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Ramanujan's Quarterly Reports

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Abstract. This is a faithful reproduction of Ramanujan's Quarterly Reports from a handwritten copy by T.A. Satagopan in 1925.

Keywords. Ramanujan's quarterly reports, August 1913 to March 1914, Satagopan's handwritten copy
2010 Mathematics Subject Classification. 01A70, 11-03

1. History of the Quarterly Reports

After marrying Janaki in 1909, Ramanujan moved to Madras in 1910 to secure employment but also with the goal of seeking recognition for his mathematical discoveries. In 1911, Ramanujan began to publish both papers and problems in the *Journal of the Indian Mathematical Society.* His emerging fame reached the English astronomer Sir Gilbert Walker, working at an observatory in Madras. In a letter to the University of Madras dated February 26, 1913 [6, p. 51], he wrote, "The University would be justified in enabling S. Ramanujan for a few years *at least* to spend the whole of his time on mathematics, without any anxiety as to his livelihood." The Board of Studies at the University of Madras agreed to this request, and its chairman, Professor B. Hanumantha Rao, wrote a letter to the Vice-Chancellor on March 25, 1913, with the recommendation that Ramanujan be awarded a scholarship of 75 rupees per month [6, p. 76]. The approval was swift, and Ramanujan was awarded a scholarship commencing on May 1, 1913.

A stipulation in the scholarship required Ramanujan to write quarterly reports describing his research to the Board of Studies in Mathematics. Ramanujan wrote three of these quarterly reports, dated 5th August 1913, 7th November 1913, and 9th March, 1914, before he departed for England on March 17, 1914. Unfortunately, these reports were either destroyed or misplaced at the University of Madras; they have never been found. Fortunately, in 1925, T. A. Satagopan made a handwritten copy of the reports totalling 51 pages. This copy was sent to G. H. Hardy and is now at the library at Trinity College, Cambridge. Also on file at Trinity College is a second copy of the reports made by G. N. Watson. Hardy used the reports in writing Chapter 11 of his book [8] on Ramanujan's work. The reports have never been published in their entirety. However, a complete description of their contents was published by this editor in the *Bulletin of the London Mathematical Society* [4]. A shorter, somewhat less technical account was written by the same author for the *American Mathematical Monthly* [3].

Some of the material in the *Quarterly Reports* can also be found in Chapters 3 and 4 of Ramanujan's second notebook [2], [1], [5, Chapters 3, 4]. In contrast to his notebooks [11], Ramanujan provided proofs in his reports. His proofs are mostly formal and not rigorous. In discussing perhaps the most important result from the reports, in a paper published in 1937, Hardy [7, p. 150] remarked, "There is one particularly interesting formula . . . of which he was especially fond and made continual use. . . . had not 'really' proved any of the formulae which I have quoted. It was impossible that he should have done so because the 'natural' conditions involve ideas of which he knew nothing in 1914, and which he had hardly absorbed before his death". (Hardy's paper [7] is reprinted in Chapter 1 of [8].) The aforementioned theorem provides a method for evaluating large classes of definite integrals. After arriving in England, Ramanujan evidently was informed by Hardy that his proof

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was not rigorous, for in a paper [9], [10, pp. 53–58] offering several integral evaluations written in 1915, Ramanujan remarked, "My own proofs of the above results make use of a general formula, the truth of which depends on conditions which I have not yet investigated completely. A direct proof depending on Cauchy's theorem will be found in Mr. Hardy's note which follows this paper". Readers of Ramanujan's *Quarterly Reports* will find many interesting applications of his ingenious "general formula."

References

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[10] S. Ramanujan, *Collected Papers*, Chelsea, New York, 1962.

[11] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957; second edition with improved photography, 2012.

2. Notes on Transcribing the Quarterly Reports

We have attempted to reproduce Ramanujan's style as much as possible. But it should be emphasized that we have prepared our transmission of the *Quarterly Reports* from a handwritten copy made at the University of Madras in 1925. Presumably, the copyist, T. A. Satagopan, had a background in mathematics. Out of a deep respect to Ramanujan, we *guess* that Satagopan attempted to adhere to Ramanujan's style as much as possible. However, we have made some exceptions. For example, Ramanujan frequently wrote mathematical symbols, especially integrals, within his text. A long series of equalities on multiple lines can sometimes be shortened by distributing them on fewer lines. For typographical and aesthetical reasons, we have often put these mathematical expressions in display modes. In a few instances, we have put Ramanujan's words in sans-serif font to indicate that we were unable to apply latex to reproduce Ramanujan's style.

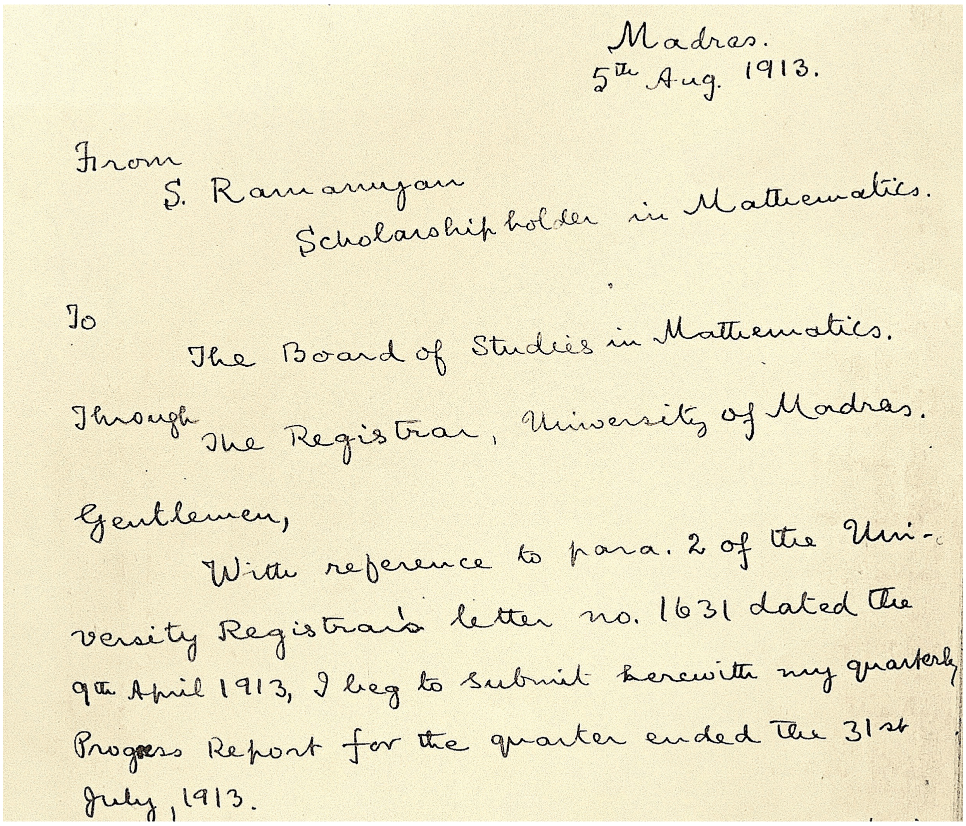
In general, we have not changed Ramanujan's punctuation, but for clarity, we have occasionally inserted commas and periods. Ramanujan's results and proofs are not organized consistently, with Theorem, Cor., Art., (a), (b), (i), (ii), etc., 1, 2, 3, etc., (1), (2), (3), etc. being employed, but we have kept his designations.

We have adhered to Ramanujan's notation, but with two notable exceptions. We write $!$ in place of the now antiquated $\underline{|n}$. which was utilized by Ramanujan. In his third *Quarterly Report*, he used the notation $\Pi(x)$ for $\Gamma(x+1)$. Since the latter notation is almost now universally used, we have adopted the Γ-function notation.

We are grateful to Jaebum Sohn, who helped prepare the Quarterly Reports, and to Shivajee Gupta, who constructed the two geometric figures.

3. Ramanujan's Quarterly Reports

Sample image from Satagopan's copy



Madras.
5th Aug. 1913.

From
S. Ramanujan
Scholarship holder in Mathematics.

To
The Board of Studies in Mathematics.

Through The Registrar, University of Madras.

Gentlemen,
With reference to para. 2 of the University Registrar's letter no. 1631 dated the 9th April 1913, I beg to submit herewith my quarterly Progress Report for the quarter ended the 31st July, 1913.

Madras
5th Aug. 1913

From S. Ramanujan,

Scholarship holder in Mathematics

To The Board of Studies in Mathematics

Through The Registrar, University of Madras.

Gentlemen,

With reference to para. 2 of the University Registrar's letter no. 1631 dated the 9th April 1913, I beg to submit herewith my quarterly Progress Report for the quarter ended the 31st July, 1913. The Progress Report is merely the exposition of a new theorem I have discovered in Integral Calculus. At present there are many definite integrals the values of which we know to be finite but still not possible of evaluation by the present known methods . This theorem will be an instrument by which at least some of the definite integrals whose values are at present not known can be evaluated. For instance, the integral treated in Ex(V) <u>note</u> Art. 5. in the paper, Mr. G. H. Hardy M.A., F.R.S. of Trinity

College, Cambridge, considers to be "new and interesting." Similarly, the integral connected with the Besselian function of the n^{th} order which at present requires many complicated manipulations to evaluate can be readily inferred from the theorem given in the paper. I have also utilised this theorem in definite integrals for the expansion of functions which can now be ordinarily done by Lagrange's, Burmann's, or Abel's theorems. For instance, the expansions marked as examples nos. (3) and (4), Art. 6, in the second part of the paper.

The investigations I have made on the basis of this theorem are not all contained in the attached paper. There is ample scope for new and interesting results out of this theorem. This paper may be considered the first instalment of the results I have got out of the theorem. Other new results, based on the theorem I shall communicate in my later reports.

I beg to submit this, my maiden attempt, and I humbly request that the Members of the Board will make allowance for any defect which they may notice to my want of usual training which is now undergone by college students and view sympathetically my humble effort in the attached paper.

I beg to remain,
Gentlemen
Your obedient servant
S. Ramanujan

1. Subject of the paper

If $F(x)$ be a function capable of expansion in positive integral powers of x, then

(A) the value of

$$\int_0^\infty x^{n-1}F(x)dx$$

can be found from the coefficient of x^n in the expansion of $F(x)$ and conversely.

(B) the expansion of $F(x)$ in powers of x can be found if the value of the integral

$$\int_0^\infty x^{n-1}F(x)dx$$

be known.

2. A:—

Let the expansion of $F(x)$ be

$$\phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \cdots,$$

then

$$\int_0^\infty x^{n-1}F(x)dx = \Gamma(n)\phi(-n).$$

Dem. We know

$$\int_0^\infty e^{-mx}x^{n-1}dx = \frac{\Gamma(n)}{m^n}.$$

By giving the values $1, r, r^2, r^3, \ldots$ to m on both the sides, multiplying the results by

$$f(a), \quad \frac{hf'(a)}{1!}, \quad \frac{h^2f''(a)}{2!}, \quad \frac{h^3f'''(a)}{3!}, \ldots$$

and adding up all these results, we have

$$\begin{aligned}
&f(a)\int_0^\infty e^{-x}x^{n-1}dx+\frac{h}{1!}f'(a)\int_0^\infty e^{-rx}x^{n-1}dx+\frac{h^2}{2!}f''(a)\int_0^\infty e^{-r^2x}x^{n-1}dx\\
&+\frac{h^3}{3!}f'''(a)\int_0^\infty e^{-r^3x}x^{n-1}dx+\cdots\\
=&\Gamma(n)\left\{f(a)+\frac{h}{r^n}\frac{f'(a)}{1!}+\frac{h^2}{r^{2n}}\frac{f''(a)}{2!}+\frac{h^3}{r^{3n}}\frac{f'''(a)}{3!}+\cdots\right\}.
\end{aligned}$$

Expanding $e^{-x}, e^{-rx}, e^{-r^2x},\dots$ on the left side in ascending powers of x and collecting all the terms that contain the same powers of x, we have by applying Taylor's Theorem,

$$\begin{aligned}
\int_0^\infty x^{n-1}\left\{f(a+h)-\frac{x}{1!}f(a+rh)+\frac{x^2}{2!}f(a+r^2h)-\cdots\right\}dx\\
=\Gamma(n)\,f\left(a+\frac{h}{r^n}\right).
\end{aligned}$$

Now let us suppose $f(a+hr^n)=\phi(n)$ treating a, h and r to be constants. Then we see that $f(a+\frac{h}{r^n})=\phi(-n)$ and also $f(a+h), f(a+rh), f(a+r^2h),\dots$ are respectively equal to $\phi(0),\phi(1),\phi(2),\dots$. Substituting these results in the above one we have,

$$\int_0^\infty x^{n-1}\left\{\phi(0)-\frac{x}{1!}\phi(1)+\frac{x^2}{2!}\phi(2)-\cdots\right\}dx=\Gamma(n)\phi(-n).$$

Q. E. D.

3. <u>When valid?</u>

The above theorem is legitimate if the following conditions are satisfied.

(a) As already stated, $F(x)$ should be capable of expansion in positive integral powers of x.

(b) $F(x)$ should be finite and continuous between the limits 0 and ∞ but not necessarily at 0 and ∞.

(c) n should be positive.

(d) $x^nF(x)$ should vanish when x becomes infinite.

The first two conditions are evident from the nature of the integral itself.

The 3rd condition is necessary because we have used the Eulerian Integral

$$\int_0^\infty x^{n-1}e^{-x}dx=\Gamma(n)$$

which is true only when n is positive.

The 4th condition is also necessary; for, if when $x=\infty$, $x^nF(x)$ does not vanish but be finite, say equal to a, then the greatest term in the expansion of $x^{n-1}F(x)$ is $\frac{a}{x}$ and consequently the greatest term in

$$\int_a x^{n-1}F(x)dx$$

is $a\log x$ which is infinite when $x=\infty$. Hence we see that, if when $x=\infty$, $x^nF(x)$ is finite, then

$$\int_0^\infty x^{n-1}F(x)dx$$

is infinite; and much more so it will be if $x^nF(x)$ is itself infinite when x becomes infinite.

Although the first three conditions are necessary in case of oscillating functions such as the circular, Besselian and other functions, yet the fourth condition differs for different functions we take.

