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AN Ω-RESULT RELATED TO $r_4(n)$

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§ 1. INTRODUCTION.

Let

$$g(n) = \sum_{d|n} h(d) \tag{1}$$

where h is a multiplicative function such that $\sum_{d=1}^{\infty} h(d)$ is convergent and $h(d) = O(\frac{1}{d})$. Let

$$M_0(x) = x \sum_{d=1}^{\infty} \frac{h(d)}{d}$$

$$M_1(x) = \frac{x^2}{2} \sum_{d=1}^{\infty} \frac{h(d)}{d}$$

$$R_0(x) = \sum_{n \leq x} g(n) - M_0(x)$$

and

$$R_1(x) = \sum_{n \leq x} ng(n) - M_1(x) \tag{2}$$

Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k squares. If we put $h(d) = \frac{\alpha(d)}{d}$ where

$$\begin{aligned}\alpha(d) &= -3 \text{ if } 4 \mid d \\ &= +1 \text{ if } 4 \nmid d, \text{ then we have} \\ g(n) &= \tfrac{r_4(n)}{8n}.\end{aligned}$$

(See page 205, Hua [5] for an equivalent expression).

Let $P_k(x)$ be the error term defined by

$$\sum_{n \leq x} r_k(n) = \frac{(\pi x)^{\frac{k}{2}}}{\Gamma(\frac{k+1}{2})} + P_k(x). \tag{3}$$

Szego [7] showed that, if $k \equiv 2, 3, 4$ $(mod 8)$, then

$$P_k(x) = \Omega_-((x \log x)^{\frac{(k-1)}{4}}) \tag{4}$$

and if $k \equiv 6, 7, 8$ $(mod\ 8)$, then

$$P_k(x) = \Omega_+((x \log x)^{\frac{(k-1)}{4}}) \tag{5}$$

For the particular case $k = 2$, the result had been proved by Hardy [4] and the best Ω_- result to date is due to Hafner [3] which is

$$P_2(x) = \Omega_-((x \log x)^{\frac{1}{4}} (\log\log x)^{\frac{\log 2}{4}} exp(-B(\log\log\log x)^{\frac{1}{2}})) \tag{6}$$

The best Ω_+ result to date is due to Corradi and Katai [1], namely

$$P_2(x) = \Omega_+(x^{\frac{1}{4}} exp(c(\log\log x)^{\frac{1}{4}} (\log\log\log x)^{-\frac{3}{4}})) \tag{7}$$

(for eg. see Grosswald page 21, [2]).

Our object of this paper is to consider the case $k = 4$ and we prove the following theorems.

THEOREM 1. *We have*

$$P_4(x) = \Omega_+(x \log\log x).$$

REMARK 1. Our treatment is inspired by a paper of Montgomery [6]. We also observe that an elementary proof of a theorem of Montgomery (Theorem

1 of [6]) follows from our treatment which we state as

THEOREM 2. *If* $g(n) = \frac{\varphi(n)}{n}$ *where* $\varphi(n)$ *is the Euler's totient function, then we have*

$$\frac{R_1(x)}{x} - R_0(x) \ll exp(-c\sqrt{\log x})$$

for some $c > 0$.

REMARK 2. We feel that the treatment can be applied to some other interesting arithmetic functions. We intend to take them up in a further work.

§ 2. NOTATION.

1) $\{x\}$ denotes the fractional part of x.
2) $[x]$ denotes the integral part of x
3) (a, b) denotes the greatest common divisor of a and b.
4) $f(x) = O(g(x))$ or $f(x) \ll g(x)$ denotes that there exists a positive constant A such that $| f(x) | < A\, g(x)$, where $g(x)$ is real.

§ 3. SOME LEMMAS.

LEMMA 3.1. *We have*

$$R_0(x) = -x\sum_{d>x}\frac{h(d)}{d} - \sum_{d\leq x}h(d)\{\frac{x}{d}\}$$

PROOF. We have,

$$\begin{aligned} R_0(x) &= \sum_{n\leq x} g(n) - M_0(x) \\ &= \sum_{n\leq x}\sum_{d|n} h(d) - M_0(x) \\ &= \sum_{d\leq x} h(d)(\tfrac{x}{d} - \{\tfrac{x}{d}\}) - M_0(x) \\ &= -x\sum_{d>x}\tfrac{h(d)}{d} - \sum_{d\leq x} h(d)\{\tfrac{x}{d}\} \\ &\quad \text{(by the definition of} M_0(x)) \end{aligned}$$

which proves the lemma.

