cases} \zeta(s, a) & \text{if } \operatorname{Re}(a) > 1, \\ \zeta(s, 2-a) & \text{if } \operatorname{Re}(a) < 1. \end{cases}$$

In fact, the Hurwitz zeta function is the constant term in the Taylor series expansion of $\zeta_w(s, a)$ around $w = 0$.

In [DK21, Theorem 1.1.3], it was shown that $\zeta_w(s, a)$ has meromorphic continuation in the region $\operatorname{Re}(s) > -1$ with a simple pole at $s = 1$. To record this result in the following theorem, we need the error function $\operatorname{erf}(w)$ and the imaginary error function $\operatorname{erfi}(w)$:

$$\operatorname{erf}(w) := \frac{2}{\sqrt{\pi}} \int_0^w e^{-t^2} dt = \frac{2w}{\sqrt{\pi}} e^{-w^2} {}_1F_1\left(1; \frac{3}{2}; w^2\right), \quad (1.10)$$

$$\operatorname{erfi}(w) := \frac{2}{\sqrt{\pi}} \int_0^w e^{t^2} dt = \frac{2w}{\sqrt{\pi}} e^{w^2} {}_1F_1\left(1; \frac{3}{2}; -w^2\right). \quad (1.11)$$

Theorem 1.2. *Let² $a > 0$ and $w \in \mathbb{C}$. The generalized Hurwitz zeta function $\zeta_w(s, a)$ can be*

²There is a typo in the statement of Theorem 1.1.3 of [DK21]. The condition should be $a > 0$ and not $0 < a < 1$.

analytically continued to $\text{Re}(s) > -1$ except for a simple pole at $s = 1$. The residue at this pole is

$$e^{\frac{w^2}{4}} {}_1F_1^2\left(1; \frac{3}{2}; -\frac{w^2}{4}\right) = \frac{\pi}{w^2} e^{-\frac{w^2}{4}} \text{erfi}^2\left(\frac{w}{2}\right), \tag{1.12}$$

where $\text{erfi}(w)$ is defined in (1.11). Moreover near $s = 1$, we have

$$\zeta_w(s, a + 1) = \frac{e^{\frac{w^2}{4}} {}_1F_1^2\left(1; \frac{3}{2}; -\frac{w^2}{4}\right)}{s - 1} - \psi_w(a + 1) + O_{w,a}(|s - 1|), \tag{1.13}$$

where $\psi_w(a)$ is a new generalization of the digamma function $\psi(a)$ defined by

$$\psi_w(a) := \frac{4}{w^2 \sqrt{\pi}} \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-(u^2+v^2+x)}}{u} \sin(wv) \sinh(wu) \left(\frac{1}{x} - \frac{J_0\left((a-1)\frac{vx}{u}\right)}{1 - e^{-x}}\right) dx dudv,$$

with $J_z(x)$ being the Bessel function of the first kind [Wat66, p. 40].

Although the definition of $\psi_w(a)$ looks complicated, it is pleasing to see that Ramanujan’s formula (1.5) can be extended in the setting of $\psi_w(a)$. This is done next and is the main result of this paper.

Theorem 1.3. Let $w \in \mathbb{C}$ and $x > 0$. Let $\text{erf}(w)$ be defined in (1.10). Define

$$\lambda_w(x) := \psi_w(x + 1) - \frac{1}{2x} \mathcal{C}(w) - \mathcal{C}(iw) \log(x) - \frac{1}{2} \mathcal{B}(iw), \tag{1.14}$$

where

$$\mathcal{C}(w) := \frac{\pi}{w^2} e^{\frac{w^2}{4}} \text{erf}^2\left(\frac{w}{2}\right), \tag{1.15}$$

and

$$\mathcal{B}(w) := \frac{\sqrt{\pi}}{w} \text{erf}\left(\frac{w}{2}\right) \sum_{n=0}^\infty \frac{(w^2/4)^n}{(3/2)_n} (\psi(n + 1) + \gamma). \tag{1.16}$$

Then for any positive integers α, β such that $\alpha\beta = 1$,

$$\begin{aligned} & \sqrt{\alpha} \left\{ \frac{\gamma - \log(2\pi\alpha)}{2\alpha} \mathcal{C}(w) + \frac{1}{2\alpha} \mathcal{B}(w) + \sum_{m=1}^\infty \lambda_w(m\alpha) \right\} \\ &= \sqrt{\beta} \left\{ \frac{\gamma - \log(2\pi\beta)}{2\beta} \mathcal{C}(iw) + \frac{1}{2\beta} \mathcal{B}(iw) + \sum_{m=1}^\infty \lambda_{iw}(m\beta) \right\} \\ &= -\frac{e^{\frac{w^2}{4}}}{2\pi^{\frac{3}{2}}} \int_0^\infty \left| \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\Xi^2\left(\frac{t}{2}\right)}{1+t^2} \left(\alpha^{\frac{it}{2}} {}_1F_1^2\left(\frac{3+it}{4}; \frac{3}{2}; -\frac{w^2}{4}\right) + \alpha^{-\frac{it}{2}} {}_1F_1^2\left(\frac{3-it}{4}; \frac{3}{2}; -\frac{w^2}{4}\right) \right) dt. \end{aligned} \tag{1.17}$$

Note that the modular relation in the first equality given above is of the form $F(\alpha, w) = F(\beta, iw)$, and hence is analogous to the general theta transformation formula (1.4). See [Di13] for more details.