4. Generalization:—

(a) The theorem can be used not only in case of Integrals having the limits 0 and ∞ but also in case of Integrals having any two limits; for any Integral

$$\int_{\alpha}^{\beta} \psi(x)dx$$

may be transformed to an Integral of the form

$$\int_{0}^{\infty} F(x)dx$$

by suitable substitutions such as $\frac{x-\alpha}{\beta-x} = y$, etc.

(b) According to the condition 3(a), $F(x)$ may include all algebraic functions and all transcendental functions which can be expanded in ascending powers of x, such as $\cos x$, $\sin x$, e^{-x}, $\tan^{-1} x$, $\log(1+x)$, etc.; but if $F(x)$ contained transcendentals of the form $\log x$, etc. which cannot be expressed in powers of x, we can substitute e^y etc. for x and then apply our theorem.

(c) Similarly by suitable substitutions all fractional powers also may be removed.

5. Examples:—

The extreme importance of this theorem in the Integral Calculus is illustrated by the following few examples:

(i) Let us take the integral

$$\int_0^\infty \frac{x^{m-1}}{1+x^n}dx.$$

Changing x to $y^{1/n}$ we have the integral reduced to

$$\frac{1}{n}\int_0^\infty \frac{y^{\frac{m}{n}-1}}{1+y}dy.$$

The conditions to be satisfied are:—

$$\begin{aligned}
&\frac{m}{n} \text{ should be positive} \ldots \quad \text{by 3(c)}\\
&\frac{y^{m/n}}{1+y} = 0 \text{ when } y = \infty \ldots \quad \text{by 3(d)}\\
&\text{i.e. } \frac{m}{n} \text{ should lie between 0 and 1.}\\
&\text{or } m \text{ should lie between 0 and } n.
\end{aligned}$$

Now

$$\frac{1}{n}\int_0^\infty \frac{y^{\frac{m}{n}-1}}{1+y}dy = \frac{1}{n}\int_0^\infty y^{\frac{m}{n}-1}\left\{\phi(0) - \frac{y}{1!}\phi(1) + \frac{y^2}{2!}\phi(2) - \cdots\right\}dy,$$

where $\phi(t) = \Gamma(1+t)$.

$$\begin{aligned}
\text{Hence the integral} &= \frac{1}{n}\Gamma\left(\frac{m}{n}\right)\phi\left(-\frac{m}{n}\right) \quad \text{by 2.}\\
&= \frac{1}{n}\Gamma\left(\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right) = \frac{\pi}{n\sin\frac{\pi m}{n}}.
\end{aligned}$$

Thus we see that if m lies between 0 and n

$$\int_0^\infty \frac{x^{m-1}}{1+x^n}dx = \frac{\pi}{n\sin\frac{\pi m}{n}}.$$

(ii)

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

if m and n are positive.

Changing x to $\frac{y}{1+y}$ the integral becomes

$$\int_0^\infty y^{m-1}(1+y)^{-m-n}dy.$$

The conditions to be satisfied are

$$\begin{gathered} m \text{ should be positive by 3(c)} \\ \frac{y^m}{(1+y)^{m+n}} = 0 \text{ when } y=\infty \text{ by 3(d)} \\ \text{i.e. } n \text{ should also be positive.} \end{gathered}$$

Now

$$\int_0^\infty y^{m-1}(1+y)^{-m-n}dy = \int_0^\infty y^{m-1}\left\{\phi(0) - \frac{y}{1!}\phi(1) + \frac{y^2}{2!}\phi(2) - \cdots\right\}dy,$$

where

$$\phi(t) = \frac{\Gamma(m+n+t)}{\Gamma(m+n)}.$$

$$\text{Hence the integral } = \Gamma(m)\phi(-m) \quad \text{(by 2)} \quad = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

(iii) When n lies between 0 and 1,

$$\int_0^\infty x^{n-1}\cos px\;dx = \frac{\Gamma(n)}{p^n}\cos\frac{\pi n}{2}.$$

Changing x to $\sqrt{y}$, we have

$$\begin{aligned} \int_0^\infty x^{n-1}\cos px\;dx =& \frac{1}{2}\int_0^\infty y^{\frac{n}{2}-1}\left(1 - \frac{p^2}{2!}y + \frac{p^4}{4!}y^2 - \cdots\right)dy \\ =& \frac{1}{2}\int_0^\infty y^{\frac{n}{2}-1}\left\{\phi(0) - \frac{y}{1!}\phi(1) + \frac{y^2}{2!}\phi(2) - \cdots\right\}dy \\ & \left(\text{where } \phi(t) = \frac{p^{2t}\Gamma(t+1)}{\Gamma(2t+1)}.\right) \\ =& \frac{1}{2}\Gamma\left(\frac{n}{2}\right)\phi\left(-\frac{n}{2}\right) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(1-\frac{n}{2}\right)}{2p^n\Gamma(1-n)} \\ =& \frac{\Gamma(n)\cos\frac{\pi n}{2}}{p^n}. \end{aligned}$$

In a similar manner on differentiating the above result with regard to p and then changing n to $n-1$ on both sides, we have,

When n lies between 0 and 1

$$\int_0^\infty x^{n-1}\sin px\;dx = \frac{\Gamma(n)}{p^n}\sin\frac{\pi n}{2}.$$

(iv) Similarly taking the Besselian Function of the n^{th} order, we have, If p lies between 0 and $\frac{2n+3}{2}$,

$$\int_0^\infty x^{p-1}\left\{1-\frac{x^2}{2}\cdot\frac{1}{2n+2}+\frac{x^4}{2\cdot 4}\cdot\frac{1}{2n+2}\cdot\frac{1}{2n+4}-\cdots\right\}dx$$
$$=\frac{2^{p-1}\Gamma\left(\frac{p}{2}\right)\Gamma(n+1)}{\Gamma\left(n+1-\frac{p}{2}\right)}.$$

The above result is immediately got by changing x to $\sqrt{y}$ and applying our theorem.

(v) If n is positive and a less than unity, then

$$\int_0^\infty \frac{x^{n-1}dx}{(1+x)(1+ax)(1+a^2x)(1+a^3x)\cdots}$$
$$=\frac{\pi}{\sin \pi n}\cdot\frac{1-a^{1-n}}{1-a}\cdot\frac{1-a^{2-n}}{1-a^2}\cdot\frac{1-a^{3-n}}{1-a^3}\cdot\frac{1-a^{4-n}}{1-a^4}\cdots.$$

Dem. Suppose

$$\frac{1}{(1+x)(1+ax)(1+a^2x)(1+a^3x)\cdots}$$
$$=1-A_1x+A_2x^2-A_3x^3+A_4x^4-\cdots.$$

Then changing x to ax, we see that

$$\begin{aligned}\frac{1}{(1+ax)(1+a^2x)(1+a^3x)\cdots}&=1-A_1ax+A_2a^2x^2-\cdots\\&=(1+x)\left(1-A_1x+A_2x^2-A_3x^3+\cdots\right).\end{aligned}$$

Hence,

$$A_1=\frac{1}{1-a};\qquad A_2=\frac{1}{(1-a)(1-a^2)};\qquad A_3=\frac{1}{(1-a)(1-a^2)(1-a^3)}$$

and generally

$$A_r=\frac{1}{(1-a)(1-a^2)(1-a^3)\cdots(1-a^r)}.$$

Hence the integral

$$=\int_0^\infty x^{n-1}\left\{\phi(0)-\frac{x}{1!}\phi(1)+\frac{x^2}{2!}\phi(2)-\cdots\right\}dx,$$

$$\left[\text{where}\qquad \phi(t)=\Gamma(t+1)\frac{1-a^{1+t}}{1-a}\cdot\frac{1-a^{2+t}}{1-a^2}\cdot\frac{1-a^{3+t}}{1-a^3}\cdots\right]$$
$$\begin{aligned}&=\Gamma(n)\phi(-n)\qquad\text{by 2.}\\&=\Gamma(n)\Gamma(1-n)\frac{1-a^{1-n}}{1-a}\cdot\frac{1-a^{2-n}}{1-a^2}\cdot\frac{1-a^{3-n}}{1-a^3}\cdots\\&=\frac{\pi}{\sin\pi n}\frac{1-a^{1-n}}{1-a}\cdot\frac{1-a^{2-n}}{1-a^2}\cdot\frac{1-a^{3-n}}{1-a^3}\cdots\end{aligned}$$

Note. If n is any positive integer the value of the above integral assumes an indeterminate form and the value may be evaluated by writing $n+h$ instead of n and ultimately making h vanish; examples.—

(a) When $n=1$, the above becomes

$$\int_0^\infty \frac{dx}{(1+x)(1+ax)(1+a^2x)(1+a^3x)\cdots}=-\log a.$$

(b) When $n = 2$, the result is

$$\int_0^\infty \frac{dx}{(1+\sqrt{x})(1+a\sqrt{x})(1+a^2\sqrt{x})\cdots} = -\frac{2(1-a)}{a}\log a.$$

(c) When $n = 3$, the result is

$$\int_0^\infty \frac{dx}{(1+\sqrt[3]{x})(1+a\sqrt[3]{x})(1+a^2\sqrt[3]{x})\cdots} = -\frac{3(1-a)(1-a^2)}{a^3}\log a.$$

and so on.

But if n is any fraction we can actually substitute that value; example:—
(d) when $n = \frac{1}{2}$, the result is

$$\int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)(1+a^4x^2)(1+a^6x^2)\cdots}$$
$$= \frac{\pi}{2}\frac{1-a}{1-a^2}\cdot\frac{1-a^3}{1-a^4}\cdot\frac{1-a^5}{1-a^6}\cdot\frac{1-a^7}{1-a^8}\cdots$$

which I have found to be equal to

$$\frac{\pi}{2(1+a+a^3+a^6+a^{10}+a^{15}+\cdots)}$$

where the general term in the denominator is

$$a^{n(n-1)/2}.$$

(vi) If a is positive, m less than 1 and n greater than $-m$, then

$$\int_0^\infty \frac{\Gamma(x+a)}{\Gamma(x+a+n+1)}\frac{dx}{x^m}$$
$$= \frac{\pi \csc \pi m}{\Gamma(n+1)}\left\{\frac{1}{a^m} - \frac{n}{1!}\frac{1}{(a+1)^m} + \frac{n(n-1)}{2!}\frac{1}{(a+2)^m} - \cdots\right\}$$

<u>Dem</u>. We know that

$$\int_0^1 z^{x+a-1}(1-z)^n dz = \frac{\Gamma(x+a)\Gamma(n+1)}{\Gamma(x+a+n+1)}.$$

Expanding the left side, we have

$$\int_0^1 z^{x+a-1}\left\{1 - \frac{n}{1!}z + \frac{n(n-1)}{2!}z^2 - \cdots\right\}dz$$
$$= \frac{1}{a+x} - \frac{n}{1!}\cdot\frac{1}{a+x+1} + \frac{n(n-1)}{2!}\cdot\frac{1}{a+2+x} - \cdots .$$

Therefore

$$\int_0^\infty \frac{\Gamma(x+a)\Gamma(n+1)}{x^m\Gamma(x+a+n+1)}dx$$
$$= \int_0^\infty x^{-m}\left\{\frac{1}{a+x} - \frac{n}{1!}\frac{1}{a+x+1} + \frac{n(n-1)}{2!}\frac{1}{a+2+x} - \cdots\right\}dx.$$

Now expanding the terms within the integral on the right side, we see that the above integral

$$= \int_0^\infty x^{-m}\left\{\phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \cdots\right\}dx.$$

$$\left[\text{where } \phi(t) = \Gamma(t+1)\left\{\frac{1}{a^{t+1}} - \frac{n}{1!}\frac{1}{(a+1)^{t+1}} + \frac{n(n-1)}{2!}\frac{1}{(a+2)^{t+1}} - \cdots\right\}\right]$$
$$= \Gamma(1-m)\phi(m-1) = \Gamma(1-m)\Gamma(m)\left\{\frac{1}{a^m} - \frac{n}{1!}\frac{1}{(a+1)^m} + \cdots\right\}$$
$$= \pi \csc \pi m\left\{\frac{1}{a^m} - \frac{n}{1!}\frac{1}{(a+1)^m} + \cdots\right\}$$

Hence
$$\int_0^\infty \frac{\Gamma(x+a)}{\Gamma(x+a+n+1)}\frac{dx}{x^m} = \frac{\pi \csc \pi m}{\Gamma(n+1)}\left\{\frac{1}{a^m} - \cdots\right\}.$$

6. B. Converse of A:—

Expansions of functions by using the theorem.

examples:——

(i) Required the expansion of $\left(\frac{2}{1+\sqrt{1+4x}}\right)^n$ in ascending powers of x.

Let the expansion of $\left(\frac{2}{1+\sqrt{1+4x}}\right)^n$ be
$$\phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \cdots.$$

Then
$$\int_0^\infty x^{p-1}\left\{\phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \cdots\right\}dx$$
$$= \Gamma(p)\phi(-p)$$
$$= \int_0^\infty x^{p-1}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx \qquad (\text{Substitute } x = y + y^2)$$
$$= \int_0^\infty y^{p-1}(1+y)^{p-n-1}(1+2y)dy \qquad \left(\text{Substitute } y = \frac{z}{1-z}\right)$$
$$= \int_0^1 z^{p-1}(1-z)^{n-2p-1}(1+z)dz$$
$$= \int_0^1 z^p(1-z)^{n-2p-1}dz + \int_0^1 z^{p-1}(1-z)^{n-2p-1}dz$$
$$= n\frac{\Gamma(p)\Gamma(n-2p)}{\Gamma(n-p+1)}.$$

Hence
$$\phi(-p) = n\frac{\Gamma(n-2p)}{\Gamma(n-p+1)} \qquad \text{or} \qquad \phi(p) = n\frac{\Gamma(n+2p)}{\Gamma(n+p+1)},$$

i.e.
$$\phi(0) = 1; \quad \phi(1) = n; \quad \phi(2) = n(n+3); \quad \phi(3) = n(n+4)(n+5);$$
$$\phi(4) = n(n+5)(n+6)(n+7); \quad \text{and so on.}$$

Hence
$$\left(\frac{2}{1+\sqrt{1+4x}}\right)^n = 1 - \frac{n}{1!}x + \frac{n(n+3)}{2!}x^2 - \frac{n(n+4)(n+5)}{3!}x^3 + \frac{n(n+5)(n+6)(n+7)}{4!}x^4 - \cdots.$$

(ii) Required the expansion of $\dfrac{1}{(x+\sqrt{1+x^2})^n}$ in ascending powers of x.