LEMMA 3.2. *If $b, r(>0)$ are integers such that $(b,r)=1$ and β is a real number, then we have*

$$\sum_{n=1}^{r}\{\frac{bn}{r}+\beta\}=\frac{r-1}{2}+\{r\beta\} \qquad (3.2.1)$$

PROOF. We note that both sides are periodic in β with period $\frac{1}{r}$. So we assume that $0\le\beta<\frac{1}{r}$. We can also assume that $b=1$. If $\beta=0$, then the left hand side of (3.2.1) $=\sum_{n=1}^{r-1}\frac{n}{r}=\frac{r-1}{2}$. If $0<\beta<\frac{1}{r}$, then we have

$$\begin{aligned}\sum_{n=1}^{r}\{\tfrac{n}{r}+\beta\} &= \tfrac{1}{r}\cdot\tfrac{r(r-1)}{2}+r\beta\\ &= \tfrac{r-1}{2}+\{r\beta\}\end{aligned}$$

which proves the lemma.

LEMMA 3.3. *With notation as in Lemma 3.2, for any integer N, we have*

$$\sum_{n=1}^{N}\{\frac{nb}{r}+\beta\}=\frac{N}{r}\{r\beta\}+\frac{N}{2}(\frac{r-1}{r})+O(r).$$

PROOF. We have $N=Qr+R$ for some $0\le R<r$. Therefore from Lemma 3.2, we have

$$\begin{aligned}\sum_{n=1}^{N}\{\tfrac{nb}{r}+\beta\} &= Q\{r\beta\}+Q(\tfrac{r-1}{2})+\sum_{n=1}^{R}\{\tfrac{nb}{r}+\beta\}\\ &= \tfrac{N}{r}\{r\beta\}+\tfrac{N(r-1)}{2r}+O(r).\end{aligned}$$

which proves the lemma.

LEMMA 3.4. *We have*

$$\frac{R_1(x)}{x}-R_0(x)=\frac{x}{2}\sum_{d>x}\frac{h(d)}{d}++\frac{1}{2}\sum_{d\le x}h(d)-\frac{1}{2x}\sum_{d\le x}h(d)\cdot d(\{\frac{x}{d}\}-\{\frac{x}{d}\}^2).$$

PROOF. We have

$$\begin{aligned}
R_1(x) &= \sum_{n\le x} ng(n) - M_1(x)\\
&= \sum_{d_1d_2\le x} h(d_1)\cdot d_1d_2 - M_1(x)\\
&= \sum_{d_1\le x} d_1h(d_1)\cdot \sum_{d_2\le[\frac{x}{d_1}]} d_2 - M_1(x)\\
&= \tfrac{1}{2}\sum_{d_1\le x} h(d_1)\cdot d_1(\tfrac{x}{d_1} - \{\tfrac{x}{d_1}\})(\tfrac{x}{d_1}+1-\{\tfrac{x}{d_1}\}) - M_1(x)\\
&= \tfrac{1}{2}\sum_{d\le x} h(d)\cdot d(\tfrac{x^2}{d^2} + \tfrac{x}{d} - 2\tfrac{x}{d}\{\tfrac{x}{d}\} - \{\tfrac{x}{d}\} + \{\tfrac{x}{d}\}^2) - M_1(x)\\
&= -\tfrac{x^2}{2}\sum_{d>x}\tfrac{h(d)}{d} + \tfrac{x}{2}\sum_{d\le x} h(d) - x\sum_{d\le x} h(d)\{\tfrac{x}{d}\}\\
&\quad -\tfrac{1}{2}\sum_{d\le x} h(d)\cdot d\{\tfrac{x}{d}\} + \tfrac{1}{2}\sum_{d\le x} h(d)d\{\tfrac{x}{d}\}^2.
\end{aligned}$$

From Lemma 3.1, we have

$$-\sum_{d\le x} h(d)\{\frac{x}{d}\} = R_0(x) + x\sum_{d>x}\frac{h(d)}{d}$$

and so we have

$$\begin{aligned}
\tfrac{R_1(x)}{x} - R_0(x) &= \tfrac{x}{2}\sum_{d>x}\tfrac{h(d)}{d} + \tfrac{1}{2}\sum_{d\le x} h(d)\\
&\quad -\tfrac{1}{2x}\sum_{d\le x} h(d)d(\{\tfrac{x}{d}\} - \{\tfrac{x}{d}\}^2)
\end{aligned}$$

which completes the proof.