Theorem 1.3 gives Ramanujan’s formula (1.6) as a special case:

Corollary 1.4. Equation (1.6) holds true.

Asymptotic estimates for integrals containing Riemann Ξ -function are useful in the study of the moments of the Riemann zeta function. For example, Hardy and Littlewood [HaLi16, p. 151, Section 2.4] found the asymptotic formula for the second moment of the Riemann zeta function through the asymptotic estimate of the integral

$$\int_0^\infty \left\{ \frac{\Xi(t)}{1+t^2} \right\}^2 e^{2\alpha t} dt, \quad \text{as } \alpha \rightarrow \infty.$$

See [DaHi22] for a more recent application of such integrals. An asymptotic result for a smoothly weighted second moment of the Riemann zeta function, also containing the cosine term as in (1.6) and generalizing a result of Hardy, was obtained by Maier [Mai58, Equation (9)], namely, he showed that for coprime integers p and q ,

$$\lim_{\delta \rightarrow 0} \frac{\delta}{\log(\frac{1}{\delta})} \int_0^\infty e^{-\delta t} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \cos(t \log(p/q)) dt = (pq)^{-1/2}.$$

We next derive an asymptotic estimate of a general integral containing Riemann Ξ -function.

Theorem 1.5. *Let $A_w(z)$ be defined by*

$$A_w(z) := \frac{\sqrt{\pi}}{w} \operatorname{erf}\left(\frac{w}{2}\right) {}_1F_1\left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) = e^{-\frac{w^2}{4}} {}_1F_1\left(1; \frac{3}{2}; \frac{w^2}{4}\right) {}_1F_1\left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right). \quad (1.18)$$

Let $m \in \mathbb{N}$, $w \in \mathbb{C}$ and $-1 < \operatorname{Re}(z) < 1$. Then as $\alpha \rightarrow \infty$, we have

$$\begin{aligned} & \int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right) \Gamma\left(\frac{z-1-it}{4}\right) \Xi\left(\frac{t+iz}{2}\right) \Xi\left(\frac{t-iz}{2}\right) \frac{\Delta_2\left(\alpha, \frac{z}{2}, w, \frac{1+it}{2}\right)}{(z+1)^2 + t^2} dt \\ &= -\frac{\Gamma(z+1)}{2^{z-1}\pi^{\frac{z-3}{2}}} \left(\frac{\zeta(z+1)A_w(z)}{2\alpha^{\frac{z+1}{2}}} + \frac{\zeta(z)A_w(-z)}{z\alpha^{\frac{1-z}{2}}} \right) \\ & \quad - \frac{e^{-\frac{w^2}{4}} \alpha^{\frac{1-z}{2}}}{2^{z-2}\pi^{\frac{z-3}{2}}} \sum_{k=1}^{m-1} \frac{(-1)^k \Gamma(z+2k)}{(2\pi\alpha)^{2k}} \zeta(2k) \zeta(z+2k) {}_1F_1\left(\frac{z+1}{2} + k; \frac{3}{2}; \frac{w^2}{4}\right) {}_1F_1\left(\frac{1}{2} + k; \frac{3}{2}; \frac{w^2}{4}\right) \\ & \quad + O_{z,m,w}\left(\alpha^{-\frac{1}{2}\operatorname{Re}(z)-2m}\right), \end{aligned} \quad (1.19)$$

where,

$$\begin{aligned} \Delta_2(x, z, w, s) &:= \omega(x, z, w, s) + \omega(x, z, w, 1-s), \\ \omega(x, z, w, s) &:= e^{\frac{w^2}{4}} x^{\frac{1}{2}-s} {}_1F_1\left(1 - \frac{s+z}{2}; \frac{3}{2}; -\frac{w^2}{4}\right) {}_1F_1\left(1 - \frac{s-z}{2}; \frac{3}{2}; -\frac{w^2}{4}\right). \end{aligned} \quad (1.20)$$

The special case $w = 0$ of the above theorem is a known result [DRZ17, Theorem 1.10]:

Corollary 1.6. *Let $-1 < \operatorname{Re}(z) < 1$ and $m \in \mathbb{N}$. As $\alpha \rightarrow \infty$,*

$$\begin{aligned} & \int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right) \Gamma\left(\frac{z-1-it}{4}\right) \Xi\left(\frac{t+iz}{2}\right) \Xi\left(\frac{t-iz}{2}\right) \frac{\cos\left(\frac{t}{2} \log(\alpha)\right)}{(z+1)^2 + t^2} dt \\ &= -\frac{\Gamma(z+1)}{2^{z-1}\pi^{\frac{z-3}{2}}} \left(\frac{\zeta(z+1)}{2\alpha^{\frac{z+1}{2}}} + \frac{\zeta(z)}{z\alpha^{\frac{1-z}{2}}} \right) - \frac{\alpha^{\frac{1-z}{2}}}{2^{z-2}\pi^{\frac{z-3}{2}}} \sum_{k=1}^{m-1} \frac{(-1)^k \Gamma(z+2k)}{(2\pi\alpha)^{2k}} \zeta(2k) \zeta(z+2k) \\ & \quad + O_{z,m}\left(\alpha^{-\frac{1}{2}\operatorname{Re}(z)-2m}\right). \end{aligned} \quad (1.21)$$

Moreover, letting $z = 0$ in Theorem 1.5 gives the asymptotic estimate of the integral appearing in Theorem 1.3.

