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A REMARK ON GOLDBACH'S PROBLEM II

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For $A > 0$, *even* integer $N(> 1)$, set

$$E_A(N) = \sum_{q \leq (\log N)^A} \frac{\mu^2(q)\, c_q(-N)}{\phi^2(q)}$$

Here, $c_q(n)$ is the Ramanujan's sum ; one evaluation is

(1) $$c_q(-N) = \frac{\mu(q/(q, N))\, \phi(q)}{\phi(q/(q, N))}$$

R. Balasubramanian and C. J. Mozzochi proved (cf. § 6 of [1]) the following theorem.

Theorem.

For any fixed $A(>0)$, the relation

(2) $$E_A(N) \sim E_\infty(N), \; N \to \infty,$$

is false.

Now we give an alternative proof of this theorem based on

(3) $E_\infty(N) - E_A(N) = O(E_\infty(N)/A)$, $E_\infty(N) \asymp N/\phi(N)$,

proved in [2].

Remark.

In view of the above theorem, it is necessary to state the Theorem in [2] with the factor $N/\phi(N)$ in the hypothesis; i. e., the bound

$$\delta_B \frac{N}{(\log N)^2} \text{ replaced by } \delta_B \frac{N}{(\log N)^2} \frac{N}{\phi(N)}.$$

This can be done with no change in the proof there.

We require also the following

Lemma.

Let $\alpha > 0$. Then, as $x \to \infty$,

$$|\{n < x : \mu(n) \neq 0,\ p \mid n \Rightarrow p \leqslant x^{\alpha}\}| > c_{\alpha}\, x,$$

holds with some $c_{\alpha} > 0$.

Remark

Actually, the above number has a well-known asymptotic formula. However, we give here a direct simple proof of this Lemma, as it is sufficient for our present purpose.

Proof of the Lemma

Clearly, it suffices to prove the Lemma for a sequence $\alpha = \alpha_k \to 0$ (as $k \to \infty$). We choose $\alpha_k = 3/(5k+4)$; $k = 1, 2, \dots$. Consider $0 < \frac{2}{3}\alpha_k = \delta_0 < \dots < \delta_R = \alpha_k$ with $R = 2k+1$; θ defined through $\delta_j = \delta_0 + \theta j$ $(j = 0, 1, \dots, R)$.

Note that

$$\delta_1 + \dots + \delta_{R-1} = (R-1)\delta_0 + \frac{R-1}{2} R\theta$$

$$= \frac{R-1}{2}(2\delta_0 + \delta_R - \delta_0),$$

and on inserting the values of R and δ_R in terms of k we see that

(4) $$1 - 2\delta_0 = \delta_1 + \dots + \delta_{R-1} < 1 - (\delta_R + \delta)$$

with any fixed $(0<)\ \delta < \frac{1}{2}\delta_0$.

Let q^{*} denote a typical product $\prod_{j=1}^{R} p_j$ with primes p_j

satisfying $x^{\delta_{j-1}} < p_j < x^{\delta_j}$. Now, from (4), $x^{1-\delta} > q^* > x^{1-\delta_0}$. Letting $q' < x/q^*$ run through square-free values, we see that $q^*q' < x$ are distinct square-free numbers, and their number is

$$\gg \sum \frac{x}{q^*} = x \prod_{j=1}^{R} \left(\sum \frac{1}{p_j} \right) > c_k' x$$

with some $c'_k > 0$. This completes the proof of the Lemma.

Proof of the Theorem.

In view of (3), it is sufficient to consider a lower bound for

$$(3') \qquad E_{A,A'}(N) = E_{A'}(N) - E_A(N)$$

with a suitably large A' ($> 2A$, say). Now, we *restrict* N to the sequence

$$(5) \qquad N_m = \prod_{p < \log m} p, \qquad (\log N_m \sim \log m)$$

and note that if a square-free $q \nmid N$, then $q / (q, N) \gg \log N$. Thus, by (1), we see the contribution of $q \nmid N$ to $E_{A,A'}(N)$ is

$$O\left(\sum{}' \frac{\mu^2(q)}{\phi(q)} \frac{1}{\sqrt{\log N}} \right), \text{ say,}$$

where ' denotes the restriction on q in (3'). So this contribution is

$$(6) \qquad O_{A,A'}(\log \log N / \sqrt{\log N}) = o(E_\infty(N)), \; N \to \infty,$$

since, for N in (5), $E_\infty(N) \asymp \log \log N$, by (3). Next, the (remaining) contribution of $q \mid N$ to (3') is

$$\sum_{\substack{q | N \\ (\log N)^A < q \leq (\log N)^{A'}}} \frac{\mu^2(q)}{\phi(q)} > \sum_{\substack{q | N \\ (\log N)^A < q \leq (\log N)^{2A}}} \frac{\mu^2(q)}{q}$$

and this is easily seen (by (5) and Lemma) to be $> c^*_A \log \log N$. The proof is completed by a suitably large choice of A' (in terms of c^*_A) via (3) and (6).

References

[1] R. Balasubramanian and C. J. Mozzochi, *Siegel zeros and the Goldbach Problem*, J. Number Theory, 16 (1983), 311-332.

[2] S. Srinivasan, *A Remark on Goldbach Problem*, J. Number Theory, 12 (1980), 116-121.

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