LEMMA 3.5. *If there exists a function $G(x)$ such that $G(x)$ and $\frac{x}{G(x)}$ are increasing functions of x, then we have*

$$R_0(x) = -\sum_{d\le y} h(d)\{\frac{x}{d}\} + O(1) \text{ for } y \ge \frac{x}{G(x)}.$$

PROOF. Since $\sum_{d>y} h(d)\{\frac{x}{d}\} = O(1)$ for $y \ge \frac{x}{G(x)}$, the lemma follows from Lemma 3.1.

LEMMA 3.6. *For integers $q \approx G(N)$ with $G(N)$ as in Lemma 3.5,*

$\beta = q$ *and* $y = \frac{(N+1)q}{G(N)}$, *we have*

$$\sum_{n=1}^{N} R_0(nq+\beta) = N \sum_{\substack{e \le y \\ p|e \Rightarrow p|q}} h(e)\frac{(e,q)}{e}\left(\frac{1}{2} - \left\{\frac{\beta}{(e,q)}\right\}\right) \sum_{\substack{f \le \frac{y}{e} \\ (f,q)=1}} \frac{h(f)}{f} + O(N)$$

PROOF. We have

$$\sum_{n=1}^{N} R_0(nq+\beta)$$

$$= -\sum_{n=1}^{N}\sum_{d \le y} h(d)\left\{\frac{nq+\beta}{d}\right\} + O(N)$$

(Since $y = \frac{(N+1)q}{G(N)} > \frac{nq+\beta}{G(nq)}$)

$$= -\sum_{d \le y} h(d) \sum_{n=1}^{N} \left\{\frac{nq+\beta}{d}\right\} + O(N)$$

$$= -\sum_{d \le y} h(d) \cdot \sum_{n=1}^{N} \left\{\frac{q/(d,q)}{d/(d,q)} \cdot n + \beta\right\} + O(N)$$

$$= -\sum_{d \le y} h(d)\left(\frac{N}{d}(d,q)\left\{\frac{\beta}{(d,q)}\right\} + \frac{N(d,q)}{d}\left(\frac{\frac{d}{(d,q)}-1}{2}\right) + O\left(\frac{d}{(d,q)}\right)\right) + O(N)$$

(by Lemma 3.3)

$$= -\sum_{d \le y} h(d)\frac{N(d,q)}{d}\left(\left\{\frac{\beta}{(d,q)}\right\} - \frac{1}{2}\right) + O(N)$$

$$= -N \sum_{\substack{e \le y \\ p|e \Rightarrow p|q}} h(e)\frac{(e,q)}{e}\left(\left\{\frac{\beta}{(e,q)}\right\} - \frac{1}{2}\right) \sum_{\substack{f \le \frac{y}{e} \\ (f,q)=1}} \frac{h(f)}{f} + O(N)$$

(by writing $d = ef$ where $p \mid e \Rightarrow p \mid q$ and $(f,q) = 1$)

which proves the lemma.

LEMMA 3.7. *We have*

$$\sum_{n \le x} \frac{\alpha(n)}{n} = 2 \log 2 + O\left(\frac{1}{x}\right)$$

where $\alpha(n)$ *is as defined in* § 1.

PROOF. We have

$$\begin{aligned}\sum_{n\le x}\tfrac{\alpha(n)}{n} &= -\sum_{4n\le x}\tfrac{3}{4n}+\sum_{n\le x}\tfrac{1}{n}-\sum_{4n\le x}\tfrac{1}{4n}\\ &= \sum_{n\le x}\tfrac{1}{n}-\sum_{n\le[\frac{x}{4}]}\tfrac{1}{n}\\ &= 2\log 2+O(\tfrac{1}{x})\end{aligned}$$

which proves the lemma.

LEMMA 3.8. *If* $h(n)=\frac{\alpha(n)}{n}$ *and so* $g(n)=\frac{r_4(n)}{8n}$, *then we have*

$$\frac{R_1(x)}{x}-R_0(x)=O(1).$$

PROOF. Since $\sum_{d=1}^{\infty}\frac{\alpha(d)}{d}$ is convergent and $|\sum_{d\le x}\alpha(d)|\le 3$ for all x, the lemma follows from Lemmas 3.4 and 3.7.