Corollary 1.7. *Let $\mathcal{C}(w)$ and $\mathcal{B}(w)$ be defined in (1.15) and (1.16) respectively. Let $w \in \mathbb{C}$ and $m \in \mathbb{N}$. As $\alpha \rightarrow \infty$, we have*

$$\begin{aligned} & \int_0^\infty \left| \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\Xi^2\left(\frac{t}{2}\right)}{1+t^2} \left(\alpha^{\frac{it}{2}} {}_1F_1^2\left(\frac{3+it}{4}; \frac{3}{2}; -\frac{w^2}{4}\right) + \alpha^{-\frac{it}{2}} {}_1F_1^2\left(\frac{3-it}{4}; \frac{3}{2}; -\frac{w^2}{4}\right) \right) dt \\ &= -2e^{-\frac{w^2}{4}} \pi^{\frac{3}{2}} \sqrt{\alpha} \left(\frac{\gamma - \log(2\pi\alpha)}{2\alpha} \mathcal{C}(w) + \frac{1}{2\alpha} \mathcal{B}(w) \right) \\ &\quad - 4e^{-\frac{w^2}{2}} \pi^{\frac{3}{2}} \sqrt{\alpha} \sum_{k=1}^{m-1} \frac{(-1)^k}{(2\pi\alpha)^{2k}} \Gamma(2k) \zeta^2(2k) {}_1F_1^2\left(k + \frac{1}{2}; \frac{3}{2}; \frac{w^2}{4}\right) + O_{m,w}(\alpha^{-2m}). \end{aligned} \tag{1.22}$$

Putting $w = 0$ in Corollary 1.7 then gives us a known result due to Oloa [Olo10, Equation (1.5)].

Remark 1.8. *At first glance, letting $z = 0$ in Theorem 1.5 seems problematic. However, the function $\frac{1}{2}\alpha^{\frac{-z-1}{2}} \zeta(z+1)A_w(z) + \frac{1}{z}\alpha^{\frac{z-1}{2}} \zeta(z)A_w(-z)$ has a removable singularity at $z = 0$. See (3.48).*

It is important to note that the results on asymptotics stated above are usually obtained by employing Watson’s lemma [Tem96, p. 32]; see for example [DRZ17]. However, in the general setting with the parameter w which we are considering here, Watson’s lemma is inapplicable. Nevertheless, we are able to obtain an asymptotic estimate for the integral given in (1.19), and that too by elementary means. See Section 3. for details.

This paper is organized as follows. Section 2. is devoted to proving Theorem 1.3. Theorem 1.5 and its corollaries are proved in Section 3..

2. Proof of Theorem 1.3

We obtain (1.17) as a special case of a more general result derived in [DK21, Theorems 1.1.5, 1.1.6], which is stated next. It will be clear from the proof that deriving (1.17) from a generalization is not straightforward.

Theorem 2.1. *Let $A_w(z)$ and Δ_2 be defined in (1.18) and (1.20) respectively. Let $\zeta_w(s, a)$ be the generalized Hurwitz zeta function defined in (1.9). Let $w \in \mathbb{C}$, $-1 < \text{Re}(z) < 1$ and $x > 0$. Define $\varphi_w(z, x)$ by*

$$\varphi_w(z, x) := \zeta_w(z + 1, x + 1) + \frac{1}{2}A_w(z)x^{-z-1} - A_{iw}(-z)\frac{x^{-z}}{z}. \tag{2.23}$$

Then for $\alpha, \beta > 0$ and $\alpha\beta = 1$, we have

$$\begin{aligned} & \alpha^{\frac{z+1}{2}} \left(\sum_{m=1}^\infty \varphi_w(z, m\alpha) - \frac{\zeta(z+1)A_w(z)}{2\alpha^{z+1}} - \frac{\zeta(z)A_w(-z)}{\alpha z} \right) \\ &= \beta^{\frac{z+1}{2}} \left(\sum_{m=1}^\infty \varphi_{iw}(z, m\beta) - \frac{\zeta(z+1)A_{iw}(z)}{2\beta^{z+1}} - \frac{\zeta(z)A_{iw}(-z)}{\beta z} \right) \\ &= \frac{2^{z-1}\pi^{\frac{z-3}{2}}}{\Gamma(z+1)} \int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right) \Gamma\left(\frac{z-1-it}{4}\right) \Xi\left(\frac{t+iz}{2}\right) \Xi\left(\frac{t-iz}{2}\right) \frac{\Delta_2\left(\alpha, \frac{z}{2}, w, \frac{1+it}{2}\right)}{(z+1)^2+t^2} dt. \end{aligned} \tag{2.24}$$