Let the expansion be

$$\phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \cdots,$$

then

$$\begin{aligned}
&\int_0^\infty x^{p-1}\left\{\phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \frac{x^3}{3!}\phi(3) + \cdots\right\} dx \\
=&\Gamma(p)\phi(-p) = \int_0^\infty \frac{x^{p-1}}{(x+\sqrt{1+x^2})^n} dx \qquad \left(\text{put } x+\sqrt{1+x^2} = \frac{1}{y}\right) \\
=&\frac{1}{2^p}\int_0^1 (1-y^2)^{p-1} y^{n-p+1}\left(1+\frac{1}{y^2}\right) dy; \qquad (\text{put } y = \sqrt{z}) \\
=&\frac{1}{2^{p+1}}\int_0^1 (1-z)^{p-1} z^{\frac{n-p}{2}}\left(1+\frac{1}{z}\right) dz \\
=&\frac{n}{2^{p+1}} \frac{\Gamma(p)\Gamma\left(\dfrac{n-p}{2}\right)}{\Gamma\left(\dfrac{n+p}{2}+1\right)}.
\end{aligned}$$

Hence

$$\phi(p) = n\, 2^{p-1} \frac{\Gamma\left(\dfrac{n+p}{2}\right)}{\Gamma\left(\dfrac{n-p}{2}+1\right)}$$

so that

$$\phi(0) = 1; \quad \phi(1) = n; \quad \phi(2) = n^2; \quad \phi(3) = n(n^2-1^2); \quad \phi(4) = n^2(n^2-2^2);$$
$$\phi(5) = n(n^2-1^2)(n^2-3^2); \quad \phi(6) = n^2(n^2-2^2)(n^2-4^2); \quad \text{etc.}$$

Hence

$$\begin{aligned}
\frac{1}{(x+\sqrt{1+x^2})^n} =&1 - \frac{n}{1!}x + \frac{n^2}{2!}x^2 - \frac{n(n^2-1^2)}{3!}x^3 \\
&+ \frac{n^2(n^2-2^2)}{4!}x^4 - \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 + \cdots.
\end{aligned}$$

<u>Cor</u>. Changing x to ix, and n to in, we have

$$e^{n\sin^{-1}x} = 1 + \frac{n}{1!}x + \frac{n^2}{2!}x^2 + \frac{n(n^2+1^2)}{3!}x^3 + \cdots.$$

(3) Given $\frac{\log x}{x} = -a$, required the expansion of x^n in ascending powers of a.

Let the expansion be

$$\phi(0) - \frac{a}{1!}\phi(1) + \frac{a^2}{2!}\phi(2) - \cdots,$$

then

$$\int_0^\infty a^{p-1}\left\{\phi(0)-\frac{a}{1!}\phi(1)+\frac{a^2}{2!}\phi(2)-\cdots\right\}da=\Gamma(p)\phi(-p)$$
$$=\int_0^\infty a^{p-1}x^n da=\int_0^1\left(-\frac{\log x}{x}\right)^{p-1}x^n\frac{1-\log x}{x^2}dx$$
$$=\int_0^\infty y^{p-1}(1+y)e^{-y(n-p)}dy,\qquad \text{by changing } x \text{ to } e^{-y}$$
$$=\frac{n\Gamma(p)}{(n-p)^{p+1}}.$$

Hence $\phi(p)=n(n+p)^{p-1}$ so that

$$\phi(0)=1;\qquad \phi(1)=n;\qquad \phi(2)=n(n+2);\qquad \phi(3)=n(n+3)^2;\ldots$$

Hence

$$x^n=1-\frac{n}{1!}a+\frac{n(n+2)}{2!}a^2-\frac{n(n+3)^2}{3!}a^3+\frac{n(n+4)^3}{4!}a^4-\frac{n(n+5)^4}{5!}a^5+\cdots.$$

(4) Given $aqx^p+x^q=1$; required the expansion of x^n in ascending powers of a.
Let the expansion be

$$\phi(0)-\frac{a}{1!}\phi(1)+\frac{a^2}{2!}\phi(2)-\cdots,$$

then

$$\int_0^\infty a^{r-1}\left\{\phi(0)-\frac{a}{1!}\phi(1)+\frac{a^2}{2!}\phi(2)-\frac{a^3}{3!}\phi(3)+\cdots\right\}da$$
$$=\Gamma(r)\phi(-r)=\int_0^\infty a^{r-1}x^n da\qquad\left(\text{substitute } \frac{1-x^q}{qx^p} \text{ for } a\right)$$
$$=\int_0^1 x^n\left(\frac{1-x^q}{qx^p}\right)^{r-1}\left\{\frac{p}{x}\cdot\frac{1-x^q}{qx^p}+x^{q-p-1}\right\}dx\qquad(\text{put } x=y^{1/q})$$
$$=\int_0^1\frac{1}{q}y^{\frac{n+1}{q}-1}\left(\frac{1-y}{qy^{p/q}}\right)^{r-1}\left(\frac{p}{y^{1/q}}\cdot\frac{1-y}{qy^{p/q}}+y^{\frac{q-p-1}{q}}\right)dy$$
$$=\frac{p}{q^{r+1}}\int_0^1 y^{\frac{n-pr}{q}-1}(1-y)^r dy+\frac{1}{q^r}\int_0^1 y^{\frac{n-pr}{q}}(1-y)^{r-1}dy$$
$$=\frac{n}{q^{r+1}}\frac{\Gamma(r)\Gamma\left(\frac{n-pr}{q}\right)}{\Gamma\left(\frac{n-pr}{q}+r+1\right)}.$$

Hence

$$\phi(r)=nq^{r-1}\frac{\Gamma\left(\frac{n+pr}{q}\right)}{\Gamma\left(\frac{n+pr}{q}-r+1\right)}\qquad(\text{so that } \phi(0)=1)$$
$$=nq^{r-1}\left\{\left(\frac{n+pr}{q}-1\right)\left(\frac{n+pr}{q}-2\right)\left(\frac{n+pr}{q}-3\right)\cdots \text{ to } r-1 \text{ factors}\right\}$$
$$=n(n+pr-q)(n+pr-2q)(n+pr-3q)\cdots \text{ to } r \text{ factors}.$$

Hence we see that: if $aqx^p + x^q = 1$, then

$$\begin{aligned} x^n = &1 - \frac{n}{1!}a + \frac{n(n+2p-q)}{2!}a^2 - \frac{n(n+3p-q)(n+3p-2q)}{3!}a^3 \\ &+ \frac{n(n+4p-q)(n+4p-2q)(n+4p-3q)}{4!}a^4 \\ &- \frac{n(n+5p-q)(n+5p-2q)(n+5p-3q)(n+5p-4q)}{5!}a^5 + \cdots \end{aligned}$$

S. Ramanujan
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As further examples of the Theorem in the previous progress report, let us have one on integrals and another on expansions.

(a) If r lies between 0 and 1

$$\int_0^\infty x^{r-1}\left\{1^m - 2^m\frac{x}{1!} + 3^m\frac{x^2}{2!} - 4^m\frac{x^3}{3!} + \cdots\right\}dx = \Gamma(r)(1-r)^m. \tag{1}$$

Let us verify the result for a few values of m by the known methods.

When $m = 1$, the integral reduces to

$$\int_0^\infty x^{r-1}e^{-x}(1-x)dx = \Gamma(r) - \Gamma(r+1) = \Gamma(r)(1-r).$$

When $m = 2$, the integral is

$$\int_0^\infty x^{r-1}e^{-x}(1-3x+x^2)dx = \Gamma(r) - 3\Gamma(r+1) + \Gamma(r+2) = \Gamma(r)(1-r)^2.$$

It is very difficult to prove the result by the known methods for non-integral values of m; e.g.,

$$\int_0^\infty \left(1^\pi - 2^\pi\frac{x^9}{1!} + 3^\pi\frac{x^{18}}{2!} - 4^\pi\frac{x^{27}}{3!} + \cdots\right)dx = \frac{2^{3\pi}}{3^{2\pi}}\Gamma\left(\frac{10}{9}\right).$$

Cor. Putting $r = \frac{1}{2}$, and changing x to x^2 in (1), we have

$$\int_0^\infty \left(1^m - 2^m\frac{x^2}{1!} + 3^m\frac{x^4}{2!} - 4^m\frac{x^6}{3!} + \cdots\right)dx = \frac{\sqrt{\pi}}{2^{m+1}}.$$

(b) Let us try to find the expansion of e^{ax} in ascending powers of $e^{-bx}\dfrac{\sin cx}{c}$.

First let us expand e^{-ax} in ascending powers of $\dfrac{\sinh cx}{c}e^{-bx}$ and then change a to $-a$ and c to ci.

Let

$$e^{-ax} = \phi(0) - \phi(1)\frac{y}{1!} + \phi(2)\frac{y^2}{2!} - \cdots$$

where

$$y = e^{-bx}\frac{\sinh cx}{c}.$$

Then, we have by the theorem in the previous progress report:

$$\begin{aligned}
&\int_0^\infty y^{n-1}\left\{\phi(0) - \frac{y}{1!}\phi(1) + \frac{y^2}{2!}\phi(2) - \cdots\right\}dy = \Gamma(n)\phi(-n)\\
=&\int_0^\infty \left(e^{-bx}\frac{e^{cx}-e^{-cx}}{2c}\right)^{n-1} e^{-ax}\cdot\frac{(c-b)e^{(c-b)x} + (c+b)e^{-(c+b)x}}{2c}dx\\
=&\frac{c-b}{(2c)^n}\int_0^\infty e^{-\{a+n(b-c)\}x}(1-e^{-2cx})^{n-1}dx\\
&+\frac{c+b}{(2c)^n}\int_0^\infty e^{-\{a+n(b-c)+2c\}x}(1-e^{-2cx})^{n-1}dx.
\end{aligned}$$

But we know that

$$\int_0^\infty e^{-px}(1-e^{-qx})^{n-1}dx = \int_0^1 z^{p/q-1}(1-z)^{n-1}\frac{dz}{q} = \frac{\Gamma(n)\Gamma(\frac{p}{q})}{q\Gamma(\frac{p}{q}+n)}.$$

Hence the previous integral

$$\begin{aligned}
=&\frac{c-b}{(2c)^{n+1}}\frac{\Gamma(n)\Gamma\left(\dfrac{a+n(b-c)}{2c}\right)}{\Gamma\left(n+\dfrac{a+n(b-c)}{2c}\right)} + \frac{c+b}{(2c)^{n+1}}\frac{\Gamma(n)\Gamma\left(\dfrac{a+n(b-c)+2c}{2c}\right)}{\Gamma\left(n+\dfrac{a+n(b-c)+2c}{2c}\right)}\\
=&\frac{a\Gamma(n)\Gamma\left(\dfrac{a+n(b-c)}{2c}\right)}{(2c)^{n+1}\Gamma\left(\dfrac{a+n(b+c)}{2c}+1\right)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(n) =&\frac{a(2c)^{n-1}\Gamma\left(\dfrac{a+n(c-b)}{2c}\right)}{\Gamma\left(\dfrac{a-n(b+c)}{2c}+1\right)}\\
=&a\left\{a-n(b+c)+2c\right\}\left\{a-n(b+c)+4c\right\}\left\{a-n(b+c)+6c\right\}\\
&\times\left\{a-n(b+c)+8c\right\}\cdots\left\{a-n(b-c)-2c\right\}
\end{aligned}$$

and also $\phi(0)=1$. Hence we have,

$$\begin{aligned}
e^{-ax} =&1-\frac{a}{1!}e^{-bx}\frac{\sinh cx}{c}+\frac{a(a-2b)}{2!}\left(e^{-bx}\frac{\sinh cx}{c}\right)^2\\
&-\frac{a\left\{(a-3b)^2-c^2\right\}}{3!}\left(e^{-bx}\frac{\sinh cx}{c}\right)^3\\
&+\frac{a(a-4b)\left\{(a-4b)^2-2^2c^2\right\}}{4!}\left(e^{-bx}\frac{\sinh cx}{c}\right)^4-\cdots.
\end{aligned}$$

Now changing a to $-a$ and c to ci on both the sides of the above result we have,

$$\begin{aligned}
e^{ax} =&1+\frac{a}{1!}e^{-bx}\frac{\sin cx}{c}+\frac{a(a+2b)}{2!}e^{-2bx}\left(\frac{\sin cx}{c}\right)^2\\
&+\frac{a\left\{(a+3b)^2+c^2\right\}}{3!}e^{-3bx}\left(\frac{\sin cx}{c}\right)^3\\
&+\frac{a(a+4b)\left\{(a+4b)^2+2^2c^2\right\}}{4!}e^{-4bx}\left(\frac{\sin cx}{c}\right)^4\\
&+\frac{\left\{a(a+5b)^2+c^2\right\}\left\{(a+5b)^2+3^2c^2\right\}}{5!}e^{-5bx}\left(\frac{\sin cx}{c}\right)^5\\
&+\frac{a(a+6b)\left\{(a+6b)^2+2^2c^2\right\}\left\{(a+6b)^2+4^2c^2\right\}}{6!}e^{-6bx}\left(\frac{\sin cx}{c}\right)^6+\cdots. \qquad \text{——(2)}
\end{aligned}$$

2. By comparing the n^{th} and the $(n+1)^{\text{th}}$ terms of the above expansion it can be proved that it is convergent if $\left|e^{-bx\frac{\sin cx}{c}}\right|$ is not greater than $\dfrac{e^{-\frac{b}{c}\tan^{-1}\frac{c}{b}}}{\sqrt{b^2+c^2}}$; but the least value of x which makes $e^{-bx}\dfrac{\sin cx}{c}$ equal to $\dfrac{e^{-\frac{b}{c}\tan^{-1}\frac{c}{b}}}{\sqrt{b^2+c^2}}$ is $\dfrac{1}{c}\tan^{-1}\dfrac{c}{b}$.

