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A TRIVIAL REMARK ON GOLDBACH CONJECTURE

BY

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§ 1. **INTRODUCTION.** In [3] S. Srinivasan has proved the following conditional theorem. (We write $e(x) = exp(2\pi ix)$).

THEOREM. *Let $B > 10$ be arbitrary. Suppose there is a function δ_B tending to zero as $B \to \infty$, such that*

$$\left| \int_{\mathcal{M}_3} S^2(\alpha)e(-N\alpha)d\alpha \right| < \delta_B \frac{N}{(\log N)^2}\left(\frac{N}{\varphi(N)}\right),$$

where $S(\alpha) = \sum_{p\leq X} e(p\alpha)(p :$ primes$)$, $N = [X]$ and $\mathcal{M}_3$ is the part of the minor arcs corresponding to denominators q,

$$Q(\log X)^{-2} < q < Q,$$

with $Q = X(\log X)^{-B}$. Then the number $r(N)$ of representations of N as a sum of two odd primes satisfies

$$r(N) = (1 + o(1))S_0(N)T(N)$$

as $N \to \infty$, *where* $T(N) = \sum(\log m_1 \log m_2)^{-1}$ *the summation being over integers* $m_1 > 1, m_2 > 1$ *and* $m_1 + m_2 = N$, *and*

$$S_0(N) = \prod_{p \nmid N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p|N}\left(1 + \frac{1}{p-1}\right).$$

The analogous statement holds good for the twin-primes conjecture also.

In this note we prove the theorem above with $Q(\log X)^{-2} < q < Q$ replaced by $Q(\log X)^{-1-\varepsilon} < q < Q$ where $\varepsilon > 0$ is any small constant. (In conversation with Srinivasan I came to know that he also knew this result. In fact it follows by his conditions on A, B, C. However my treatment is different.)

We now describe his basic and supplementary intervals. Let

$$L = \log X, Q = X\ L^{-B}, Q_1 = L^A, Q_0 = X\ L^{-(B+C)}$$

where the constants A, B, C satisfy

$$A > 3(B+C), B > 1, C > 1.$$

Let $\mathcal{M}_1$, be the union of α-intervals defined by

$$\left|\ \alpha - \frac{h}{q}\ \right| < \frac{1}{Q}, 0 \leq h < q, (h,q) = 1, \text{ and } q \leq Q_1.$$

These intervals are disjoint and

$$\int_{\mathcal{M}_1} S^2(\alpha)e(-N\ \alpha)d\ \alpha = T(N)S_0(N) + O\left(\frac{1}{A}\ \frac{N}{Q(N)}\ \frac{N}{(\log N)^2}\right).$$

$\mathcal{M}_2$ is in union of the α-intervals defined by

$$\mathcal{M}_2 = \bigcup_{Q_1 \leq q \leq Q_0}\ \bigcup_{\substack{h=0 \\ (h,q)=1}}^{q-1} \mathcal{M}(h,q)$$

where $\mathcal{M}(h\ q) = [\frac{h}{q} - (q\ Q)^{-1}, \frac{h}{q} + (q\ Q)^{-1}]$.

Let

$$I = [-Q^{-1}, 1 - Q^{-1}] \text{ and } \mathcal{M} = \bigcup_{1 \le q \le Q} \bigcup_{\substack{h=0 \\ (h,q)=1}}^{q-1} \mathcal{M}(h, q).$$

We know that $\mathcal{M} = I$ and we have to get an asymptotic formula for

$$\int_I S^2(\alpha) e(-N\alpha) d\alpha.$$

We put $\mathcal{M}_3 = I - (\mathcal{M}_1 \cup \mathcal{M}_2)$ and we get the result of Srinivasan. (To sum up, the basic intervals treatment of Srinivasan follows Prachar (see [2] p.182) and that of Gallagher (see [1], Lemma 7) and his supplementary intervals treatment involves a simple but new idea). We give in the next section our new treatment of supplementary intervals.

§ 2. NEW TREATMENT OF SUPPLEMENTARY INTERVALS.

In Lemmas 2 to 5 constants implied by the Vinogradov symbol $\ll$ depend on the integer constant k.

LEMMA 1. *Let v and γ be any two real numbers and $f(x)$ a k times continuously differentiable function defined in $v \le x \le v + k\gamma$. Then*

$$\int_0^\gamma \cdots \int_0^\gamma f^k(v + u_1 + \cdots + u_k) du_1 \cdots du_k$$

$$= f(v + k\gamma) - \binom{k}{1} f(v + (k-1)\gamma) + - \cdots + (-1)^k f(v).$$

PROOF. The lemma follows by induction on k.

