An $\Omega$-result related to $r_4(n)$.
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§ 1. INTRODUCTION.

Let

\[ g(n) = \sum_{d|n} h(d) \]  

(1)

where \( h \) is a multiplicative function such that \( \sum_{d=1}^{\infty} h(d) \) is convergent and \( h(d) = O\left(\frac{1}{d}\right) \). Let

\[ M_0(x) = x \sum_{d=1}^{\infty} \frac{h(d)}{d} \]

\[ M_1(x) = \frac{x^2}{2} \sum_{d=1}^{\infty} \frac{h(d)}{d} \]

\[ R_0(x) = \sum_{n \leq x} g(n) - M_0(x) \]

and

\[ R_1(x) = \sum_{n \leq x} ng(n) - M_1(x) \]  

(2)
Let $r_k(n)$ denote the number of representations of the positive integer $n$ as a sum of $k$ squares. If we put $h(d) = \frac{\alpha(d)}{d}$ where

\[
\alpha(d) = \begin{cases} 
-3 & \text{if } 4 \mid d \\
+1 & \text{if } 4 \nmid d,
\end{cases}
\]

then we have

\[
g(n) = \frac{r_4(n)}{8n}.
\]

(See page 205, Hua [5] for an equivalent expression).

Let $P_k(z)$ be the error term defined by

\[
\sum_{n \leq z} r_k(n) = \frac{(\pi z)^{\frac{3}{2}}}{\Gamma\left(\frac{k+1}{2}\right)} + P_k(z).
\]

Szegö [7] showed that, if $k \equiv 2, 3, 4 \pmod{8}$, then

\[
P_k(z) = \Omega_-(\langle z \log z \rangle^{\frac{(k-1)}{4}})
\]

and if $k \equiv 6, 7, 8 \pmod{8}$, then

\[
P_k(z) = \Omega_+(\langle z \log z \rangle^{\frac{(k-1)}{4}})
\]

For the particular case $k = 2$, the result had been proved by Hardy [4] and the best $\Omega_-$ result to date is due to Hafner [3] which is

\[
P_2(z) = \Omega_-(\langle z \log z \rangle^{\frac{1}{4}}(\log \log z)^{\frac{1}{2}} \exp(-B(\log \log \log z)^{\frac{1}{2}}))
\]

The best $\Omega_+$ result to date is due to Cramér and Katai [1], namely

\[
P_2(z) = \Omega_+(x^{\frac{1}{4}} \exp(c(\log \log z)^{\frac{1}{2}}))
\]

(for eg. see Grosswald page 21, [2]).

Our object of this paper is to consider the case $k = 4$ and we prove the following theorems.

**Theorem 1.** We have

\[
P_4(z) = \Omega_+(z \log \log z).
\]

**Remark 1.** Our treatment is inspired by a paper of Montgomery [6]. We also observe that an elementary proof of a theorem of Montgomery (Theorem...
1 of [6]) follows from our treatment which we state as

**THEOREM 2.** If \( g(n) = \frac{\varphi(n)}{n} \) where \( \varphi(n) \) is the Euler's totient function, then we have

\[
\frac{R_1(x)}{x} - R_0(x) \ll \exp(-c\sqrt{\log x})
\]

for some \( c > 0 \).

**REMARK 2.** We feel that the treatment can be applied to some other interesting arithmetic functions. We intend to take them up in a further work.

§ 2. NOTATION.

1) \( \{ z \} \) denotes the fractional part of \( z \).

2) \( [z] \) denotes the integral part of \( z \).

3) \( (a, b) \) denotes the greatest common divisor of \( a \) and \( b \).

4) \( f(x) = O(g(x)) \) or \( f(x) \ll g(x) \) denotes that there exists a positive constant \( A \) such that \( |f(x)| < A g(x) \), where \( g(x) \) is real.

§ 3. SOME LEMMAS.