Therefore the expansion (2) is legitimate only when x is not greater than $\frac{1}{c}\tan^{-1}\frac{c}{b}$ though the series is convergent for many higher values of x. ——— (3)

As particular cases of (2), we have,

(a) When $b = 0$,

$$
\begin{aligned}
e^{ax} =& 1 + \frac{a}{1!}\frac{\sin cx}{c} + \frac{a^2}{2!}\left(\frac{\sin cx}{c}\right)^2 + \frac{a(a^2+c^2)}{3!}\left(\frac{\sin cx}{c}\right)^3 \\
&+ \frac{a^2(a^2+2^2c^2)}{4!}\left(\frac{\sin cx}{c}\right)^4 \\
&+ \frac{a(a^2+c^2)(a^2+3^2c^2)}{5!}\left(\frac{\sin cx}{c}\right)^5 + \cdots ,
\end{aligned}
$$

from which we have, as particular cases, the expansions of $\cos ax, \sin ax, x, x^2, x^3, \ldots$ in ascending powers of $\frac{\sin cx}{c}$.

(b) When $c = 0$, we have

$$
e^{ax} = 1 + \frac{a}{1!}xe^{-bx} + \frac{a(a+2b)}{2!}x^2e^{-2bx} + \frac{a(a+3b)^2}{3!}x^3e^{-3bx} + \cdots
$$

from which Abel's Theorem and all the expansions connected with it can be found.

(c) Changing a to ai and b to bi in (2) and separating the real and imaginary parts we have,

$$
\begin{aligned}
\cos ax =& 1 + \frac{a}{1!}\sin bx\,\frac{\sin cx}{c} - \frac{a(a+2b)}{2!}\cos 2bx\left(\frac{\sin cx}{c}\right)^2 \\
&- \frac{a\left\{(a+3b)^2 - c^2\right\}}{3!}\sin 3bx\left(\frac{\sin cx}{c}\right)^3 \\
&+ \frac{a(a+4b)\left\{(a+4b)^2 - 2^2c^2\right\}}{4!}\cos 4bx\left(\frac{\sin cx}{c}\right)^4 + \cdots .
\end{aligned}
$$

Similarly for $\sin ax$ also.

(d) Taking the coefficient of a on both sides in (2) we have,

$$
\begin{aligned}
x =& \frac{e^{-bx}}{1!}\frac{\sin cx}{c} + \frac{b}{1!}e^{-2bx}\left(\frac{\sin cx}{c}\right)^2 \\
&+ \frac{(3b)^2+c^2}{3!}e^{-3bx}\left(\frac{\sin cx}{c}\right)^3 + \frac{b\left\{(4b)^2+(2c)^2\right\}}{3!}e^{-4bx}\left(\frac{\sin cx}{c}\right)^4 \\
&+ \frac{\left\{(5b)^2+c^2\right\}\left\{(5b)^2+(3c)^2\right\}}{5!}e^{-5bx}\left(\frac{\sin cx}{c}\right)^5 \\
&+ \frac{b\left\{(6b)^2+(2c)^2\right\}\left\{(6b)^2+(4c)^2\right\}}{6!}e^{-6bx}\left(\frac{\sin cx}{c}\right)^6 \\
&+ \cdots \qquad \text{———} (4)
\end{aligned}
$$

Similarly we can expand $x^2, x^3, \ldots$ also in powers of $e^{-bx}\sin cx$.

3. Not only can a number of expansions be derived as particular cases from (2) but also the solution of many transcendental equations. As an example, let us consider the equation

$$
1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \frac{x^{12}}{12!} - \cdots = 0.
$$

This equation has an infinite number of real roots, as will be seen later, and all its imaginary roots can be found by multiplying all the real roots by ω and ω^2, the imaginary cube roots of unity.

Now,

$$1-\frac{x^3}{3!}+\frac{x^6}{6!}-\cdots=\frac{e^{-x}+e^{-x\omega}+e^{-x\omega^2}}{3}=\frac{e^{-x}+2e^{x/2}\cos\dfrac{x\sqrt{3}}{2}}{3}=0.$$

Hence

$$\cos\frac{x\sqrt{3}}{2}=-\frac{1}{2}e^{-3x/2} \tag{5}$$

We can easily see that there are an infinite number of intersections of the two curves $y=\cos\frac{x\sqrt{3}}{2}$ and $y=-\frac{1}{2}e^{-3x/2}$ and the values of x corresponding to these intersections are all positive and are very near to the roots of $\cos\frac{x\sqrt{3}}{2}$.

As the roots of $\cos\frac{x\sqrt{3}}{2}$ are $\frac{\pi n}{\sqrt{3}}$ where n is odd, let $x=\left(\frac{\pi n}{\sqrt{3}}-z\right)$, n being any odd positive integer. Substituting this value of x in the equation (5) we have,

$$\cos\left(\frac{\pi n}{2}-\frac{z\sqrt{3}}{2}\right)=-\frac{1}{2}e^{-\pi n\sqrt{3}/2+3z/2}$$

that is

$$(-1)^{(n-1)/2}\sin\frac{z\sqrt{3}}{2}=-\frac{1}{2}e^{-\pi n\sqrt{3}/2+3z/2}.$$

Let h stand for $e^{-\pi n\sqrt{3}/2}$. Then we see that

$$(-1)^{(n+1)/2}\frac{h}{2}=e^{-3z/2}\sin\frac{z\sqrt{3}}{2}.$$

Now expand z in ascending powers of $e^{-3z/2}\sin\frac{z\sqrt{3}}{2}$ by using (4) and then substitute $(-1)^{(n+1)/2}\frac{h}{2}$ for $e^{-3z/2}\sin\frac{z\sqrt{3}}{2}$ in the expansion.

Thus we see that

$$z=\frac{(-1)^{(n+1)/2}}{\sqrt{3}}\cdot\frac{h}{1!}+\frac{h^2}{2\cdot 1!}+\frac{(-1)^{(n+1)/2}}{\sqrt{3}}\cdot\frac{7h^3}{3!}+\cdots.$$

Hence

$$\begin{aligned}x=&\frac{\pi n}{\sqrt{3}}-z=\frac{\pi n}{\sqrt{3}}-\frac{1}{2}\left\{\frac{h^2}{1!}+\frac{13}{3!}h^4+\frac{28\cdot 31}{5!}h^6\right.\\&\left.+\frac{49\cdot 52\cdot 57}{7!}h^8+\frac{76\cdot 79\cdot 84\cdot 91}{9!}h^{10}+\cdots\right\}\\&+\frac{(-1)^{(n-1)/2}}{\sqrt{3}}\left\{\frac{h}{1!}+\frac{7}{3!}h^3+\frac{19\cdot 21}{5!}h^5+\frac{37\cdot 39\cdot 43}{7!}h^7\right.\\&\left.+\frac{61\cdot 63\cdot 67\cdot 73}{9!}h^9+\frac{91\cdot 93\cdot 97\cdot 103\cdot 111}{11!}h^{11}+\cdots\right\}\end{aligned}$$

where $h=e^{-\pi n\sqrt{3}/2}$ and n any odd positive integer.

The series is convergent if $h<e^{-\pi\sqrt{3}/6}$ by (3) and fortunately the first value of h itself is $e^{-\pi\sqrt{3}/2}$ which is by far less than $e^{-\pi\sqrt{3}/6}$. When n becomes greater and greater, h rapidly diminishes and consequently the series becomes more and more rapidly convergent.

4. Let us write for the sake of simplicity the Theorem in the first progress report viz.

$$\int_0^\infty x^{n-1}\left\{\phi(0)-\frac{x}{1!}\phi(1)+\frac{x^2}{2!}\phi(2)-\cdots\right\}dx=\Gamma(n)\phi(-n)$$

in the form

$$\int_0^\infty x^{n-1}\left\{\mu_0-\mu_1\frac{x}{1!}+\mu_2\frac{x^2}{2!}-\cdots\right\}dx=\Gamma(n)\mu_{-n}.$$

Theorem II. If $\phi(0)=\psi(0)$ and $\phi(\infty)=\psi(\infty)$, then

$$\int_0^\infty \frac{\phi(ax)-\psi(bx)}{x}dx=\{\phi(0)-\phi(\infty)\}\left\{\log\frac{b}{a}+\left[\frac{d\log\dfrac{\text{coeff of } x^n \text{ in } \psi(x)}{\text{coeff of } x^n \text{ in } \phi(x)}}{dn}\right]_{n=0}\right\}.$$

<u>Dem</u>. By Theorem I we have,

$$\int_0^\infty x^{n-1}\left\{u_0-u_1\frac{ax}{1!}+u_2\frac{a^2x^2}{2!}-\cdots\right\}dx=\Gamma(n)a^{-n}u_{-n}$$

and

$$\int_0^\infty x^{n-1}\left\{v_0-v_1\frac{bx}{1!}+v_2\frac{b^2x^2}{2!}-\cdots\right\}dx=\Gamma(n)b^{-n}v_{-n}.$$

Let

$$u_0-u_1\frac{x}{1!}+u_2\frac{x^2}{2!}-\cdots$$

be denoted by $\phi_1(x)$ and

$$v_0-v_1\frac{x}{1!}+v_2\frac{x^2}{2!}-\cdots$$

be denoted by $\psi_1(x)$ then we see that

$$\begin{aligned}\int_0^\infty x^{n-1}\left\{\phi_1(ax)-\psi_1(bx)\right\}dx =&\Gamma(n)\left(a^{-n}u_{-n}-b^{-n}v_{-n}\right)\\ =&\Gamma(n+1)\frac{a^{-n}u_{-n}-b^{-n}v_{-n}}{n}.\end{aligned}$$

Now suppose that $\phi_1(0)=\psi_1(0)$ so that $u_0=v_0$ and let n become indefinitely small and ultimately vanish, then we have,

$$\begin{aligned}&\int_0^\infty \frac{\{\phi_1(ax)-\psi_1(bx)\}}{x}dx=\left[\frac{a^{-n}u_{-n}-b^{-n}v_{-n}}{n}\right]_{n=0}\\
&=\left[\frac{b^nv_n-a^nu_n}{n}\right]_{n=0}, \qquad \text{by changing } n \text{ to } -n\\
&=\left[\frac{d(b^nv_n-a^nu_n)}{dn}\right]_{n=0}, \qquad \left(\text{since the above is of the form } \frac{0}{0}\right)\\
&=\left[v_nb^n\log b-u_na^n\log a+b^n\frac{dv_n}{dn}-a^n\frac{du_n}{dn}\right]_{n=0}\\
&=v_0\log b-u_0\log a+\left[\frac{dv_n}{dn}-\frac{du_n}{dn}\right]_{n=0}\\
&=v_0\log b-u_0\log a+\left[v_n\frac{d\log v_n}{dn}-u_n\frac{d\log u_n}{dn}\right]_{n=0}\\
&=v_0\log b-u_0\log a+v_0\left[\frac{d\log v_n}{dn}\right]_{n=0}-u_0\left[\frac{d\log u_n}{dn}\right]_{n=0}.\end{aligned}$$

But $u_0 = v_0 = \phi_1(0)$. Hence the above reduces to

$$\phi_1(0)\left\{\log\frac{b}{a} + \left[\frac{d\log\dfrac{v_n}{u_n}}{dn}\right]_{n=0}\right\}$$

But

$$\frac{v_n}{u_n} = \frac{\text{coefft of } x^n \text{ in } \psi_1(x)}{\text{coefft of } x^n \text{ in } \phi_1(x)}.$$

Hence we have: if $\phi_1(0) = \psi_1(0)$, then

$$\int_0^\infty \frac{\phi_1(ax)-\psi_1(bx)}{x}dx = \phi_1(0)\left\{\log\frac{b}{a} + \left[\frac{d\log\dfrac{\text{coefft of } x^n \text{ in } \psi_1(x)}{\text{coefft of } x^n \text{ in } \phi_1(x)}}{dn}\right]_{n=0}\right\}.$$

But according to the 4th condition of validity of Theorem I given in the first Progress Report $\phi_1(x)$ and $\psi_1(x)$ should vanish when x becomes infinitely great.

Now let $\phi(x)$ and $\psi(x)$ be two functions which do not necessarily vanish when $x = \infty$ but let them approach the same finite limit when x becomes infinite so that $\phi(\infty) = \psi(\infty)$ and let $\phi(0)$ be also equal to $\psi(0)$.

Then we see that,

$$\int_0^\infty \frac{\phi(ax)-\psi(bx)}{x}dx = \int_0^\infty \frac{\{\phi(ax)-\phi(\infty)\}-\{\psi(bx)-\psi(\infty)\}}{x}dx$$
$$= \{\phi(0)-\phi(\infty)\}\left\{\log\frac{b}{a} + \left[\frac{d\log\dfrac{\text{coefft of } x^n \text{ in } \psi(x)}{\text{coefft of } x^n \text{ in } \phi(x)}}{dn}\right]_{n=0}\right\}$$

since $\{\phi(ax)-\phi(\infty)\}$ and $\{\psi(bx)-\psi(\infty)\}$ vanish when $x = \infty$ and also $\phi(0)-\phi(\infty) = \psi(0)-\psi(\infty)$.
Q.E.D.

5. Examples of Theorem II.

(1)

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = \{\phi(0)-\phi(\infty)\}\log\frac{b}{a}.$$

This result, which is known as "Frullani's Theorem" (given in Williamson's Integral Calculus) can very easily be got from Theorem II by supposing ϕ and ψ to be the same function.