LEMMA 2. *Let Q_0, Q_1, Q and q be positive integers with*

$$Q_1 \le q \le Q_0 \le Q.$$

Let a be an integer satisfying $1 \le a \le q, (a, q) = 1, \gamma$ a real number satisfying $|\gamma| \le \frac{1}{q Q_0}$, k a fixed natural number, $v = \frac{a}{q} + \beta$ with $0 \le |\beta| \le \frac{1}{q Q}$. Put $F(x) = \sum_{p \le N} e(p\, x)$ where N is any natural number and $f(x) = |F(x)|^2$. Then

$$f(v) \ll \frac{N^{k+2}}{(\log N)^2 (q\, Q_0)^k} + \sum_{j=1}^{k} f(v + j\, \gamma).$$

PROOF. The first term on the RHS comes from a trivial estimation of the multiple integral in Lemma 1 and the second term is obvious.

LEMMA 3. *We have,*

$$\frac{2f(v)}{q\,Q_0} = \int_{|\gamma|\leq\frac{1}{q\,Q_0}} f(v)d\,\gamma$$

$$\ll \frac{N^{k+2}}{(\log N)^2(q\,Q_0)^{k+1}} + \int_{|w|\leq\frac{k+1}{q\,Q_0}} f(\frac{a}{q}+w)d\,w.$$

PROOF. Follows from Lemma 2 on integration with respect to γ since for $1 \leq j \leq k$ we have

$$\int_{|\gamma|\leq\frac{1}{q\,Q_0}} f(\frac{a}{q}+\beta+j\gamma)d\,\gamma \ll \int_{|w|\leq\frac{k+1}{q\,Q_0}} f(\frac{a}{q}+w)d\,w$$

(on putting $\beta + j\,\gamma = w$).

LEMMA 4. *Any fixed interval*

$$\frac{a}{q} \pm \frac{k+1}{q\,Q_0}(1 \leq a \leq q, (a,q) = 1, 1 \leq q \leq Q_0)$$

intersects at most $O(1)$ other intervals and so

$$\sum_{q\leq Q_0}\sum_{a}\int_{|\beta|\leq\frac{1}{q\,Q}}\int_{|w|\leq\frac{k+1}{q\,Q_0}} f(\frac{a}{q}+w)d\,w$$

$$\ll \left(\int_0^1 f(x)dx\right)\frac{Q_0}{Q} \ll \frac{N}{\log N}(\frac{Q_0}{Q}).$$

PROOF. Follows by the remark that the innermost integral on the LHS is independent of β.

LEMMA 5. *We have*

$$\sum_{Q_1\leq q\leq Q_0}\sum_{a}\int_{|\beta|\leq\frac{1}{q\,Q}} f(\frac{a}{q}+\beta)d\beta$$

$$\ll \frac{N^2}{Q(\log N)^2}\left(\frac{N}{Q_0\,Q_1}\right)^k + \frac{N}{\log N}\left(\frac{Q_0}{Q}\right).$$

PROOF. By Lemma 3, we have,

$$\sum_a f\left(\frac{a}{q}+\beta\right) \ll \frac{N^{k+2} q\, Q_0}{(\log N)^2 (q\, Q_0)^{k+1}} + q\, Q_0 \sum_a \int_{|\gamma| \le \frac{k+1}{q\, Q_0}} f\left(\frac{a}{q}+w\right) d\, w.$$

Integrating with respect to β in $|\beta| \le \frac{1}{q\,Q}$ and summing over q in

$$Q_1 \le q \le Q_0$$

we obtain

$$\sum_{Q_1 \le q \le Q_0} \sum_a \int_{|\beta| \le \frac{1}{q\,Q}} f\left(\frac{a}{q}+\beta\right) d\beta$$

$$\frac{N^{k+2}}{(\log N)^2} \sum_{q \ge Q_1} \frac{q\, Q_0}{(q\, Q_0)^{k+1}}\left(\frac{1}{q\, Q}\right) + \left(\frac{Q_0}{Q}\right)\left(\frac{N}{\log N}\right)$$

$$\ll \frac{N^2}{Q(\log N)^2}\left(\frac{N}{Q_0\,Q_1}\right)^k + \left(\frac{Q_0}{Q}\right)\left(\frac{N}{\log N}\right).$$

This proves the lemma.

§ 3. **REMARKS.** The basic intervals are already investigated by S. Srinivasan [3]. The application of the results of §2 to estimation of a trivial portion of supplementaryintervals is clear. For instance we can take

$$Q = \frac{N}{(\log N)^A}, Q_0 = N(\log N)^{-A-1-\varepsilon}, Q_1 = (\log N)^{A+1+2\varepsilon}, k \ge k_0(\varepsilon, A).$$

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