**LEMMA 3.1.** We have

\[
R_0(x) = -x \sum_{d > x} \frac{h(d)}{d} - \sum_{d \leq x} h(d) \{ \frac{x}{d} \}
\]

**PROOF.** We have,

\[
R_0(x) = \sum_{n \leq x} g(n) - M_0(x) \\
= \sum_{n \leq x} \sum_{d \mid n} h(d) - M_0(x) \\
= \sum_{d \leq x} h(d) \{ \frac{x}{d} \} - \sum_{d \leq x} h(d) \{ \frac{x}{d} \} - M_0(x) \\
= -x \sum_{d > x} \frac{h(d)}{d} - \sum_{d \leq x} h(d) \{ \frac{x}{d} \} \\
\text{(by the definition of } M_0(x) \text{)}
\]

which proves the lemma.
LEMMA 3.2. If \( b, r(>0) \) are integers such that \((b,r) = 1\) and \( \beta \) is a real number, then we have

\[
\sum_{n=1}^{r} \left\{ \frac{bn}{r} + \beta \right\} = \frac{r-1}{2} + \{r\beta\} \tag{3.2.1}
\]

PROOF. We note that both sides are periodic in \( \beta \) with period \( \frac{1}{r} \). So we assume that \( 0 \leq \beta < \frac{1}{r} \). We can also assume that \( b = 1 \). If \( \beta = 0 \), then the left hand side of (3.2.1) = \( \sum_{n=1}^{r-1} \). If \( 0 < \beta < \frac{1}{r} \), then we have

\[
\sum_{n=1}^{r} \left\{ \frac{n}{r} + \beta \right\} = \frac{1}{r} \cdot \frac{r(r-1)}{2} + r\beta = \frac{r-1}{2} + \{r\beta\}
\]

which proves the lemma.

LEMMA 3.3. With notation as in Lemma 3.2, for any integer \( N \), we have

\[
\sum_{n=1}^{N} \left\{ \frac{nb}{r} + \beta \right\} = \frac{N}{r} \{r\beta\} + \frac{N-1}{2} (\frac{r-1}{r}) + O(r).
\]

PROOF. We have \( N = Qr + R \) for some \( 0 \leq R < r \). Therefore from Lemma 3.2, we have

\[
\sum_{n=1}^{N} \left\{ \frac{nb}{r} + \beta \right\} = Q\{r\beta\} + Q(\frac{r-1}{2}) + \sum_{n=1}^{R} \left\{ \frac{nb}{r} + \beta \right\} = \frac{N}{r} \{r\beta\} + \frac{N(r-1)}{2r} + O(r).
\]

which proves the lemma.

LEMMA 3.4. We have

\[
\frac{R_{1}(x)}{x} - R_{0}(x) = \frac{x}{2} \sum_{d|x} \frac{h(d)}{d} + \frac{1}{2} \sum_{d \leq x} h(d) - \frac{1}{2x} \sum_{d \leq x} h(d) \cdot d\left(\frac{x}{d}\right) - \left\{\frac{x}{d}\right\}^{2}.
\]
PROOF. We have

\[
R_1(z) = \sum_{n \leq z} n g(n) - M_1(z)
\]

\[
= \sum_{d_1 d_2 \leq z} h(d_1) \cdot d_1 d_2 - M_1(z)
\]

\[
= \sum_{d_1 \leq z} d_1 h(d_1) \cdot \sum_{d_2 \leq \lfloor \frac{z}{d_1} \rfloor} d_2 - M_1(z)
\]

\[
= \frac{1}{2} \sum_{d_1 \leq z} h(d_1) \cdot d_1 (\frac{z}{d_1} - \{ \frac{z}{d_1} \})(\frac{z}{d_1} + 1 - \{ \frac{z}{d_1} \}) - M_1(z)
\]

\[
= \frac{1}{2} \sum_{d \leq z} h(d) \cdot d(\frac{z^2}{d^2} + \frac{z}{d} - 2 \{ \frac{z}{d} \} - \{ \frac{z}{d} \} + \{ \frac{z}{d} \}^2) - M_1(z)
\]

\[
= -\frac{z}{2} \sum_{d \leq z} \frac{h(d)}{d} + \frac{z}{2} \sum_{d \leq z} h(d) - \frac{z}{2} \sum_{d \leq z} h(d)\{ \frac{z}{d} \}
\]

\[
- \frac{1}{2} \sum_{d \leq z} h(d) \cdot d\{ \frac{z}{d} \}^2 + \frac{1}{2} \sum_{d \leq z} h(d)d\{ \frac{z}{d} \}^2.
\]