(2)

$$\int_0^\infty \frac{(1+ax)^{-p}-(1+bx)^{-q}}{x}dx$$
$$= \log\frac{b}{a} + \left[\frac{d\log\dfrac{\Gamma(q+n)\Gamma(p)}{\Gamma(p+n)\Gamma(q)}}{dn}\right]_{n=0}$$

which reduces to

$$\log\frac{b}{a} + \frac{1}{p} - \frac{1}{q} + \frac{1}{p+1} - \frac{1}{q+1} + \frac{1}{p+2} - \frac{1}{q+2} + \cdots.$$

In particular cases we have,

$$\begin{aligned}
\int_0^\infty \frac{(1+ax)^{-5}-(1+bx)^{-7}}{x}dx =& \frac{11}{30}+\log\frac{b}{a}.\\
\int_0^\infty \frac{(1+ax)^{-p}-(1+bx)^{-p-1}}{x}dx =& \frac{1}{p}+\log\frac{b}{a}.\\
\int_0^\infty \frac{(1+ax)^{-p-1}-(1+bx)^{-p-5}}{x}dx =& \frac{2(2p+5)(p^2+5p+5)}{(p+1)(p+2)(p+3)(p+4)}+\log\frac{b}{a}.\\
\int_0^\infty \frac{(1+ax)^{-1/4}-(1+bx)^{-3/4}}{x}dx =& \pi+\log\frac{b}{a},
\end{aligned}$$

and so on.

(3)

$$\begin{aligned}
&\int_0^\infty \frac{(1+ax)^{-p}-\left(\dfrac{2}{1+\sqrt{1+4bx}}\right)^q}{x}dx\\
&=\log\frac{b}{a}+\left[\frac{d\log\dfrac{q\Gamma(q+2n)\Gamma(p)}{\Gamma(q+n+1)\Gamma(p+n)}}{dn}\right]_{n=0}\\
&=\log\frac{b}{a}-\frac{1}{q}+\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{p+1}-\frac{1}{q+1}+\cdots\right).
\end{aligned}$$

<u>Cor</u>.

$$\int_0^\infty \frac{(1+ax)^{-p}-\left(\dfrac{2}{1+\sqrt{1+4bx}}\right)^p}{x}dx=\log\frac{b}{a}-\frac{1}{p}.$$

(4)

$$\begin{aligned}
&\int_0^\infty \frac{(1+ax)^{-p}-(bx+\sqrt{1+b^2x^2})^{-2q}}{x}dx\\
&=\log\frac{b}{a}+\left[\frac{d\log\dfrac{2^nq\Gamma(q+\frac{n}{2})\Gamma(p)}{\Gamma(q+1-\frac{n}{2})\Gamma(p+n)}}{dn}\right]_{n=0}\\
&=\frac{1}{2}\psi(q)+\frac{1}{2}\psi(q+1)-\psi(p)+\log 2+\log\frac{b}{a}\\
&=\log\frac{2b}{a}+\frac{1}{2q}+\frac{1}{p}-\frac{1}{q}+\frac{1}{p+1}-\frac{1}{q+1}+\cdots.
\end{aligned}$$

<u>Cor</u>.

$$\int_0^\infty \frac{(1+ax)^{-p}-(bx+\sqrt{1+b^2x^2})^{-2p}}{x}dx=\log\frac{2b}{a}+\frac{1}{2p}.$$

Similarly many other Integrals, such as

$$\int_0^\infty \frac{e^{-ax}-(1+bx)^{-p}}{x}dx,\qquad \int_0^\infty \frac{e^{-ax}\cos^5 bx-\left(\dfrac{2}{1+\sqrt{1+4cx}}\right)^p}{x}dx$$

etc. can be found from Theorem II.
(5) If

$$p\phi(0)+q\psi(0)+rf(0)+s\chi(0)+\cdots=0,$$

and

$$p\phi(\infty)+q\psi(\infty)+rf(\infty)+s\chi(\infty)+\cdots=0,$$

then

$$\int_0^\infty \frac{p\phi(kx)+q\psi(\ell x)+rf(mx)+s\chi(nx)+\cdots}{x}dx$$

can be found from Theorem II.

6. Corollaries to Theorem I. (1)

$$\int_0^\infty x^{n-1}\left(u_0-u_2\frac{x^2}{2!}+u_4\frac{x^4}{4!}-\cdots\right)dx=\Gamma(n)u_{-n}\cos\frac{\pi n}{2},$$

n being any positive quantity.

<u>Dem</u>. By Theorem I we have,

$$\int_0^\infty x^{n-1}\left(u_0-u_2\frac{x}{1!}+u_4\frac{x^2}{2!}-\cdots\right)dx=\Gamma(n)u_{-2n}.$$

Changing x to x^2 and n to $\frac{n}{2}$, we have,

$$\int_0^\infty x^{n-1}\left(u_0-u_2\frac{x^2}{1!}+u_4\frac{x^4}{2!}-\cdots\right)dx=\frac{1}{2}\Gamma\left(\frac{n}{2}\right)u_{-n}.$$

Now changing u_n to $u_n\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}$ we have

$$\begin{aligned}\int_0^\infty x^{n-1}\left(u_0-u_2\frac{x^2}{2!}+u_4\frac{x^4}{4!}-\cdots\right)dx=&\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(-\frac{n}{2}\right)}{2\Gamma(-n)}u_{-n}\\=&\Gamma(n)u_{-n}\cos\frac{\pi n}{2}.\end{aligned}$$

(2) If n lies between 0 and 1,

$$\int_0^\infty x^{n-1}\left(u_0-u_1x+u_2x^2-u_3x^3+\cdots\right)dx=\frac{\pi u_{-n}}{\sin\pi n}.$$

<u>Dem</u>. Changing u_n to $u_n\Gamma(n+1)$ in Theorem I we can get the result.

(3) If n lies between 0 and 2,

$$\int_0^\infty x^{n-1}\left(u_0-u_2x^2+u_4x^4-u_6x^6+\cdots\right)dx=\frac{\pi u_{-n}}{2\sin\frac{\pi n}{2}}.$$

<u>Dem</u>. Change x to x^2, n to $\frac{n}{2}$, and u_t to u_{2t} in Cor. 2.

Similarly by changing x to x^2 and n to $\frac{n}{r}$, we have

(4) If n lies between 0 and r,

$$\int_0^\infty x^{n-1}\left(u_0-u_rx^r+u_{2r}x^{2r}-u_{3r}x^{3r}+\cdots\right)dx=\frac{\pi u_{-n}}{r\sin\frac{\pi n}{r}}.$$

(5)

$$\int_0^\infty \left(a_0 - a_1\frac{x}{1!} + a_2\frac{x^2}{2!} - a_3\frac{x^3}{3!} + \cdots\right)\cos nx\,dx$$
$$= a_{-1} - n^2a_{-3} + n^4a_{-5} - n^6a_{-7} + \cdots .$$

Dem. Expand $\cos nx$ and integrate separately.

(6) In a similar manner we have,

$$\int_0^\infty \left(a_0 - a_1\frac{x}{1!} + a_2\frac{x^2}{2!} - \cdots\right)\left(b_0 - b_2\frac{n^2x^2}{2!} + b_4\frac{n^4x^4}{4!} - \cdots\right)dx$$
$$=b_0a_{-1} - n^2b_2a_{-3} + n^4b_4a_{-5} - n^6b_6a_{-7} + \cdots .$$

(7)

$$\int_0^\infty \left(a_0 - a_2x^2 + a_4x^4 - a_6x^6 + \cdots\right)\cos nx\,dx$$
$$=\frac{\pi}{2}\left(a_{-1} - \frac{n}{1!}a_{-2} + \frac{n^2}{2!}a_{-3} - \frac{n^3}{3!}a_{-4} + \cdots\right).$$

Dem. Here we should not expand $\cos nx$ and integrate separately as in Cor. (5) for the integral

$$\int_0^\infty x^{n-1}\left(a_0 - a_2x^2 + a_4x^4 - \cdots\right)dx = \frac{\pi a_{-n}}{2\sin\frac{\pi n}{2}}$$

is true only when n lies between 0 and 2. Now

$$\left(a_0 - a_2x^2 + a_4x^4 - \cdots\right)\cos nx$$
$$= a_0 - x^2\left(a_2 + a_0\frac{n^2}{2!}\right) + x^4\left(a_4 + a_2\frac{n^2}{2!} + a_0\frac{n^4}{4!}\right) - \cdots .$$

Let

$$u_r = a_r + a_{r-2}\frac{n^2}{2!} + a_{r-4}\frac{n^4}{4!} + \cdots \quad \text{to infinity}$$
$$-a_{-2}\frac{n^{r+2}}{(r+2)!} - a_{-4}\frac{n^{r+4}}{(r+4)!} - \cdots .$$

Then we see that,

$$\int_0^\infty \left(a_0 - a_2x^2 + a_4x^4 - a_6x^6 + \cdots\right)\cos nx\,dx$$
$$= \int_0^\infty \left(u_0 - u_2x^2 + u_4x^4 - u_6x^6 + \cdots\right)dx = \frac{\pi}{2}u_{-1} \quad \text{by Cor. (3)}$$
$$= \frac{\pi}{2}\left(a_{-1} - \frac{n}{1!}a_{-2} + \frac{n^2}{2!}a_{-3} - \frac{n^3}{3!}a_{-4} + \cdots\right).$$

(8) Similarly we can prove that

$$\int_0^\infty \left(a_0 - a_2x^2 + a_4x^4 - \cdots\right)\left(b_0 - b_2\frac{n^2x^2}{2!} + b_4\frac{n^4x^4}{4!} - \cdots\right)dx$$
$$= \frac{\pi}{2}\left(b_0a_{-1} - \frac{n}{1!}b_1a_{-2} + \frac{n^2}{2!}b_2a_{-3} - \cdots\right).$$

7. Theorem III If

$$\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$$

then

$$\int_0^\infty \psi(x)\cos nx\,dx = \frac{\pi}{2}\phi(n),$$

and also: if

$$\int_0^\infty \phi(x)\sin nx\,dx = \psi(n)$$

then

$$\int_0^\infty \psi(x)\sin nx\,dx = \frac{\pi}{2}\phi(n).$$

Dem. By Cor. (5) to Theorem I, we have,

$$\int_0^\infty \left(a_0 - a_1\frac{x}{1!} + a_2\frac{x^2}{2!} - \cdots\right)\cos nx\,dx = a_{-1} - n^2a_{-3} + n^4a_{-5} - n^6a_{-7} + \cdots$$

and by Cor. (7) we have

$$\int_0^\infty \left(a_{-1} - x^2a_{-3} + x^4a_{-5} - \cdots\right)\cos nx\,dx = \frac{\pi}{2}\left(a_0 - a_1\frac{n}{1!} + a_2\frac{n^2}{2!} - \cdots\right).$$

Let

$$a_0 - a_1\frac{x}{1!} + a_2\frac{x^2}{2!} - \cdots$$

be denoted by $\phi(x)$ and

$$a_{-1} - n^2a_{-3} + n^4a_{-5} - \cdots$$

by $\psi(n)$. Then we evidently see that

$$\int_0^\infty \psi(x)\cos nx\,dx = \frac{\pi}{2}\phi(n).$$

Thus the first part of the theorem is proved.

In a similar manner the 2^{nd} part also can be proved.

Q. E. D.

Examples:—-

(1)

$$\int_0^\infty e^{-x}\cos nx\,dx = \frac{1}{1+n^2};$$

Hence

$$\int_0^\infty \frac{\cos nx}{1+x^2}dx = \frac{\pi}{2}e^{-n}.$$

(2)

$$\int_0^\infty x^{p-1}e^{-ax}\cos nx\,dx = \frac{\Gamma(p)\cos\left(p\,\tan^{-1}\frac{n}{a}\right)}{(a^2+n^2)^{p/2}}.$$

Hence

$$\Gamma(p)\int_0^\infty \frac{\cos\left(p\,\tan^{-1}\frac{x}{a}\right)}{(a^2+x^2)^{p/2}}\cos nx\,dx = \frac{\pi}{2}n^{p-1}e^{-an}.$$

(3)

$$\int_0^\infty x^{p-1}e^{-ax}\sin nx\,dx = \frac{\Gamma(p)\sin\left(p\,\tan^{-1}\frac{n}{a}\right)}{(a^2+n^2)^{p/2}}.$$

Hence

$$\Gamma(p)\int_0^\infty \frac{\sin\left(p\,\tan^{-1}\frac{x}{a}\right)}{(a^2+x^2)^{p/2}}\sin nx\,dx = \frac{\pi}{2}n^{p-1}e^{-an}.$$

(4)

$$\int_0^\infty e^{-ax^2}\cos 2nx\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}e^{-n^2/a}.$$

Changing a to $a+bi$, separating the real and imaginary parts and then putting $a=0$, we have

$$\int_0^\infty \cos 2nx\,\cos bx^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{b}}\cos\left(\frac{\pi}{4}-\frac{n^2}{b}\right)$$

and

$$\int_0^\infty \cos 2nx\,\sin bx^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{b}}\sin\left(\frac{\pi}{4}-\frac{n^2}{b}\right).$$

Changing x to $\sqrt{x}$ we have

$$\int_0^\infty \frac{\cos 2\sqrt{ax}\,\cos nx}{\sqrt{x}}dx = \sqrt{\frac{\pi}{n}}\cos\left(\frac{\pi}{4}-\frac{a}{n}\right)$$

and

$$\int_0^\infty \frac{\cos 2\sqrt{ax}\,\sin nx}{\sqrt{x}}dx = \sqrt{\frac{\pi}{n}}\sin\left(\frac{\pi}{4}-\frac{a}{n}\right).$$

Hence by Theorem III, we have

$$\int_0^\infty \frac{\cos\left(\frac{\pi}{4}-\frac{a}{x}\right)}{\sqrt{x}}\cos nx\,dx = \frac{\sqrt{\pi}}{2\sqrt{n}}\cos(2\sqrt{an})$$

and

$$\int_0^\infty \frac{\sin\left(\frac{\pi}{4}-\frac{a}{x}\right)}{\sqrt{x}}\sin nx\,dx = \frac{\sqrt{\pi}}{2\sqrt{n}}\cos(2\sqrt{an}).$$

S. Ramanujan
7^{th} Nov. 1913.

Madras
9^{th} March 1914

From

S. Ramanujan
Scholarship holder in Mathematics

To The Board of Studies in Mathematics

through The Registrar of the University of Madras

Gentlemen,

In continuation of my second progress report, dated 7^{th} Nov. 1913, I beg to submit herewith my third progress report for the period ended the 28^{th} Feb. 1914.