LEMMA 3.9. *If* $h(n)=\frac{\alpha(n)}{n}$ *and so* $g(n)=\frac{r_4(n)}{n}$, *then we have*

$$R_0(x)=-\sum_{d\le y}\frac{\alpha(d)}{d}\{\frac{x}{d}\}+O(1)$$

uniformly for $x\ge 2, y\ge\sqrt{x}$.

PROOF. From Lemma 3.1, we have

$$\begin{aligned}R_0(x) &= -x\sum_{d>x}\tfrac{\alpha(d)}{d^2}-\sum_{d\le x}\tfrac{\alpha(d)}{d}\{\tfrac{x}{d}\}\\ &= -\sum_{d\le x}\tfrac{\alpha(d)}{d}\{\tfrac{x}{d}\}+O(1)\\ &\quad(\text{Since }\sum_{d>x}\tfrac{\alpha(d)}{d^2}=O(\tfrac{1}{x})).\end{aligned}$$

So it is enough to show that

$$\sum_{y\le d\le x}\frac{\alpha(d)}{d}\{\frac{x}{d}\}=O(1)\text{ for }\sqrt{x}\le y\le x.$$

We choose k such that $1\le k\le\frac{x}{y}$ and in $\frac{x}{k+1}<d\le\frac{x}{k}$, the function $\{\frac{x}{d}\}$ is monotone. Therefore we have

$$\sum_{\frac{x}{k+1}<d\le\frac{x}{k}}\frac{\alpha(d)}{d}\{\frac{x}{d}\}=O(\frac{k}{x})\text{ (by Lemma 3.7)}$$

Now summing up for k in $1 \le k \le \frac{x}{y}$, we get

$$\begin{aligned}\sum_{y\le d\le x} \tfrac{\alpha(d)}{d}\{\tfrac{x}{d}\} &= O(\tfrac{1}{x}\sum_{1\le k\le \frac{x}{y}} k)\\ &= O(\tfrac{1}{x}\cdot\tfrac{x^2}{y^2})\\ &= O(1) \text{ (since } y\ge\sqrt{x})\end{aligned}$$

which proves the lemma.

§ 4. PROOF OF THEOREM 1.

We take $h(n) = \frac{\alpha(n)}{n}$. Therefore from Lemma 3.6 and 3.9 we have

$$\sum_{n=1}^{N} R_0(nq+\beta) = N \sum_{\substack{e\le y\\ p|e\Rightarrow p|q}} \frac{\alpha(e)}{e}\frac{(e,q)}{e}(\frac{1}{2}-\{\frac{\beta}{(e,q)}\}) \sum_{\substack{f\le y/e\\ (f,q)=1}} \frac{\alpha(f)}{f^2} + O(N) \quad (4.1)$$

for $q \simeq \sqrt{N}, \beta \le q$ and $y = \frac{(N+1)q}{\sqrt{N}} (= O(N))$. Since

$$\sum_{f\ge\frac{y}{e}} \frac{\alpha(f)}{f^2} = O(\sum_{f\ge\frac{y}{e}}\frac{1}{f^2}) = O(\frac{e}{y}),$$

we have

$$\begin{aligned}&N\sum_{e\le y}\tfrac{\alpha(e)}{e}\tfrac{(e,q)}{e}(\tfrac{1}{2}-\{\tfrac{\beta}{(e,q)}\})\sum_{f\ge\frac{y}{e}}\tfrac{\alpha(f)}{f^2}\\ &\ll N\sum_{e\le y}\tfrac{|\alpha(e)|}{e}\;\tfrac{(e,q)}{e}\mid\tfrac{e}{y}\mid\\ &\ll \tfrac{N}{y}\sum_{e\le y}\tfrac{(e,q)}{e}\\ &= O(N). \qquad (4.2)\end{aligned}$$

Therefore from (4.1) and (4.2) we have

$$\sum_{n=1}^{N} R_0(nq+\beta) = N(\prod_{p|q}(1+\frac{\alpha(p)}{p^2}+\frac{\alpha(p^2)}{p^4}+\cdots)) \sum_{\substack{e\le y\\ p|e\Rightarrow p|q}} \frac{\alpha(e)}{e}(e,q)(\frac{1}{2}-\{\frac{\beta}{(e,q)}\})+O(N) \quad (4.3)$$