From Lemma 3.1, we have

\[
- \sum_{d \leq z} h(d)\{ \frac{z}{d} \} = R_0(z) + z \sum_{d > z} \frac{h(d)}{d}
\]

and so we have

\[
\frac{R_1(z)}{z} - R_0(z) = \frac{z}{2} \sum_{d > z} \frac{h(d)}{d} + \frac{1}{2} \sum_{d \leq z} h(d)
\]

\[
- \frac{1}{2} \sum_{d \leq z} h(d)d(\{ \frac{z}{d} \} - \{ \frac{z}{d} \}^2)
\]

which completes the proof.

**Lemma 3.5.** If there exists a function \( G(z) \) such that \( G(z) \) and \( \frac{z}{G(z)} \) are increasing functions of \( z \), then we have

\[
R_0(z) = - \sum_{d \leq y} h(d)\{ \frac{z}{d} \} + O(1) \text{ for } y \geq \frac{z}{G(z)}.
\]

**Proof.** Since \( \sum_{d > y} h(d)\{ \frac{z}{d} \} = O(1) \text{ for } y \geq \frac{z}{G(z)} \), the lemma follows from Lemma 3.1.

**Lemma 3.6.** For integers \( q \approx G(N) \) with \( G(N) \) as in Lemma 3.5,
\[ \beta = q \quad \text{and} \quad y = \frac{(N+1)q}{\alpha(N)}, \quad \text{we have} \]

\[ \sum_{n=1}^{N} R_0(nq + \beta) = N \sum_{\substack{e \in \mathbb{F}^* \quad p | e \Rightarrow p \mid q \quad \text{and} \quad (f, q) = 1}} h(e) \frac{1}{e} \left( \frac{1}{2} - \frac{\beta}{(e, q)} \right) \sum_{f \leq \frac{1}{2}} \frac{h(f)}{f} + O(N) \]

**PROOF.** We have

\[ \sum_{n=1}^{N} R_0(nq + \beta) \]

\[ = -\sum_{n=1}^{N} \sum_{d \leq y} h(d) \left\{ \frac{naq + \beta}{d} \right\} + O(N) \]

(Since \( y = \frac{(N+1)q}{\alpha(N)} > \frac{naq + \beta}{\alpha(nq)} \))

\[ = -\sum_{d \leq y} h(d) \sum_{n=1}^{N} \left\{ \frac{naq + \beta}{d} \right\} + O(N) \]

\[ = -\sum_{d \leq y} h(d) \cdot \sum_{n=1}^{N} \left\{ \frac{q}{d} \left( \frac{d}{d,q} \right) \cdot n + \beta \right\} + O(N) \]

\[ = -\sum_{d \leq y} h(d) \left( \frac{N}{d} (d, q) \left\{ \frac{\beta}{(d,q)} \right\} + \frac{N(d,q)}{d} \left( \frac{d}{d,q} \right) - 1 \right) + O(N) \]

(by Lemma 3.3)

\[ = -\sum_{d \leq y} h(d) \frac{N(d,q)}{d} \left( \left\{ \frac{\beta}{(d,q)} \right\} - \frac{1}{2} \right) + O(N) \]

\[ = -N \sum_{\substack{e \in \mathbb{F}^* \quad p | e \Rightarrow p \mid q \quad \text{and} \quad (f, q) = 1}} h(e) \frac{\left\{ \frac{\beta}{(e,q)} \right\} - \frac{1}{2}}{e} \sum_{f \leq \frac{1}{2}} \frac{h(f)}{f} + O(N) \]

(by writing \( d = ef \) where \( p \mid e \Rightarrow p \mid q \) and \( (f, q) = 1 \))

which proves the lemma.