I beg to remain
gentlemen,
Your obedient servant
S. Ramanujan

1. The validity of the Theorem given in the first report, viz.,

$$\int_0^\infty x^{n-1}\left(u_0 - u_1\frac{x}{1!} + u_2\frac{x^2}{2!} - \cdots\right)dx = \Gamma(n)u_{-n}$$

depends upon

(a) The nature of n

(b) The nature of the function (of x) viz.,

$$u_0 - u_1\frac{x}{1!} + u_2\frac{x^2}{2!} - u_3\frac{x^3}{3!} + \cdots$$

(c) The nature of the function (of n) u_n.

(a) and (b) have already been discussed in the 1^{st} report, (c) is the most important of all as the theorems in the 2^{nd} report as well as many other results are derived from the above theorem.

Since n can be fractional u_n should be a continuous function of n for all values of n.

u_n should not be infinite for all finite positive integral values of n.

2. In spite of all these conditions there is another difficulty, viz., that of ascertaining u_n for fractional values of n from the values of u_n for positive integral values of n only.

But an infinite no. of functions may have the same values $u_0, u_1, u_2, u_3, \ldots$ as shown below.

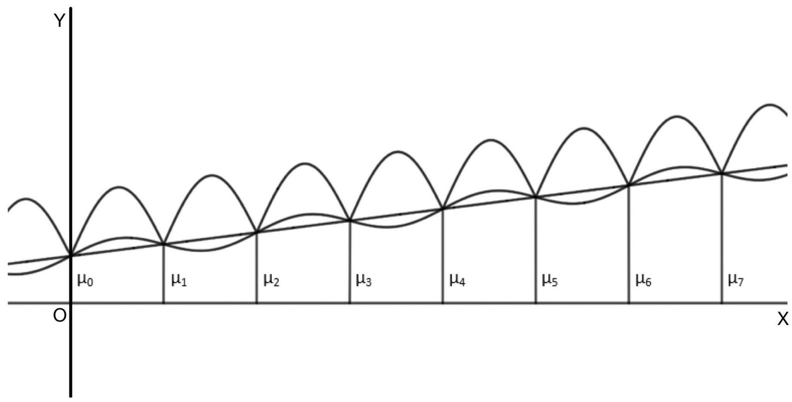

Figure 1: Graphical Illustration

Now let us try to ascertain which of these functions we should take for our theorem as the value of the integral cannot be indeterminate.

First let us see what becomes of u_n when $u_0 = 0, u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0, \ldots$. Here we evidently see that the value of the integral is always zero for all values of n integral or fractional. Therefore u_n in this case should always be zero and not a function of n vanishing only for the values $0, 1, 2, 3, 4, \ldots$ of n, such as the functions $f(n) \sin \pi n$, $\frac{f(n)}{\Gamma(-n)}$, etc. where $f(n)$ is any arbitrary function not becoming infinite for positive integral values of n. Hence we infer that u_n in the integral we have taken should not contain any function of n vanishing for positive, integral values of n; as an example $n + \sin \pi n$ should not be supposed to be u_n as it contains a function $\sin \pi n$ which vanishes for $0, 1, 2, 3, 4, \ldots$ of n.

3. There can be only one such function having the values $u_0, u_1, u_2, u_3, \ldots$ corresponding to the values of $0, 1, 2, 3, \ldots$ of n.

For, if possible let there be two such functions u_n and v_n different from each other, and yet having the same values $u_0, u_1, u_2, u_3, \ldots$ when $n = 0, 1, 2, 3, \ldots$. Then we see that $v_n = u_n + (v_n - u_n)$ of which $v_n - u_n = 0$ when $n = 0, 1, 2, 3, 4, \ldots$ which contradicts our hypothesis. Hence we conclude there cannot be more than one such function.

Thus we see that, whenever we want such a unique function passing through $u_0, u_1, u_2, u_3, \ldots$ first we should find a continuous function passing through $u_0, u_1, u_2, u_3, \ldots$ and then remove from it functions vanishing for positive integral values of n (if any) and the function left is the unique function required.

Examples:— If $u_0 = 0, u_1 = 1, u_2 = 4, u_3 = 9$, etc., the squares of natural numbers, then the unique u_n is n^2 and not any other function such as $n^2 + \sin^2 \pi n$.

If $u_0, u_1, u_2, \ldots$ are $1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 7$, etc., then

$$u_n = \frac{n + 1 + \cos \frac{\pi n}{2}}{2}.$$

The unique u_n is not only useful in the present theorem on integrals but also in all fractional order of functions, differentiations, etc.

Fractional Order of Functions

4. Let $FF(x)$ be denoted by $F^2(x)$, $FFF(x)$ by $F^3(x)$ and so on. Then $F^n(x)$ for all values of n is the unique function passing through

$$F^0(x), \quad F^1(x), \quad F^2(x), \quad F^3(x), \ldots; \quad \text{e.g.,}$$

let $F(x) = ax^p$, then we see that

$$F^0(x) = x; \quad F^1(x) = ax^p; \quad F^2(x) = a^{p+1} x^{p^2}; \quad F^3(x) = a^{p^2+p+1} x^{p^3};$$

and so on.

Hence

$$F^n(x) = a^{\frac{p^n - 1}{p-1}} x^{p^n}$$

which is unique, so that

$$F^{1/2}(x) = a^{\frac{1}{1+\sqrt{p}}} x^{\sqrt{p}}.$$

5. The meaning of $F^m(x)$ when m is not an integer can be got from the identity $F^m(F^n(x)) = F^{m+n}(x)$. Since $F^{1/2} F^{1/2}(x) = F(x)$, we see that $F^{1/2}(x)$ is a function operated upon by the same function producing the original function $F(x)$; e.g., let

$$\psi(x) = a^{\frac{1}{1+\sqrt{p}}} x^{\sqrt{p}},$$

then

$$\psi\psi(x) = a^{\frac{1}{1+\sqrt{p}}}\left(a^{\frac{1}{1+\sqrt{p}}}x^{\sqrt{p}}\right)^{\sqrt{p}} = ax^p.$$

Hence if $F(x) = ax^p$, then

$$F^{1/2}(x) = a^{\frac{1}{1+\sqrt{p}}}x^{\sqrt{p}}.$$

Similarly, $F^{p/q}(x)$ may be defined and $F^n(x)$ may be extended to irrational values of n such as

$$F^{\sqrt{2}}(x); \quad F^{\log 2}(x); \quad F^e(x); \quad F^\pi(x); \quad \text{etc.}$$

6. Theorem.
If

$$F^m(x) = \phi(x)$$

and

$$F^n(x) = \psi(x),$$

and

$$f^m(x) = \chi^{-1}\phi\{\chi(x)\},$$

then

$$f^n(x) = \chi^{-1}\psi\{\chi(x)\}.$$

Dem. Since

$$f^m(x) = \chi^{-1}\phi\{\chi(x)\},$$

we see that

$$f^{2m}(x) = \chi^{-1}\phi^2\{\chi(x)\}$$

and generally

$$f^n(x) = \chi^{-1}\phi^{n/m}\{\chi(x)\}.$$

But

$$\phi^{n/m}\{\chi(x)\} = F^n\{\chi(x)\} = \psi\{\chi(x)\}.$$

Hence

$$f^n(x) = \chi^{-1}\psi\{\chi(x)\}.$$

Q.E.D.

Examples.
(a) If

$$F(x) = \sqrt[m]{a(x^m+1)^p - 1},$$

then

$$F^n(x) = \sqrt[m]{a^{\frac{p^n-1}{p-1}}(x^m+1)^{p^n} - 1}.$$

Dem. Let $\chi(x) = x^m + 1$, so that $\chi^{-1}(x) = \sqrt[m]{x-1}$ and $\phi(x) = ax^p$, then we see that

$$F(x) = \chi^{-1}\phi\{\chi(x)\}$$

so that

$$\begin{aligned} F^n(x) =& \chi^{-1}\phi^n\{\chi(x)\} \\ =& \chi^{-1}\left[a^{\frac{p^n-1}{p-1}}\{\chi(x)\}^{p^n}\right] \qquad \text{by the above theorem} \\ =& \sqrt[m]{a^{\frac{p^n-1}{p-1}}(x^m+1)^{p^n}-1}. \end{aligned}$$

(b) If $F(x) = x^2 - 2$, then

$$\begin{aligned} F^{1/2}(x) =& \left(\frac{x+\sqrt{x^2-4}}{2}\right)^{\sqrt{2}} + \left(\frac{x-\sqrt{x^2-4}}{2}\right)^{\sqrt{2}}, \\ F^{\frac{\log 3}{\log 2}}(x) =& x^3 - 3x, \end{aligned}$$

and

$$F^{\frac{\log 5}{\log 2}} = x^5 - 5x^3 + 5x.$$

Dem. Let $x = y + \dfrac{1}{y}$, then we see that

$$F^0(x) = y + \frac{1}{y}; \quad F^1(x) = y^2 + \frac{1}{y^2}; \quad F^2(x) = y^4 + \frac{1}{y^4}; \quad \text{etc.}$$

Hence

$$F^n(x) = y^{2^n} + y^{-2^n} = \left(\frac{x+\sqrt{x^2-4}}{2}\right)^{2^n} + \left(\frac{x-\sqrt{x^2-4}}{2}\right)^{-2^n},$$

so that $F^{1/2}(x)$ etc. $=$ the required result.
(c) Find $F^n(x)$ when

$$\begin{aligned} \text{(i)} \quad F(x) =& x^2 + 2x, \\ \text{(ii)} \quad F(x) =& x^2 - 2x, \\ \text{(iii)} \quad F(x) =& x^2 + 4x. \end{aligned}$$

(i) Let $\phi(x) = x^2$ and $\chi(x) = x + 1$ so that $\chi^{-1}(x) = x - 1$ then we see that

$$F(x) = \chi^{-1}\phi\{\chi(x)\}.$$

Hence we have

$$F^n(x) = \chi^{-1}\phi^n\{\chi(x)\} = (x+1)^{2^n} - 1.$$

(ii) Let $\phi(x) = x^2 - 2$, and $\chi(x) = x - 1$, so that $\chi^{-1}(x) = x + 1$, then we see that

$$F(x) = \chi^{-1}\phi\{\chi(x)\}.$$

Hence we have

$$\begin{aligned} F^n(x) =& \chi^{-1}\phi^n\{\chi(x)\} \quad \text{which by (b)} \\ =& 1 + \left(\frac{x-1+\sqrt{x^2-2x-3}}{2}\right)^{2^n} + \left(\frac{x-1-\sqrt{x^2-2x-3}}{2}\right)^{-2^n}. \end{aligned}$$

(iii) Let $\phi(x) = x^2 - 2$, and $\chi(x) = x + 2$ so that $\chi^{-1}(x) = x - 2$, then we see that

$$F(x) = \chi^{-1}\phi\{\chi(x)\}.$$

Hence we have

$$\begin{aligned} F^n(x) =& \chi^{-1}\phi^n\{\chi(x)\} \quad \text{which by (b)} \\ =& \left\{\left(\frac{\sqrt{x+4}+\sqrt{x}}{2}\right)^{2^n} - \left(\frac{\sqrt{x+4}-\sqrt{x}}{2}\right)^{-2^n}\right\}^2. \end{aligned}$$

(d) If

$$1 + \sqrt{F^{\log 2}(x)} = \sqrt{\frac{1 - F^{\log 2}(x)}{1 - x}}$$

then

$$1 + 2\sqrt{\frac{F^{\log 3}(x)}{x}} = \sqrt{\frac{1 - F^{\log 3}(x)}{1 - x}}.$$

<u>Dem</u>. Let

$$F^{\log 2}(x) = \left(\frac{1 - y^2}{1 + y^2}\right)^2,$$

then we see that $\sqrt{1 - x} = y$ or $x = 1 - y^2$ from which we can prove as in the previous example that

$$F^{n \log 2}(x) = \frac{4}{\left(\dfrac{1+y}{1-y}\right)^{2^n} + \left(\dfrac{1-y}{1+y}\right)^{2^n} + 2}.$$

Now putting $n = \dfrac{\log 3}{\log 2}$, we have

$$F^{\log 3}(x) = \frac{4(1 - y^2)^3}{\{(1+y)^3 + (1-y)^3\}^2} = \frac{(1 - y^2)^3}{(1 + 3y^2)^2},$$

so that

$$1 - F^{\log 3}(x) = \frac{y^2(3 + y^2)^2}{(1 + 3y^2)^2}.$$

Hence we have

$$1 + 2\sqrt{\frac{F^{\log 3}(x)}{1 - y^2}} = \sqrt{\frac{1 - F^{\log 3}(x)}{y^2}}$$

that is

$$1 + 2\sqrt{\frac{F^{\log 3}(x)}{x}} = \sqrt{\frac{1 - F^{\log 3}(x)}{1 - x}}.$$

<u>Orders of Infinity</u>

7. Using the notation used by Mr. G. H. Hardy, viz., $\succ$ which means 'is of an order higher than' and $\prec$ which means 'is of an order lower than' we have when x becomes infinite,

$$x \prec x^2 \prec x^3 \prec x^4 \quad \text{and so on.}$$

$$e^{x^2} \succ x^x \succ x! \succ e^x \succ x^n \quad \text{however great } n \text{ may be.}$$

$$e^x \prec e^{e^x} \prec e^{e^{e^x}} \quad \text{and so on.}$$

There are functions which defy all exponential scales; for example, the series,

$$1 + \frac{e^x}{2^3} + \frac{e^{e^x}}{2^{3^4}} + \frac{e^{e^{e^x}}}{2^{3^{4^5}}} + \frac{e^{e^{e^{e^x}}}}{2^{3^{4^{5^6}}}} + \cdots$$

(which is convergent for all values of x)

$$\succ e^{e^{e^{\cdot^{\cdot^{\cdot^{e^x}}}}}}$$

however great n may be, and where there are n e's. We need not consider the higher orders than x and lower orders than x separately as the higher orders are inverse functions of the lower orders; e.g., e^x is the inverse of $\log x$; e^{e^x} is the inverse of $\log\log x$ or $\log^2 x$ and so on.

$$\begin{aligned} \log x &\prec \sqrt[n]{x} \qquad \text{however great } n \text{ may be} \\ \log^2 x \text{ or } \log\log x &\prec \sqrt[n]{\log x} \qquad \text{"} \qquad \text{"} \qquad \text{"} \\ \log^3 x &\prec \sqrt[n]{\log^2 x} \qquad \text{"} \qquad \text{"} \qquad \text{"}. \end{aligned}$$

and so on.

