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A TRIVIAL REMARK ON GOLDBACH CONJECTURE

BY

K. RAMACHANDRA

§ 1. INTRODUCTION. In [3] S. Srinivasan has proved the following conditional theorem. (We write $e(x) = \exp(2\pi ix)$).

THEOREM. Let $B > 10$ be arbitrary. Suppose there is a function $\delta_B$ tending to zero as $B \to \infty$, such that

$$\left| \int_{M_3} S^2(\alpha) e(-N \alpha) d\alpha \right| < \delta_B \frac{N}{(N \log N)^2} \frac{N}{\varphi(N)},$$

where $S(\alpha) = \sum_{p \leq x} e(p\alpha)(p : \text{primes}), N = [X]$ and $M_3$ is the part of the minor arcs corresponding to denominators $q$,

$$Q(\log X)^{-2} < q < Q,$$

with $Q = X(\log X)^{-B}$. Then the number $r(N)$ of representations of $N$ as a sum of two odd primes satisfies

$$r(N) = (1 + o(1))S_0(N)T(N)$$
as \( N \to \infty \), where \( T(N) = \sum (\log m_1 \log m_2)^{-1} \) the summation being over integers \( m_1 > 1, m_2 > 1 \) and \( m_1 + m_2 = N \), and

\[
S_0(N) = \prod_{p|N} \left( 1 - \frac{1}{(p-1)^2} \right) \prod_{p|N} \left( 1 + \frac{1}{p-1} \right).
\]

The analogous statement holds good for the twin-primes conjecture also.

In this note we prove the theorem above with \( Q(\log X)^{-2} < q < Q \) replaced by \( Q(\log X)^{-1-\varepsilon} < q < Q \) where \( \varepsilon > 0 \) is any small constant. (In conversation with Srinivasan I came to know that he also knew this result. In fact it follows by his conditions on \( A, B, C \). However my treatment is different.)

We now describe his basic and supplementary intervals. Let

\[
L = \log X, Q = X L^{-B}, Q_1 = L^A, Q_0 = X L^{-(B+C)}
\]

where the constants \( A, B, C \) satisfy

\[
A > 3(B + C), B > 1, C > 1.
\]

Let \( \mathcal{M}_1 \), be the union of \( \alpha \)-intervals defined by

\[
| \alpha - \frac{h}{q} | < \frac{1}{Q}, 0 \leq h < q, (h, q) = 1, \text{ and } q \leq Q_1.
\]

These intervals are disjoint and

\[
\int_{\mathcal{M}_1} S^2(\alpha) \epsilon(-N \alpha) d \alpha = T(N)S_0(N) + O\left( \frac{1}{A} \frac{N}{Q(N)} \frac{N}{(\log N)^2} \right).
\]

\( \mathcal{M}_2 \) is in union of the \( \alpha \)-intervals defined by

\[
\mathcal{M}_2 = \bigcup_{Q_1 \leq q \leq Q_0} \bigcup_{(h,q) = 1}^{q-1} \mathcal{M}(h, q)
\]

where \( \mathcal{M}(h, q) = [\frac{h}{q} - (qQ)^{-1}, \frac{h}{q} + (qQ)^{-1}] \).
Let
\[ I = [-Q^{-1}, 1 - Q^{-1}] \text{ and } \mathcal{M} = \bigcup_{1 \leq q \leq Q} \bigcup_{h=0}^{q-1} \mathcal{M}(h, q). \]

We know that \( \mathcal{M} = I \) and we have to get an asymptotic formula for
\[ \int_I S^2(\alpha)e(-N \alpha) d\alpha. \]

We put \( \mathcal{M}_3 = I - (\mathcal{M}_1 \cup \mathcal{M}_2) \) and we get the result of Srinivasan. (To sum up, the basic intervals treatment of Srinivasan follows Prachar (see [2] p.182) and that of Gallagher (see [1], Lemma 7) and his supplementary intervals treatment involves a simple but new idea). We give in the next section our new treatment of supplementary intervals.

§ 2. NEW TREATMENT OF SUPPLEMENTARY INTERVALS.
In Lemmas 2 to 5 constants implied by the Vinogradov symbol \( \ll \) depend on the integer constant \( k \).