**LEMMA 3.7.** We have

\[ \sum_{n \leq x} \frac{\alpha(n)}{n} = 2 \log 2 + O\left( \frac{1}{x} \right) \]

where \( \alpha(n) \) is as defined in \$1.\)
PROOF. We have
\[
\sum_{n \leq z} \frac{\alpha(n)}{n} = -\sum_{4n \leq z} \frac{3}{4n} + \sum_{n \leq z} \frac{1}{n} - \sum_{4n \leq z} \frac{1}{4n}
= \sum_{n \leq z} \frac{1}{n} - \sum_{n \leq \lfloor \frac{z}{4} \rfloor} \frac{1}{n}
= 2 \log 2 + O\left(\frac{1}{z}\right)
\]
which proves the lemma.

**Lemma 3.8.** If \( h(n) = \frac{\alpha(n)}{n} \) and so \( g(n) = \frac{r_4(n)}{8n} \), then we have
\[
\frac{R_1(z)}{z} - R_0(z) = O(1).
\]

**PROOF.** Since \( \sum_{d=1}^{\infty} \frac{\alpha(d)}{d} \) is convergent and \( |\sum_{d \leq z} \alpha(d)| \leq 3 \) for all \( z \), the lemma follows from Lemmas 3.4 and 3.7.

**Lemma 3.9.** If \( h(n) = \frac{\alpha(n)}{n} \) and so \( g(n) = \frac{r_4(n)}{8n} \), then we have
\[
R_0(z) = -\sum_{d \leq y} \frac{\alpha(d)}{d} \left\{ \frac{z}{d} \right\} + O(1)
\]
uniformly for \( x \geq 2, y \geq \sqrt{z} \).

**PROOF.** From Lemma 3.1, we have
\[
R_0(z) = -\sum_{d > x} \frac{\alpha(d)}{d} + \sum_{d \leq z} \frac{\alpha(d)}{d} \left\{ \frac{z}{d} \right\}
= -\sum_{d \leq z} \frac{\alpha(d)}{d} \left\{ \frac{z}{d} \right\} + O(1)
\]
(Since \( \sum_{d > x} \frac{\alpha(d)}{d} = O\left(\frac{1}{x}\right)\)).

So it is enough to show that
\[
\sum_{y \leq d \leq z} \frac{\alpha(d)}{d} \left\{ \frac{z}{d} \right\} = O(1) \text{ for } \sqrt{z} \leq y \leq z.
\]
We choose \( k \) such that \( 1 \leq k \leq \frac{y}{z} \) and in \( \frac{k}{k+1} < d \leq \frac{k}{k} \), the function \( \left\{ \frac{x}{d} \right\} \) is monotone. Therefore we have
\[
\sum_{\frac{k}{k+1} < d \leq \frac{k}{k}} \frac{\alpha(d)}{d} \left\{ \frac{z}{d} \right\} = O\left(\frac{k}{z}\right) \text{ (by Lemma 3.7)}
\]
Now summing up for \( k \) in \( 1 \leq k \leq \frac{y}{\sqrt{y}} \), we get
\[
\sum_{\frac{x}{y} \leq d \leq x} \alpha(d) \frac{1}{d} = O\left( \frac{1}{x} \sum_{1 \leq k \leq \frac{y}{\sqrt{y}}} k \right) = O\left( \frac{1}{x} \cdot \frac{y^{2}}{y^{2}} \right) = O(1) \text{ (since } y \geq \sqrt{x})
\]
which proves the lemma.

§ 4. PROOF OF THEOREM 1.