8. Now let us try to find a function defying all logarithmic scales i.e., a function $\prec \log^n x$ however great n may be.

First let us consider the function $\phi(x)$ defined by the relation $x\phi(x) = \phi(\log x)$ and $\phi(0) = 1$. Then we see that

$$\phi(1) = 1; \qquad \phi(e) = \frac{1}{e}; \qquad \phi(e^e) = \frac{1}{e^{1+e}} \qquad \text{and so on.}$$

Let

$$\phi\left(e^{e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}}\right) = u_n,$$

and where there are n e's. Then

$$u_0 = 1, \qquad u_1 = \frac{1}{e}, \qquad u_2 = \frac{1}{e^{1+e}}, \qquad \text{etc.}$$

Hence by taking the unique function u_n passing through u_0, u_1, u_2 etc. we see that $\phi(x)$ is finite and definite for all values of x and $\phi(\pm\infty) = 0$ and also $\phi(x) < 1$ when $x > 1$.

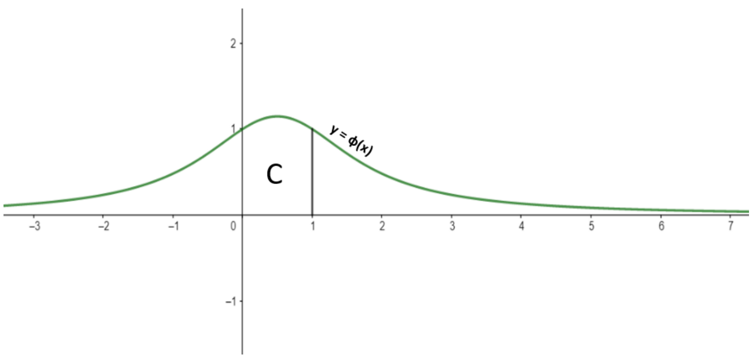

Figure 2: Graphical Illustration

Let

$$\int_0^1 \phi(x)dx = C,$$

then we see that

$$\int_1^e \phi(x)dx = \int_0^1 \frac{\phi(\log x)}{x}dx = \int_0^1 \phi(x)dx = C,$$
$$\int_e^{e^e} \phi(x)dx = \int_1^e \frac{\phi(\log x)}{x}dx = \int_1^e \phi(x)dx = C,$$

and so on. Hence

$$\int_0^\infty \phi(x)dx = \infty$$

and therefore

$$\int_0^x \phi(x)dx$$

is a function becoming infinite as x becomes ∞. Now let us see what the order of $\int \phi(x)dx$ is. When x is great,

$$\phi(x) < \frac{1}{x} \qquad \text{since} \qquad \phi(x) = \frac{\phi(\log x)}{x}.$$

Therefore

$$\int \phi(x)dx \prec \log x.$$

Similarly

$$\phi(x) = \frac{\phi(\log x)}{x} = \frac{\phi(\log\log x)}{x\log x} = \frac{\phi(\log\log\log x)}{x\log x\log\log x}, \quad \textit{etc.}$$

Hence

$$\int \phi(x)dx \prec \log^n x$$

however great n may be. Therefore $\int \phi(x)dx$ is a function becoming infinite when $x = \infty$ and defying all logarithmic scales and consequently the inverse of $\int \phi(x)dx$ defies all exponential scales.

From the above result we can easily prove that the series

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \cdots$$

is a divergent series defying all logarithmic tests and its divergency is so slow that the sum to $10^{2000000}$ terms does not exceed 5.

But there will be no wonder in the above statement if we know that even if we add up 10^{29} terms in the ordinary divergent series

$$\frac{1}{2\log 2} + \frac{1}{3\log 3} + \frac{1}{4\log 4} + \frac{1}{5\log 5} + \cdots$$

the sum will not exceed 5.

Scales of Orders of Infinity

9.

$$x, \quad \sqrt{x}, \quad \sqrt[3]{x}, \quad \sqrt[4]{x}, \ldots$$

belong to the ordinary scale.

$$\log x, \quad \sqrt[3]{\log x}, \quad \log\log x, \quad \log\log\log x, \quad \log(x + \log x),$$

etc. belong to the logarithmic scale.

Using the ϕ function of art. 8., we have

$$\phi(x), \quad \phi\phi(x) \quad \text{or} \quad \phi^2(x), \quad \phi(\log x),$$

etc. belonging to the ϕ-scale. Thus we see that the number of orders of infinity in each scale is infinite.

Theorem. The number of scales is infinite.

Dem. Let

$$f(x), \quad ff(x), \quad \text{or} \quad f^2(x), \quad \{f(x)\}^2, \qquad \sqrt{ff(x)},$$

etc. belong to the f-scale (a known scale) and let F be a function defined by the relation

$$F(x) = f^1(x)F\{f(x)\}.$$

Then we can prove, as in art. 8. that

$$\int_a^{f(a)} F(x)dx = \int_{f(a)}^{f^2(a)} F(x)dx = \int_{f^2(a)}^{f^3(a)} F(x)dx = \cdots = C \qquad \text{say.}$$

Hence

$$\int_a^{f^n(a)} F(x)dx = nC,$$

and consequently

$$\int_a^{\infty} F(x)dx = \infty.$$

Therefore $\int F(x)dx$ is a function becoming infinite as x becomes infinite, and can be proved to be (as in the previous art. 8) to be $\prec f^n(x)$ however great n may be; that is $\int F(x)dx$ defies all f-scales. Hence we see that if one scale is known another scale can be found defying the known scale and so the number of scales is infinite.

To Expand $f^r(x)$ in Ascending Powers of r

10. In arts. 4 and 5 we have given meaning to $f^r(x)$ for fractional values of r and since there is a continuity in the value of r we shall try to differentiate $f^r(x)$ with respect to r and expand it in ascending powers of r also.

(a) Let

$$f^r(x) = \psi_0(x) + \frac{r}{1!}\psi_1(x) + \frac{r^2}{2!}\psi_2(x) + \cdots,$$

where $\psi_0(x), \psi_1(x), \ldots$ do not contain r.

(b) Putting $r = 0$, we have $f^0(x) = x = \psi_0(x)$.

Again changing x to $f^h(x)$ in both sides we have

$$f^r\{f^h(x)\} = f^h(x) + \frac{r}{1!}\psi_1\{f^h(x)\} + \cdots.$$

But

$$\begin{aligned} f^r\{f^h(x)\} =& f^{r+h}(x) \qquad \text{which by Taylor's Theorem} \\ =& f^h(x) + \frac{r}{1!}\frac{df^h(x)}{dh} + \cdots. \end{aligned}$$

Hence by equating the coefficients of r we have

$$\frac{df^h(x)}{dh} = \psi_1\{f^h(x)\}.$$

Let $f^h(x) = y$ and $f^h(y) = z$, then we see that

$$\frac{dy}{dh} = \psi_1(y).$$

Therefore

$$\frac{dz}{dh} = \psi_1(y)\frac{dz}{dy}$$

i.e.,

$$\frac{df^h(y)}{dh} = \psi_1(y)\frac{df^h(y)}{dy},$$

which by changing h to r and y to x becomes
(c) Theorem.

$$\frac{df^r(x)}{dr} = \psi_1(x)\frac{df^r(x)}{dx}.$$

Again,

$$\frac{df^r(x)}{dr} = \psi_1(x) + \frac{r}{1!}\psi_2(x) + \frac{r^2}{2!}\psi_3(x) + \cdots$$

and

$$\psi_1(x)\frac{df^r(x)}{dx} = \psi_1(x) + \frac{r}{1!}\frac{d\psi_1(x)}{dx}\psi_1(x)$$
$$+\frac{r^2}{2!}\frac{d\psi_2(x)}{dx}\psi_1(x) + \frac{r^3}{3!}\frac{d\psi_3(x)}{dx}\psi_1(x) + \cdots \quad \text{by 10(a).}$$

Hence by equating the coefficient of r^{n-1} we have
Theorem.

$$\psi_n(x) = \psi_1(x)\frac{d\psi_{n-1}(x)}{dx}$$

from which we can deduce $\psi_2(x), \psi_3(x), \psi_4(x), \dots$ from $\psi_1(x)$ so that we have to determine $\psi_1(x)$ only.

Now let us consider the function in art. 9; viz., $F(x)$ defined by the relation

$$F(x) = F\{f(x)\}\frac{df(x)}{dx}.$$

Then integrating both sides, we have

$$\int_x^{f(x)} F(z)dz = C,$$

i.e., a constant whatever be the value of x. We have also proved in art. 9. that

$$\int_x^{f^r(x)} F(z)dz = rC.$$

Differentiating both sides with respect to r we have

$$F\{f^r(x)\}\frac{df^r(x)}{dr} = C.$$

But
$$\frac{df^r(x)}{dr} = \psi_1(x)\frac{df^r(x)}{dx} \qquad \text{by 10(c).}$$
Hence we have
$$\psi_1(x)\frac{df^r(x)}{dx}F\{f^r(x)\} = C.$$
But
$$\begin{aligned} F(x) =& \frac{df(x)}{dx}F\{f(x)\} = \frac{df^2(x)}{dx}F\{f^2(x)\} \\ =& \frac{df^3(x)}{dx}F\{f^3(x)\} = \cdots = \frac{df^r(x)}{dx}F\{f^r(x)\}. \end{aligned}$$
(e) Therefore
$$\psi_1(x) = \frac{C}{F(x)}.$$
Thus we have expanded $f^r(x)$ in ascending powers of r.

Fractional Order of Differentiation

11. Using the notation
$$Df(x) = f'(x), \quad D^2f(x) = f''(x), \quad \text{etc.,}$$
we can define $D^nf(x)$ for all values of n as the unique function (discussed in art. 3) passing through
$$D^0f(x), \quad D^1f(x), \quad D^2f(x), \quad D^3f(x), \ldots; \quad \text{e.g.,}$$
Let $f(x) = e^{ax}$ then we see that D^ne^{ax} is the unique function passing through
$$e^{ax}, \quad ae^{ax}, \quad a^2e^{ax}, \quad a^3e^{ax}, \quad \text{etc.}$$
Hence
$$D^n(e^{ax}) = a^ne^{ax}$$
for all values of n.

Theorem. If n is any positive quantity then
$$\int_0^\infty x^{n-1}f^{(r)}(a-x)dx = \Gamma(n)f^{(r-n)}(a)$$
where $f^{(r)}(a)$ denotes that $f(a)$ is differentiated r times. The above result is a theorem as well as a definition; for if we know $f^{(r)}(a)$ for all values of r, we can use the theorem in finding the definite integral
$$\int_0^\infty x^{n-1}f^r(a-x)dx.$$
But if we do not know $f^r(a)$ for all values of r, we can use the theorem in getting fractional order of differential coefficient, thus:–

Suppose we want the k^{th} differential coefficient of $f(a)$ where k is any quantity, fraction or integer.

Take r any integer greater than k and let $n = r - k$. Then n is a positive quantity and so we have
$$\int_0^\infty x^{n-1}f^r(a-x)dx = \Gamma(n)f^k(a).$$

Since r is an integer the left side is intelligible and so we can use the theorem as a definition of fractional differentiation.

Dem. of the Theorem.

$$\begin{aligned}\int_0^\infty x^{n-1}f^{(r)}(a-x)dx &= \int_0^\infty x^{n-1}\left\{f^r(a)-\frac{x}{1!}f^{(r+1)}(a)+\cdots\right\}dx\\ &=\Gamma(n)f^{(r-n)}(a) \qquad \text{by the Theorem on Integral.}\end{aligned}$$

Cor. Putting $n=\frac{1}{2}$, and changing x to x^2 in the above theorem, we have

$$\int_0^\infty f^{(r)}(a-x^2)dx = \frac{\sqrt{\pi}}{2}\cdot f^{(r-\frac{1}{2})}(a),$$

so that

$$\int_0^\infty f(a-x^2)dx = \frac{\sqrt{\pi}}{2}\cdot f^{(-\frac{1}{2})}(a).$$

We can get a number of difficult, new and interesting results by differentiating many known simple results to fractional number of times.

The Use of Operators

12(i) If n is any positive integer $E^n\phi(0)$ can be used for $\phi(n)$ whatever be the function ϕ. But if n is not restricted to positive integers $E^n\phi(0)=\phi(n)$ only when ϕ is the unique function discussed in (3), for

$$\begin{aligned}E^n\phi(0) &= (1+\Delta)^n\phi(0)\\ &= \left\{1+\frac{n}{1!}\Delta+\frac{n(n-1)}{2!}\Delta^2+\cdots\right\}\phi(0)\\ &= \phi(0)+\frac{n}{1!}\{\phi(1)-\phi(0)\}+\frac{n(n-1)}{2!}\{\phi(2)-2\phi(1)+\phi(0)\}+\cdots.\end{aligned}$$

Hence if we write $\phi(n)$ for $E^n\phi(0)$, we see that $\phi(n)$ is got from the values $\phi(0),\phi(1),\phi(2),\phi(3),\ldots$ only. Therefore $\phi(n)$ should be the unique function in 3.