Now, we assume $q = \prod_{p\le z} p, \beta = \prod_{2<p\le z} p = \frac{q}{2}$ where $z = [\frac{1}{2}\ log\ N]$. Hence $q \simeq \sqrt{N}$. From (4.3), we have

$$\sum_{n=1}^{N} R_0(nq+\beta) = N(\prod_{p|q}(1-\frac{1}{p^2})^{-1})\zeta(2) \sum_{\substack{e\le y\\ p|e\Rightarrow p|e}} \frac{\alpha(e)}{e^2}(e,q)(\frac{1}{2}-\{\frac{\beta}{(e,q)}\})+O(N). \quad (4.4)$$

We note that, if $2 \mid e$, then $\frac{1}{2} - \{\frac{\beta}{(e,q)}\} = 0$. If $2 \nmid e$, then $\alpha(e) = 1$ and $\frac{1}{2} - \{\frac{\beta}{(e,q)}\} = \frac{1}{2}$. Therefore we have,

$$\begin{aligned} \sum_{\substack{e \leq y \\ p \mid e \Rightarrow p \mid q}} \frac{\alpha(e)}{e^2}(e,q)\left(\frac{1}{2} - \left\{\frac{\beta}{(e,q)}\right\}\right) &\geq \sum_{\substack{e \leq y \\ p \mid e \Rightarrow p \mid q, 2 \nmid e}} \frac{1}{e^2}(e,q) \cdot \frac{1}{2} \\ &\geq \frac{1}{2}\sum_{\substack{e \mid q \\ 2 \nmid e}} \frac{1}{e} \\ &\approx log\ z \\ &\approx loglog\ N \qquad (4.5) \end{aligned}$$

From (4.4) and (4.5), we have

$$\sum_{n=1}^{N} R_0(nq+\beta) \geq N\zeta(2) \cdot \frac{1}{2} loglog\ N + O(N)$$

which implies

$$R_0(x) = \Omega_+(loglog\ x)$$

and hence from Lemma 3.8, we have

$$R_1(x) = \Omega_+(x\ loglog\ x)$$

which completes the proof of Theorem 1.

§ 5. PROOF OF THEOREM 2.

We take $h(n) = \frac{\mu(n)}{n}$. From Lemma 3.4, we have

$$\frac{R_1(x)}{x} - R_0(x) = \frac{x}{2}\sum_{d > x} \frac{\mu(d)}{d^2} + \frac{1}{2}\sum_{d \leq x} \frac{\mu(d)}{d} - \frac{1}{2x}\sum_{d \leq x} \mu(d)\left(\left\{\frac{x}{d}\right\} - \left\{\frac{x}{d}\right\}^2\right)$$

We consider the sum $\sum_{d \leq x} \mu(d)\{\frac{x}{d}\}$. For the range $d \leq xe^{-c\sqrt{\log x}}$, we use trivial estimate and get

$$\sum_{d \leq xe^{-c\sqrt{\log x}}} \mu(d)\left\{\frac{x}{d}\right\} \ll xe^{-c\sqrt{\log x}}$$

For $x \geq d \geq xe^{-c\sqrt{\log x}}$, we notice that $\{\frac{x}{d}\}$ is monotonic in $\frac{x}{k+1} < d \leq \frac{x}{k}$ for any k such that $1 \leq k \leq e^{c\sqrt{\log x}}$. Therefore we get

$$\sum_{\frac{x}{k+1}<d\leq\frac{x}{k}} \mu(d)\{\tfrac{x}{d}\} \ll \tfrac{x}{k}exp(-c\sqrt{log\tfrac{x}{k}})$$
$$\ll \tfrac{x}{k}exp(-c_1\sqrt{log\ x})$$

Now summing up over all k's in $1 \leq k \leq exp(c\sqrt{log\ x})$, we get

$$\sum_{x\ exp(-c\sqrt{log\ x})\leq d\leq x} \mu(d)\{\frac{x}{d}\} \ll exp(-c_2\sqrt{log\ x}).$$

The sum $\sum_{d\leq x}\mu(d)\{\frac{x}{d}\}^2$ can be treated similarly. Other sums are easy to deal with. Hence the Theorem 2 follows.

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