Proof of Theorem 1.3. It can be seen that the series containing $\lambda_w(m\alpha)(\lambda_w(m\beta))$ is convergent using [DK21, p. 42, Remark 9]:

$$\lambda_w(m\alpha) = \psi_w(m\alpha + 1) - \frac{1}{2m\alpha} \mathcal{C}(w) - \mathcal{C}(iw) \log(m\alpha) - \frac{1}{2} \mathcal{B}(iw) = O_w\left(\frac{1}{m^2}\right).$$

We now prove (1.17). Let $z \rightarrow 0$ in (2.24). Then the right-hand side of (2.24) is easily seen to reduce to the extreme right side of (1.17) (up to a minus sign). Thus we need only show that

$$\begin{aligned} & \lim_{z \rightarrow 0} \left(\sum_{m=1}^{\infty} \varphi_w(z, m\alpha) - \frac{\zeta(z+1)A_w(z)}{2\alpha^{z+1}} - \frac{\zeta(z)A_w(-z)}{\alpha z} \right) \\ &= - \left(\frac{\gamma - \log(2\pi\alpha)}{2\alpha} \mathcal{C}(w) + \frac{\sqrt{\pi}}{2\alpha} \mathcal{B}(w) + \sum_{m=1}^{\infty} \lambda_w(m\alpha) \right). \end{aligned} \quad (2.25)$$

We first evaluate

$$L_1(w, \alpha) := \lim_{z \rightarrow 0} \left\{ \frac{\zeta(z+1)A_w(z)}{2\alpha^{z+1}} + \frac{\zeta(z)A_w(-z)}{\alpha z} \right\} \quad (2.26)$$

$$= \frac{\sqrt{\pi}}{w} \operatorname{erf}\left(\frac{w}{2}\right) \lim_{z \rightarrow 0} \left\{ \frac{\zeta(1+z)}{2\alpha^{1+z}} {}_1F_1\left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) + \frac{\zeta(z)}{\alpha z} {}_1F_1\left(1 - \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) \right\}, \quad (2.27)$$

where in the last step we used (1.18). Note that [Tit86, p. 16, Equation (2.1.16)] near $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|), \quad (2.28)$$

whence

$$\lim_{z \rightarrow 0} \left(\zeta(z+1) - \frac{1}{z} \right) = \gamma.$$

Also, the following series expansions hold as $z \rightarrow 0$:

$$\begin{aligned} \alpha^{-z} &= 1 - z \log \alpha + \frac{z^2(\log \alpha)^2}{2!} + O(|z|^3), \\ \zeta(z) &= -\frac{1}{z} - \frac{1}{2} \log(2\pi)z + O(|z|^2), \\ {}_1F_1\left(1 \pm \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) &= \frac{\sqrt{\pi}e^{\frac{w^2}{4}} \operatorname{erf}\left(\frac{w}{2}\right)}{w} + z \frac{d}{dz} {}_1F_1\left(1 \pm \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) \Big|_{z=0} + \frac{z^2}{2!} \frac{d^2}{dz^2} {}_1F_1\left(1 \pm \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) \Big|_{z=0} \\ &\quad + O(|z|^3). \end{aligned} \quad (2.29)$$

Hence from (2.27), (2.28) and (2.29), we compute

$$\begin{aligned} L_1(w, \alpha) &= \frac{\sqrt{\pi}}{w} \operatorname{erf}\left(\frac{w}{2}\right) \lim_{z \rightarrow 0} \left\{ \frac{\left(\frac{1}{z} + \gamma - \gamma_1 z + O(|z|^2)\right)}{2\alpha} \left(1 - z \log \alpha + \frac{z^2}{2!} (\log \alpha)^2 + O_\alpha(|z|^3)\right) \right. \\ &\quad \times \left. \left(\frac{\sqrt{\pi}e^{\frac{w^2}{4}} \operatorname{erf}\left(\frac{w}{2}\right)}{w} + z \frac{d}{dz} {}_1F_1\left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) \Big|_{z=0} + O_w(|z|^2) \right) \right. \\ &\quad \left. + \frac{\left(-\frac{1}{z} - \frac{1}{2} \log(2\pi)z + O(|z|^2)\right)}{\alpha z} \left(\frac{\sqrt{\pi}e^{\frac{w^2}{4}} \operatorname{erf}\left(\frac{w}{2}\right)}{w} + z \frac{d}{dz} {}_1F_1\left(1 - \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) \Big|_{z=0} + O_w(|z|^2) \right) \right\} \\ &= \frac{\sqrt{\pi}}{w} \operatorname{erf}\left(\frac{w}{2}\right) \left\{ \frac{\sqrt{\pi}e^{\frac{w^2}{4}} \operatorname{erf}\left(\frac{w}{2}\right)}{w} \frac{\gamma}{2\alpha} - \frac{\log(2\pi\alpha)}{2\alpha} \frac{\sqrt{\pi}e^{\frac{w^2}{4}} \operatorname{erf}\left(\frac{w}{2}\right)}{w} + \frac{1}{2\alpha} \left(\frac{d}{dz} {}_1F_1\left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) \Big|_{z=0} \right. \right. \\ &\quad \left. \left. - \frac{d}{dz} {}_1F_1\left(1 - \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4}\right) \Big|_{z=0} \right) \right\}. \end{aligned} \quad (2.30)$$