**LEMMA 1.** Let \( v \) and \( \gamma \) be any two real numbers and \( f(x) \) a \( k \) times continuously differentiable function defined in \( v \leq x \leq v + k\gamma \). Then
\[ \int_0^v \cdots \int_0^v f^k(v + u_1 + \cdots + u_k) du_1 \cdots du_k = f(v + k\gamma) - \binom{k}{1} f(v + (k - 1)\gamma) - \cdots + (-1)^k f(v). \]

**PROOF.** The lemma follows by induction on \( k \).

**LEMMA 2.** Let \( Q_0, Q_1, Q \) and \( q \) be positive integers with
\[ Q_1 \leq q \leq Q_0 \leq Q. \]

Let \( a \) be an integer satisfying \( 1 \leq a \leq q \), \( (a, q) = 1 \), \( \gamma \) a real number satisfying \( |\gamma| \leq \frac{1}{q} Q_0 \), \( k \) a fixed natural number, \( v = \frac{a}{q} + \beta \) with \( 0 \leq |\beta| \leq \frac{1}{q} Q \). Put
\[ F(x) = \sum_{p \leq N} e(px) \text{ where } N \text{ is any natural number and } f(x) = \text{ } F(x) |^2. \]

Then
\[ f(v) \ll \frac{N^{k+2}}{(\log N)^2(q Q_0)^k} + \sum_{j=1}^{k} f(v + j \gamma). \]
PROOF. The first term on the RHS comes from a trivial estimation of the multiple integral in Lemma 1 and the second term is obvious.

LEMMA 3. We have,

\[
\frac{2f(v)}{q Q_0^3} = \int_{|\gamma| \leq \frac{1}{q Q_0}} f(v) d \gamma
\]

\[
\ll \frac{N^{k+2}}{(\log N)^2(q Q_0)^{k+1}} + \int_{|w| \leq \frac{k+1}{q Q_0}} f(\frac{a}{q} + w) d w.
\]

PROOF. Follows from Lemma 2 on integration with respect to \( \gamma \) since for \( 1 \leq j \leq k \) we have

\[
\int_{|\gamma| \leq \frac{1}{q Q_0}} f(\frac{a}{q} + \beta + j \gamma) d \gamma \ll \int_{|w| \leq \frac{k+1}{q Q_0}} f(\frac{a}{q} + w) d w
\]

(on putting \( \beta + j \gamma = w \)).

LEMMA 4. Any fixed interval

\[
\frac{a}{q} \pm \frac{k + 1}{q Q_0} (1 \leq a \leq q, (a, q) = 1, 1 \leq q \leq Q_0)
\]

intersects at most \( O(1) \) other intervals and so

\[
\sum_{q \leq Q_0} \sum_{a} \int_{|\beta| \leq \frac{1}{q}} \int_{|w| \leq \frac{k+1}{q Q_0}} f(\frac{a}{q} + w) d w
\]

\[
\ll \left( \int_{0}^{1} f(x) dx \right) \frac{Q_0}{Q} \ll \frac{N}{\log N} (\frac{Q_0}{Q}).
\]

PROOF. Follows by the remark that the innermost integral on the LHS is independent of \( \beta \).

LEMMA 5. We have

\[
\sum_{Q_1 \leq q \leq Q_0} \sum_{a} \int_{|\beta| \leq \frac{1}{q}} f(\frac{a}{q} + \beta) d \beta
\]
\[ \leq \frac{N^2}{Q(\log N)^2} \left( \frac{N}{Q_0 Q_1} \right)^k + \frac{N}{\log N} \left( \frac{Q_0}{Q} \right). \]

**PROOF.** By Lemma 3, we have,

\[ \sum_a f\left( \frac{a}{q} + \beta \right) \leq \frac{N^{k+2} q Q_0}{(\log N)^2 (q Q_0)^{k+1}} + q Q_0 \sum_a \int_{|\gamma| \leq \frac{1}{q Q_0}} f\left( \frac{a}{q} + \omega \right) d\omega. \]

Integrating with respect to \( \beta \) in \( |\beta| \leq \frac{1}{q Q} \) and summing over \( q \) in \( Q_1 \leq q \leq Q_0 \),

we obtain

\[ \sum_{Q_1 \leq q \leq Q_0} \sum_a \int_{|\beta| \leq \frac{1}{q Q}} f\left( \frac{a}{q} + \beta \right) d\beta \]

\[ \leq \frac{N^{k+2}}{(\log N)^2} \sum_{q \geq Q_1} \frac{q Q_0}{(q Q_0)^{k+1}} \left( \frac{1}{q Q} \right) + \left( \frac{Q_0}{Q} \right) \left( \frac{N}{\log N} \right). \]

\[ \leq \frac{N^2}{Q(\log N)^2} \left( \frac{N}{Q_0 Q_1} \right)^k + \left( \frac{Q_0}{Q} \right) \left( \frac{N}{\log N} \right). \]

This proves the lemma.

§ 3. **REMARKS.** The basic intervals are already investigated by S. Srinivasan [3]. The application of the results of §2 to estimation of a trivial portion of supplementary intervals is clear. For instance we can take

\[ Q = \frac{N}{(\log N)^A}, Q_0 = N(\log N)^{-A+1-\varepsilon}, Q_1 = (\log N)^{A+1+2\varepsilon}, k \geq k_0(\varepsilon, A). \]

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REFERENCES


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