Example.– The Euler–Maclaurin Sum Formula viz.

$$\phi(1)+\phi(2)+\cdots+\phi(n)=C+\int\phi(n)dn+\frac{1}{2}\phi(n)+\frac{B_2}{2!}\phi'(n)-\cdots$$

should be applied if $\phi(n)$ is the unique function as we have used the operators in getting the result and $\phi(n)$ is known for integral values of n in the left side while we use $\phi(n)$ and $\int\phi(n)dn$ as if it is known for all values of n.

ii. If a result is true only for real values of a quantity (say a), then the result got by using the operators for a is true only when the new function can be expressed in terms of the original function; e.g.,

We know

$$\int_0^\infty \frac{\cos ax}{1+x^2}dx=\frac{\pi}{2}e^{-a}$$

is not true for complex values of a. Hence

$$\int_0^\infty \frac{\cos Dx}{1+x^2}\phi(0)dx=\frac{\pi}{2}e^{-D}\phi(0)$$

or

$$\int_0^\infty \frac{\phi(ix)+\phi(-ix)}{2(1+x^2)}dx = \frac{\pi}{2}\phi(-1)$$

is true only when $\phi(ix)+\phi(-ix)$ can be expressed in the form

$$A + B\cos Px + C\cos Qx + \cdots$$

and not in case of any other function.
iii. But if a result is true for complex values of a then we can freely use the operators; e.g.
(a) Since

$$\int_0^\infty x^{n-1}e^{-ax}dx \qquad \text{or} \qquad \int_0^\infty x^{n-1}\sum_{s=0}^\infty \frac{(-1)^s a^s x^s}{s!}dx = \frac{\Gamma(n)}{a^n}$$

is true even when a is changed to $p+qi$, we have

$$\int_0^\infty x^{n-1}\sum_{s=0}^\infty \frac{(-1)^s E^s x^s}{s!}\phi(0)dx = \Gamma(n)E^{-n}\phi(0) \quad E \text{ operating on } \phi$$

$$\text{i.e.} \quad \int_0^\infty x^{n-1}\sum_{s=0}^\infty \frac{(-1)^s x^s \phi(s)}{s!}dx = \Gamma(n)\phi(-n)$$

which is the theorem in the first report.
Example. Since

$$J_n(x) = \sum_{s=0}^\infty \frac{(-1)^s x^{n+2s}}{2^{n+2s}s!\Gamma(n+s+1)}$$

we have

$$\int_0^\infty x^{p-n-1}J_n(ax)dx = 2^{p-n-1}a^{n-p}\frac{\Gamma(\frac{1}{2}p)}{\Gamma(n+1-\frac{1}{2}p)}.$$

(b) Since we have

$$\int_0^\infty e^{-ax}\cos nx\,dx = \frac{a}{n^2+a^2}$$

and

$$\int_0^\infty \frac{a}{x^2+a^2}\cos nx\,dx = \frac{\pi}{2}e^{-an},$$

both being true even if we change a to $p+iq$, we easily infer that

$$\int_0^\infty e^{-aDx}\cos nx\,f(b)dx = \frac{aD}{n^2+a^2D^2}f(b)$$

and

$$\int_0^\infty \frac{aD}{x^2+a^2D^2}\cos nx\,f(b)dx = \frac{\pi}{2}e^{-aDn}f(b).$$

Now let

$$e^{-aDx} f(b) = \phi(x) \qquad \text{and} \qquad \frac{aD}{x^2 + a^2 D^2} f(b) = \psi(x);$$

then we see that: if

$$\int_0^\infty \phi(x) \cos nx \, dx = \psi(n)$$

then

$$\int_0^\infty \psi(x) \cos nx \, dx = \frac{\pi}{2} \phi(n).$$

Similarly for $\sin nx$ also, which is the theorem at the end of the 2^{nd} report.
N.B. The use of operators is somewhat dangerous when the constant for which the operator is substituted is not true for complex values.
13. By splitting

$$\frac{1}{\left\{1 + \left(\frac{x}{a}\right)^2\right\} \left\{1 + \left(\frac{x}{a+1}\right)^2\right\} \left\{1 + \left(\frac{x}{a+2}\right)^2\right\} \cdots \left\{1 + \left(\frac{x}{a+n}\right)^2\right\}}$$

into partial fractions and ultimately making n infinite we can show that
(i)

$$\begin{aligned} &\frac{1}{\left\{1 + \left(\frac{x}{a}\right)^2\right\} \left\{1 + \left(\frac{x}{a+1}\right)^2\right\} \left\{1 + \left(\frac{x}{a+2}\right)^2\right\} \cdots \text{ad infinitum}} \\ &= \frac{2\Gamma(2a)}{\{\Gamma(a)\}^2} \left\{ \frac{1}{a + \frac{x^2}{a}} - \frac{2a}{1!} \frac{1}{a + 1 + \frac{x^2}{a+1}} + \frac{2a(2a+1)}{2!} \frac{1}{a + 2 + \frac{x^2}{a+2}} \right. \\ &\quad \left. - \frac{2a(2a+1)(2a+2)}{3!} \frac{1}{a + 3 + \frac{x^2}{a+3}} + \cdots \right\}. \end{aligned}$$

Hence we have

$$\text{(ii)} \qquad \int_0^\infty \frac{\cos 2nx \, dx}{\left\{1 + \left(\frac{x}{a}\right)^2\right\} \left\{1 + \left(\frac{x}{a+1}\right)^2\right\} \left\{1 + \left(\frac{x}{a+2}\right)^2\right\} \cdots} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(a + \frac{1}{2})}{\Gamma(a)} \frac{1}{\cosh^{2a} n}.$$

Hence by applying the theorem in 12(b) we have

$$\text{(iii)} \qquad \int_0^\infty \frac{\cos 2nx}{\cosh^{2a} x} dx = \frac{\sqrt{\pi}\Gamma(a)}{2\Gamma(a + \frac{1}{2}) \left\{1 + \left(\frac{n}{a}\right)^2\right\} \left\{1 + \left(\frac{n}{a+1}\right)^2\right\} \left\{1 + \left(\frac{n}{a+2}\right)^2\right\} \cdots}.$$

Example. Putting $a = \frac{5}{2}$ we have by changing n to $\frac{n}{2}$

$$\int_0^\infty \frac{\cos nx}{\cosh^5 x} dx = \frac{\pi(n^2 + 1)(n^2 + 9)}{48 \cosh \frac{\pi n}{2}}.$$

(iv) Changing a to $1-a$ in (iii) and multiplying the two results we have

$$\int_0^\infty \frac{\cos 2nx}{\cosh^{2a} x}dx \int_0^\infty \frac{\cos 2nx}{\cosh^{2(1-a)} x}dx = \frac{\pi\Gamma(a)\Gamma(1-a)}{4\Gamma(a+\frac{1}{2})\Gamma(\frac{3}{2}-a)}$$

$$\times \frac{1}{\left\{1+\left(\frac{n}{a}\right)^2\right\}\left\{1+\left(\frac{n}{a+1}\right)^2\right\}\left\{1+\left(\frac{n}{a+2}\right)^2\right\}\cdots\left\{1+\left(\frac{n}{1-a}\right)^2\right\}\left\{1+\left(\frac{n}{2-a}\right)^2\right\}\cdots}$$

$$= \frac{\pi \sin 2\pi a}{2(1-2a)(\cosh 2\pi n - \cos 2\pi a)};$$

that is

$$\int_0^\infty \frac{\cos nx}{\cosh^a x}dx \int_0^\infty \frac{\cos nx}{\cosh^{2-a} x}dx = \frac{\pi \sin \pi a}{2(1-a)(\cosh \pi n - \cos \pi a)}.$$

Similarly we can find

$$\int_0^\infty \frac{\cos nx}{\cosh^a x}dx \int_0^\infty \frac{\cos nx}{\cosh^b x}dx$$

if $a+b$ is any odd even integer.

14.

$$\int_0^h e^{-ax} \cos mx\, dx \qquad \text{is the real part of} \qquad \int_0^h e^{-(a+mi)x}dx$$

i.e. that of

$$\frac{1-e^{-(a+mi)b}}{a+mi} = \frac{a(1-e^{-ab}\cos mb)}{a^2+m^2} + \frac{me^{-ab}\sin mb}{a^2+m^2}.$$

Again we have,

$$\int_0^\infty \frac{a(1-e^{-ab}\cosh x) + xe^{-ab}\sinh x}{a^2+x^2} \cos nx\, dx = \frac{\pi}{2}e^{-an},\ \frac{\pi}{4}e^{-an},\ \text{or } 0$$

according as $n < h, = h$ or $> h$. Using the operator in the above result as in 12 iii(b) we have the Theorem. If

$$\int_0^h \phi(x) \cos mx\, dx = \psi(m)$$

then

$$\int_0^\infty \psi(x) \cos nx\, dx = \frac{\pi}{2}\phi(n),\ \frac{\pi}{4}\phi(n),\ \text{or } 0$$

according as $n < h, n = h$ or $n > h$.

Similarly for sine also.

Till now we have seen only the peculiarities of the trigonometric functions sine and cosine within a definite integral and now let us try to find out other functions having similar properties.

15. Theorem. If the functions F and f are so related that

$$\int_0^\infty F(ax)f(bx)dx = \frac{1}{a+b},$$

or

$$\int_0^\infty x^{p-1}F(x)dx \int_0^\infty x^{-p}f(x)dx = \frac{\pi}{\sin \pi p},$$

p being any positive proper fraction, then if

$$\int_0^\infty \phi(x)\frac{F(nxi)+F(-nxi)}{2}dx = \psi(n)$$

then

$$\int_0^\infty \psi(x)\frac{f(nxi)+f(-nxi)}{2}dx = \frac{\pi}{2}\phi(n).$$

If F is known, then f can be found by solving the first or the second Integral Equation.
Dem. By Cor. 6 and 8 to Theorem I given in the 2nd progress report, we have,

$$\begin{aligned}(a)\quad &\int_0^\infty \left(a_0 - a_1\frac{x}{1!} + a_2\frac{x^2}{2!} - \cdots\right)\left(b_0 - b_2\frac{n^2x^2}{2!} + b_4\frac{n^4x^4}{4!} - \cdots\right)dx\\ =&a_{-1}b_0 - a_{-3}b_2n^2 + a_{-5}b_4n^4 - a_{-7}b_6n^6 + \cdots .\end{aligned}$$

and

$$\begin{aligned}(b)\quad &\int_0^\infty \left(c_0 - c_2x^2 + c_4x^4 - \cdots\right)\left(d_0 - d_2\frac{n^2x^2}{2!} + d_4\frac{n^4x^4}{4!} - \cdots\right)dx\\ =&\frac{\pi}{2}\left(c_{-1}d_0 - \frac{n}{1!}c_{-2}d_1 + \frac{n^2}{2!}c_{-3}d_2 - \cdots\right).\end{aligned}$$

Supposing $c_r = b_r a_{-r-1}$ and $d_r = \frac{1}{b_{-r-1}}$, and

$$\begin{aligned}\phi(x) =&a_0 - a_1\frac{x}{1!} + a_2\frac{x^2}{2!} - \cdots\\ \psi(x) =&c_0 - c_2x^2 + c_4x^4 - \cdots\\ F(x) =&b_0 - b_1\frac{x}{1!} + b_2\frac{x^2}{2!} - \cdots\end{aligned}$$

and

$$f(x) =d_0 - d_1\frac{x}{1!} + d_2\frac{x^2}{2!} - \cdots ,$$

we see that, by Theorem I,

$$\int_0^\infty x^{p-1}F(x)dx = \Gamma(p)b_{-p} \qquad \text{and} \qquad \int_0^\infty x^{-p}f(x)dx = \Gamma(1-p)d_{p-1}.$$

Multiplying the two results, we have

$$\int_0^\infty x^{p-1}F(x)dx\int_0^\infty x^{-p}f(x)dx = \frac{\pi}{\sin \pi p},$$

since

$$b_{-p} = \frac{1}{d_{p-1}}$$

by supposition. Thus, the second part is proved.

Now change x into kx in $f(x)$ and multiply $F(x)$ by $f(kx)$.

Then expanding the product in ascending powers of k and integrating the terms separately by Theorem I, we have

$$\int_0^\infty F(x)f(kx)dx = \frac{1}{1+k}.$$

Now putting $k = \frac{b}{a}$ and changing x into ax, we have

$$\int_0^\infty F(ax)f(bx)dx = \frac{1}{a+b}.$$

Also (a) and (b) may be written as

$$\int_0^\infty \phi(x)\frac{F(nxi)+F(-nxi)}{2}dx = \psi(n)$$

and

$$\int_0^\infty \psi(x)\frac{f(nxi)+f(-nxi)}{2}dx = \frac{\pi}{2}\phi(n).$$

<u>Cor. 1</u> We know that

$$\int_0^\infty e^{-ax}e^{-bx}dx = \frac{1}{a+b}.$$

Here $F(x) = f(x) = e^{-x}$ when the theorem is: if

$$\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$$

then

$$\int_0^\infty \psi(x)\cos nx\,dx = \frac{\pi}{2}\phi(n).$$

<u>Cor. 2</u> We know that

$$\int_0^\infty \frac{1}{1+a^2x^2}\cdot\frac{2/\pi}{1+b^2x^2}dx = \frac{1}{a+b}.$$

Here

$$F(x) = \frac{1}{1+x^2} \qquad \text{and} \qquad f(x) = \frac{2}{\pi(1+x^2)}$$

and the theorem reduces to: if

$$\int_0^\infty \frac{\phi(x)}{1-n^2x^2}dx = \psi(n)$$

then

$$\int_0^\infty \frac{\psi(x)}{1-n^2x^2}dx = \frac{\pi^2}{4}\phi(n).$$

<u>Cor. 3</u>. In a similar manner, we can prove that: if

$$\int_0^\infty x\phi(x)J_n(\ell x)dx = \psi(\ell)$$

then

$$\int_0^\infty x\psi(x)J_n(\ell x)dx = \phi(\ell)$$

and so on.

<u>Cor. 4</u> If

$$\frac{a}{m} = \frac{n-b}{n} = p$$

and

$$\int_0^\infty F(\alpha x)f(\beta x)dx = \frac{1}{\alpha+\beta}$$

then

$$\int_0^\infty x^{a-1}F(x^m)dx \int_0^\infty x^{b-1}f(x^n)dx = \frac{\pi}{mn\,\sin\pi p}.$$

Dem. We have proved that: if

$$\int_0^\infty F(\alpha x)f(\beta x)dx = \frac{1}{\alpha+\beta}$$

then

$$\int_0^\infty x^{p-1}F(x)dx \int_0^\infty x^{-p}f(x)dx = \frac{\pi}{\sin\pi p}.$$

If we change x to x^m and x^n respectively in the above integrals we can get the result by assuming

$$p = \frac{a}{m} = \frac{n-p}{n}.$$

Cor. 5 As a particular case of the above Cor. when $m = n = 2$, and $a = b = 1$ and $p = \frac{1}{2}$ we have: if

$$\int_0^\infty F(ax)f(bx)dx = \frac{1}{a+b}$$

then

$$\int_0^\infty F(x^2)dx \int_0^\infty f(x^2)dx = \frac{\pi}{4}.$$

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