Note that

$$\begin{aligned} 2 \frac{d}{dz} {}_1F_1 \left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4} \right) &= 2 \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(1 + \frac{z}{2})_n (w^2/4)^n}{(3/2)_n n!} \\ &= 2 \sum_{n=0}^{\infty} \frac{(w^2/4)^n}{(3/2)_n n!} \frac{d}{dz} \prod_{j=0}^{n-1} \left(1 + \frac{z}{2} + j \right) \\ &= \sum_{n=0}^{\infty} \frac{(w^2/4)^n}{(3/2)_n n!} \left(1 + \frac{z}{2} \right)_n \sum_{j=0}^{n-1} \frac{1}{(1 + \frac{z}{2} + j)}, \end{aligned}$$

so that

$$2 \frac{d}{dz} {}_1F_1 \left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4} \right) \Big|_{z=0} = \sum_{n=0}^{\infty} \frac{(w^2/4)^n}{(3/2)_n} (\psi(n+1) + \gamma), \tag{2.31}$$

where we use the well-known result [Tem96, p. 54, Equation (3.10)] for $n \in \mathbb{N} \cup \{0\}$:

$$\psi(n+1) = -\gamma + \sum_{m=0}^{\infty} \left(\frac{1}{m+1} - \frac{1}{n+1+m} \right).$$

Similarly,

$$-2 \frac{d}{dz} {}_1F_1 \left(1 - \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4} \right) \Big|_{z=0} = \sum_{n=0}^{\infty} \frac{(w^2/4)^n}{(3/2)_n} (\psi(n+1) + \gamma). \tag{2.32}$$

Therefore, from (2.30), (2.31), (2.32), we have

$$\begin{aligned} L_1(w, \alpha) &= \frac{\gamma - \log(2\pi\alpha)}{2\alpha} \frac{\pi}{w^2} e^{\frac{w^2}{4}} \operatorname{erf}^2 \left(\frac{w}{2} \right) + \frac{1}{2\alpha} \frac{\sqrt{\pi}}{w} \operatorname{erf} \left(\frac{w}{2} \right) \sum_{n=0}^{\infty} \frac{(w^2/4)^n}{(3/2)_n} (\psi(n+1) + \gamma) \\ &= \frac{\gamma - \log(2\pi\alpha)}{2\alpha} \mathcal{C}(w) + \frac{1}{2\alpha} \mathcal{B}(w), \end{aligned} \tag{2.33}$$

using the definitions in (1.15) and (1.16). Now the asymptotic expansion of $\zeta_w(s, a)$ [DK21, p. 38, Theorem 5.3.1], for $-1 < \operatorname{Re}(s) < 2$,

$$\zeta_w(s, a+1) = -\frac{a^{-s}}{2} A_w(s-1) + \frac{a^{1-s}}{s-1} A_{iw}(1-s) + O_{s,w} \left(a^{-\operatorname{Re}(s)-1} \right), \text{ as } a \rightarrow \infty, \tag{2.34}$$

implies that the series in (2.24) are absolutely and uniformly convergent in a neighborhood of $z = 0$. Hence, we can take the limit inside the series on the extreme left-hand side of (2.24). This gives, using (2.23),

$$\begin{aligned} \lim_{z \rightarrow 0} \sum_{m=1}^{\infty} \varphi_w(z, m\alpha) &= \sum_{m=1}^{\infty} \lim_{z \rightarrow 0} \left\{ \zeta_w(z+1, m\alpha+1) - \frac{\sqrt{\pi} \operatorname{erfi} \left(\frac{w}{2} \right) {}_1F_1 \left(1 - \frac{z}{2}; \frac{3}{2}; -\frac{w^2}{4} \right)}{z(m\alpha)^z w} \right. \\ &\quad \left. + \frac{\sqrt{\pi} \operatorname{erf} \left(\frac{w}{2} \right) {}_1F_1 \left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4} \right)}{2(m\alpha)^{z+1} w} \right\} \\ &= \sum_{m=1}^{\infty} \left(\lim_{z \rightarrow 0} \left\{ \left(\zeta_w(z+1, m\alpha+1) - \frac{\pi e^{-\frac{w^2}{4}} \operatorname{erfi}^2 \left(\frac{w}{2} \right)}{z w^2} \right) \right. \right. \\ &\quad \left. \left. + \frac{\sqrt{\pi} \operatorname{erf} \left(\frac{w}{2} \right) {}_1F_1 \left(1 + \frac{z}{2}; \frac{3}{2}; \frac{w^2}{4} \right)}{2(m\alpha)^{z+1} w} \right\} + L_2(w, \alpha, m) \right), \end{aligned} \tag{2.35}$$

where $L_2(w, \alpha, m)$ is defined as

$$L_2(w, \alpha, m) := \lim_{z \rightarrow 0} \left(\frac{\pi e^{-\frac{w^2}{4}} \operatorname{erfi}^2\left(\frac{w}{2}\right)}{zw^2} - \frac{\sqrt{\pi} \operatorname{erfi}\left(\frac{w}{2}\right)}{z(m\alpha)^z w} {}_1F_1\left(1 - \frac{z}{2}; \frac{3}{2}; -\frac{w^2}{4}\right) \right). \tag{2.36}$$

Invoking (1.12), (1.13) in (2.35), we are led to

$$\lim_{z \rightarrow 0} \sum_{m=1}^{\infty} \varphi_w(z, m\alpha) = \sum_{m=1}^{\infty} \left\{ -\psi_w(m\alpha + 1) + \frac{\mathcal{C}(w)}{2m\alpha} + L_2(w, \alpha, m) \right\}. \tag{2.37}$$

We now use (2.30) in (2.36) to see that

$$\begin{aligned} L_2(w, \alpha, m) &= \lim_{z \rightarrow 0} \left\{ \frac{\sqrt{\pi} \operatorname{erfi}(w/2)}{w} \left[\frac{e^{-w^2/4} \sqrt{\pi} \operatorname{erfi}(w/2)}{zw} - (1 - z \log(m\alpha) + O(|z|^2)) \right. \right. \\ &\quad \left. \left. \times \left(\frac{\sqrt{\pi} e^{-\frac{w^2}{4}} \operatorname{erfi}\left(\frac{w}{2}\right)}{wz} + \frac{d}{dz} {}_1F_1\left(1 - \frac{z}{2}; \frac{3}{2}; -\frac{w^2}{4}\right) \Big|_{z=0} + O(|z|) \right) \right] \right\} \\ &= \frac{\sqrt{\pi} \operatorname{erfi}(w/2)}{w} \left(\frac{\sqrt{\pi}}{w} e^{-\frac{w^2}{4}} \operatorname{erfi}\left(\frac{w}{2}\right) \log(m\alpha) - \frac{d}{dz} {}_1F_1\left(1 - \frac{z}{2}; \frac{3}{2}; -\frac{w^2}{4}\right) \Big|_{z=0} \right) \\ &= \frac{1}{w^2} \pi e^{-\frac{w^2}{4}} \operatorname{erfi}^2(w/2) \log(m\alpha) + \frac{\sqrt{\pi} \operatorname{erfi}(w/2)}{2w} \sum_{n=0}^{\infty} \frac{(-w^2/4)^n}{(3/2)_n} (\psi(n+1) + \gamma) \\ &= \mathcal{C}(iw) \log(m\alpha) + \frac{1}{2} \mathcal{B}(iw), \end{aligned} \tag{2.38}$$

where in the penultimate step we used (2.32) and in the ultimate step (1.15) and (1.16). Substitute (2.38) in (2.37), thereby obtaining

$$\begin{aligned} \lim_{z \rightarrow 0} \sum_{m=1}^{\infty} \varphi_w(z, m\alpha) &= \sum_{m=1}^{\infty} \left\{ -\psi_w(m\alpha + 1) + \frac{\mathcal{C}(w)}{2m\alpha} + \mathcal{C}(iw) \log(m\alpha) + \frac{1}{2} \mathcal{B}(iw) \right\} \\ &= - \sum_{m=1}^{\infty} \lambda_w(m\alpha), \end{aligned} \tag{2.39}$$

which follows from (1.14). Equations (2.33) and (2.39) together prove the claim (2.25). This completes the proof of the theorem. \square

Proof of Corollary 1.4. Let $w \rightarrow 0$ on both sides of (1.17). Note that $\lim_{w \rightarrow 0} \mathcal{C}(w) = 1$ and $\lim_{w \rightarrow 0} \mathcal{B}(w) = 0$ as $\lim_{w \rightarrow 0} \frac{\operatorname{erfi}(w/2)}{w} = \frac{1}{\sqrt{\pi}}$ and $\psi(1) = -\gamma$. These facts then give $\lim_{w \rightarrow 0} \lambda_w(x) = \lambda(x)$. This proves (1.6). \square

3. Asymptotics of some integrals with $\Xi(t)$ in their integrands

Here we prove Theorem 1.5 and then recover two results from it, one of which is new.

Proof of Theorem 1.5. We start with (2.24),

$$\begin{aligned} &\frac{2^{z-1} \pi^{\frac{z-3}{2}}}{\Gamma(z+1)} \int_0^{\infty} \Gamma\left(\frac{z-1+it}{4}\right) \Gamma\left(\frac{z-1-it}{4}\right) \Xi\left(\frac{t+iz}{2}\right) \Xi\left(\frac{t-iz}{2}\right) \frac{\Delta_2\left(\alpha, \frac{z}{2}, w, \frac{1+it}{2}\right)}{(z+1)^2 + t^2} dt \\ &= \alpha^{\frac{z+1}{2}} \left(\sum_{m=1}^{\infty} \varphi_w(z, m\alpha) - \frac{\zeta(z+1)A_w(z)}{2\alpha^{z+1}} - \frac{\zeta(z)A_w(-z)}{\alpha^z} \right). \end{aligned} \tag{3.40}$$

From [DK21, p. 75, Equation (9.2.10)], we have

$$\begin{aligned} & -\frac{1}{\pi}\Gamma\left(\frac{z}{2}\right)\sum_{n=1}^{\infty}\sigma_{-z}(n)\int_0^{\infty}e^{-\frac{w^2}{2}}{}_1K_{\frac{z}{2},iw}(2\alpha x)\left({}_2F_1\left(1,\frac{z}{2};\frac{1}{2};-\frac{x^2}{\pi^2n^2}\right)-1\right)x^{\frac{z-2}{2}}dx \\ & =\frac{\alpha^{\frac{z}{2}}\Gamma(z+1)e^{-\frac{w^2}{4}}}{2^{z+1}}\sum_{m=1}^{\infty}\varphi_w(z,m\alpha), \end{aligned} \tag{3.41}$$

where ${}_1K_{z,w}(x)$ is the generalized modified Bessel function introduced in [DK21, p. 11, Equation (1.1.37)] by

$$\begin{aligned} {}_1K_{z,w}(x) & :=\frac{1}{2\pi i}\int_{(c)}\Gamma\left(\frac{1+s-z}{2}\right)\Gamma\left(\frac{1+s+z}{2}\right){}_1F_1\left(\frac{1+s-z}{2};\frac{3}{2};-\frac{w^2}{4}\right) \\ & \quad \times {}_1F_1\left(\frac{1+s+z}{2};\frac{3}{2};-\frac{w^2}{4}\right)2^{s-1}x^{-s}ds, \end{aligned} \tag{3.42}$$

where $z, w \in \mathbb{C}$, $x \in \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ and $c := \operatorname{Re}(s) > -1 \pm \operatorname{Re}(z)$. Substituting (3.41) in (3.40), we see that

$$\begin{aligned} & \frac{e^{-\frac{w^2}{4}}\pi^{\frac{z-3}{2}}}{2}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\Delta_2\left(\alpha,\frac{z}{2},w,\frac{1+it}{2}\right)}{(z+1)^2+t^2}dt \\ & =-\frac{2\sqrt{\alpha}}{\pi}\Gamma\left(\frac{z}{2}\right)\sum_{n=1}^{\infty}\sigma_{-z}(n)\int_0^{\infty}e^{-\frac{w^2}{2}}{}_1K_{\frac{z}{2},iw}(2\alpha x)\left({}_2F_1\left(1,\frac{z}{2};\frac{1}{2};-\frac{x^2}{\pi^2n^2}\right)-1\right)x^{\frac{z-2}{2}}dx \\ & \quad -\frac{\Gamma(z+1)e^{-\frac{w^2}{4}}}{2^z}\alpha^{\frac{(z+1)}{2}}\left(\frac{\zeta(z+1)A_w(z)}{2\alpha^{z+1}}+\frac{\zeta(z)A_w(-z)}{\alpha^z}\right). \end{aligned} \tag{3.43}$$

We first find the asymptotic estimate of the integral inside the sum on the right-hand side of (3.43). Note that

$$\begin{aligned} & \int_0^{\infty}{}_1K_{\frac{z}{2},iw}(2\alpha x)\left({}_2F_1\left(1,\frac{z}{2};\frac{1}{2};-\frac{x^2}{\pi^2n^2}\right)-1\right)x^{\frac{z-2}{2}}dx \\ & =\frac{1}{(2\alpha)^{z/2}}\int_0^{\infty}{}_1K_{\frac{z}{2},iw}(x)\left({}_2F_1\left(1,\frac{z}{2};\frac{1}{2};-\frac{x^2}{4\pi^2n^2\alpha^2}\right)-1\right)x^{\frac{z-2}{2}}dx. \end{aligned} \tag{3.44}$$

Using the series definition of ${}_2F_1$, for $m \in \mathbb{N}$, as $\alpha \rightarrow \infty$, we have

$${}_2F_1\left(1,\frac{z}{2};\frac{1}{2};-\frac{x^2}{4\pi^2n^2\alpha^2}\right)-1=\sum_{k=1}^{m-1}\frac{(-1)^k(z/2)_k}{(1/2)_k}\left(\frac{x^2}{4\pi^2n^2\alpha^2}\right)^k+O_z\left(\frac{x^{2m}}{\alpha^{2m}n^{2m}}\right).$$

Substituting the above result in (3.44), we see that as $\alpha \rightarrow \infty$

$$\begin{aligned} & \int_0^{\infty}{}_1K_{\frac{z}{2},iw}(2\alpha x)\left({}_2F_1\left(1,\frac{z}{2};\frac{1}{2};-\frac{x^2}{\pi^2n^2}\right)-1\right)x^{\frac{z-2}{2}}dx \\ & =\frac{1}{(2\alpha)^{z/2}}\left\{\sum_{k=1}^{m-1}\frac{(-1)^k(z/2)_k}{(1/2)_k(4\pi^2n^2\alpha^2)^k}\int_0^{\infty}x^{\frac{z}{2}+2k-1}{}_1K_{\frac{z}{2},iw}(x)dx\right. \\ & \quad \left.+O_{z,m}\left(\frac{1}{(n\alpha)^{2m}}\int_0^{\infty}x^{\frac{z}{2}+2m-1}{}_1K_{\frac{z}{2},iw}(x)dx\right)\right\} \\ & =\frac{1}{(2\alpha)^{z/2}}\left\{\sum_{k=1}^{m-1}\frac{(-1)^k(z/2)_k\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{z+1}{2}+k\right)}{(1/2)_k(4\pi^2n^2\alpha^2)^k}{}_1F_1\left(\frac{z+1}{2}+k;\frac{3}{2};\frac{w^2}{4}\right){}_1F_1\left(\frac{1}{2}+k;\frac{3}{2};\frac{w^2}{4}\right)2^{\frac{z}{2}+2k-1}\right. \\ & \quad \left.+O_{z,m,w}\left(\frac{1}{(n\alpha)^{2m}}\right)\right\}, \end{aligned} \tag{3.45}$$

where we used the definition of ${}_1K_{\frac{z}{2}, iw}(x)$ from (3.42) and the fact that integral inside the big $-O$ bound is convergent and independent of α . An application of duplication formula of gamma function in (3.45) yields

$$\begin{aligned} & \int_0^\infty {}_1K_{\frac{z}{2}, iw}(2\alpha x) \left({}_2F_1 \left(1, \frac{z}{2}; \frac{1}{2}; -\frac{x^2}{\pi^2 n^2} \right) - 1 \right) x^{\frac{z-2}{2}} dx \\ &= \frac{\pi}{\alpha^{z/2} 2^z} \sum_{k=1}^{m-1} \frac{(-1)^k \Gamma(z+2k)}{(2\pi n \alpha)^{2k} \Gamma(z/2)} {}_1F_1 \left(\frac{z+1}{2} + k; \frac{3}{2}; \frac{w^2}{4} \right) {}_1F_1 \left(\frac{1}{2} + k; \frac{3}{2}; \frac{w^2}{4} \right) + O_{z,m,w} \left(\frac{1}{n^{2m} \alpha^{2m+\frac{z}{2}}} \right), \end{aligned} \tag{3.46}$$

Equations (3.43) and (3.46) along with the fact $\sum_{n=1}^\infty \sigma_a(n) n^{-s} = \zeta(s)\zeta(s-a)$, $\text{Re}(s) > \max\{1, 1 + \text{Re}(a)\}$, imply

$$\begin{aligned} & \frac{e^{-\frac{w^2}{4}} \pi^{\frac{z-3}{2}}}{2} \int_0^\infty \Gamma \left(\frac{z-1+it}{4} \right) \Gamma \left(\frac{z-1-it}{4} \right) \Xi \left(\frac{t+iz}{2} \right) \Xi \left(\frac{t-iz}{2} \right) \frac{\Delta_2 \left(\alpha, \frac{z}{2}, w, \frac{1+it}{2} \right)}{(z+1)^2 + t^2} dt \\ &= -\frac{e^{-w^2/2}}{\alpha^{\frac{z-1}{2}} 2^{z-1}} \sum_{k=1}^{m-1} \frac{(-1)^k \Gamma(z+2k)}{(2\pi \alpha)^{2k}} \zeta(2k) \zeta(z+2k) {}_1F_1 \left(\frac{z+1}{2} + k; \frac{3}{2}; \frac{w^2}{4} \right) {}_1F_1 \left(\frac{1}{2} + k; \frac{3}{2}; \frac{w^2}{4} \right) \\ & \quad - \frac{\Gamma(z+1) e^{-\frac{w^2}{4}}}{2^z} \left(\frac{\zeta(z+1) A_w(z)}{2\alpha^{\frac{z+1}{2}}} + \frac{\zeta(z) A_w(-z)}{z\alpha^{\frac{1-z}{2}}} \right) + O_{z,m,w} \left(\frac{1}{\alpha^{2m+\frac{z}{2}}} \right). \end{aligned} \tag{3.47}$$

Finally divide both sides of the above equation by $\frac{1}{2} e^{-\frac{w^2}{4}} \pi^{\frac{z-3}{2}}$ to complete the proof of the theorem. \square

Proof of Corollary 1.6. Note that ${}_1F_1(a; c; 0) = 1$ and $\Delta_2 \left(\alpha, \frac{z}{2}, w, \frac{1+it}{2} \right) = 2 \cos \left(\frac{1}{2} t \log(\alpha) \right)$. Now (1.21) is an easy consequence of (1.19). \square

Proof of Corollary 1.7. Let $z = 0$ on both sides of (1.19). It is easy to get the left-hand side of (1.22) from the integral in (1.19) up to the factor $e^{\frac{w^2}{4}}$ coming from $\Delta_2 \left(\alpha, 0, w, \frac{1+it}{2} \right)$. Also, it is easy to take $z = 0$ in the finite sum on the right-hand side of (1.19). Now observe that

$$\begin{aligned} \lim_{z \rightarrow 0} \left\{ \frac{\zeta(z+1) A_w(z)}{2\alpha^{\frac{1+z}{2}}} + \frac{\zeta(z) A_w(-z)}{z\alpha^{\frac{1-z}{2}}} \right\} &= \lim_{z \rightarrow 0} \left\{ \alpha^{\frac{1+z}{2}} \left(\frac{\zeta(z+1) A_w(z)}{\alpha^{z+1}} + \frac{\zeta(z) A_w(-z)}{z\alpha} \right) \right\} \\ &= \sqrt{\alpha} L_1(w, \alpha) \\ &= \sqrt{\alpha} \left(\frac{\gamma - \log(2\pi\alpha)}{2\alpha} \mathcal{C}(w) + \frac{1}{2\alpha} \mathcal{B}(w) \right), \end{aligned} \tag{3.48}$$

where in the second last step we used (2.26) and in the last step (2.33) is invoked. The proof of (1.22) is then complete upon using the above facts. \square

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