We take \( h(n) = \frac{\alpha(n)}{n} \). Therefore from Lemma 3.6 and 3.9 we have
\[
\sum_{n=1}^{N} R_{0}(nq + \beta) = N \sum_{\frac{e}{y} \leq \frac{x}{y} \leq x} \frac{\alpha(e) (e, q)}{e} \left( \frac{1}{2} - \{ \frac{\beta}{(e, q)} \} \right) \sum_{f \leq \frac{y}{\sqrt{y}}} \frac{\alpha(f)}{f^{2}} + O(N) \tag{4.1}
\]
for \( q \simeq \sqrt{N}, \beta \leq q \) and \( y = \frac{(N+1)^{y}}{\sqrt{N}} (= O(N)) \). Since
\[
\sum_{f \geq \frac{y}{\sqrt{y}}} \frac{\alpha(f)}{f^{2}} = O\left( \sum_{f \geq \frac{y}{\sqrt{y}}} \frac{1}{f^{2}} \right) = O\left( \frac{e}{y} \right),
\]
we have
\[
N \sum_{e \leq y} \frac{\alpha(e) (e, q)}{e} \left( \frac{1}{2} - \{ \frac{\beta}{(e, q)} \} \right) \sum_{f \geq \frac{y}{\sqrt{y}}} \frac{\alpha(f)}{f^{2}} \leq \frac{y}{\sqrt{y}} \sum_{e \leq y} \frac{|(e, q)|}{e} \leq \frac{y}{\sqrt{y}} \sum_{e \leq y} \frac{1}{e} = O(N). \tag{4.2}
\]
Therefore from (4.1) and (4.2) we have
\[
\sum_{n=1}^{N} R_{0}(nq + \beta) = N \left( \prod_{p \mid q} \left( 1 + \frac{\alpha(p)}{p^{2}} + \frac{\alpha(p^{2})}{p^{4}} + \cdots \right) \right) \sum_{e \leq y} \frac{\alpha(e)}{e} (e, q) \left( \frac{1}{2} - \{ \frac{\beta}{(e, q)} \} \right) + O(N) \tag{4.3}
\]
Now, we assume \( q = \prod_{p \leq z} p, \beta = \prod_{2 < p \leq z} p = \frac{z}{2} \) where \( z = \lceil \frac{1}{2} \log N \rceil \). Hence
\( q \simeq \sqrt{N} \). From (4.3), we have
\[
\sum_{n=1}^{N} R_{0}(nq + \beta) = N \left( \prod_{p \mid q} \left( 1 - \frac{1}{p^{2}} \right)^{-1} \right) \zeta(2) \sum_{e \leq y} \frac{\alpha(e)}{e^{2}} (e, q) \left( \frac{1}{2} - \{ \frac{\beta}{(e, q)} \} \right) + O(N). \tag{4.4}
\]
We note that, if $2 \mid e$, then $\frac{1}{2} - \{\frac{\beta}{\{e,q\}}\} = 0$. If $2 \not\mid e$, then $\alpha(e) = 1$ and $\frac{1}{2} - \{\frac{\beta}{\{e,q\}}\} = \frac{1}{2}$. Therefore we have,

$$\sum_{\frac{e}{p} \leq \frac{\alpha(e)}{2}, p \mid \alpha(p)} \frac{\alpha(e)}{e^2} (e, g) \left( \frac{1}{2} - \{\frac{\beta}{\{e,q\}}\} \right) \geq \sum_{\frac{e}{p} \leq \frac{\alpha(e)}{2}, p \mid \alpha(p)} \frac{1}{e^2} (e, g) \cdot \frac{1}{2} \geq \frac{1}{2} \sum_{\frac{e}{p} \leq \frac{\alpha(e)}{2}, p \mid \alpha(p)} \frac{1}{e} \approx \log z \approx \log \log N \quad (4.5)$$

From (4.4) and (4.5), we have

$$\sum_{n=1}^{N} R_0(nq + \beta) \geq N \zeta(2) \cdot \frac{1}{2} \log \log N + O(N)$$

which implies

$$R_0(x) = \Omega_\ast(\log \log x)$$

and hence from Lemma 3.8, we have

$$R_1(x) = \Omega_\ast(x \log \log x)$$

which completes the proof of Theorem 1.

§ 5. PROOF OF THEOREM 2.

We take $h(n) = \frac{\mu(n)}{n}$. From Lemma 3.4, we have

$$\frac{R_1(x)}{x} - R_0(x) = \frac{x}{2} \sum_{d > x} \frac{\mu(d)}{d^2} + \frac{1}{2} \sum_{d \leq x} \frac{\mu(d)}{d} - \frac{1}{2x} \sum_{d \leq x} \mu(d) \left( \frac{2}{d} \right) - \left\{ \frac{x}{d} \right\}^2$$

We consider the sum $\sum_{d \leq x} \mu(d) \left\{ \frac{x}{d} \right\}$. For the range $d \leq xe^{-c\sqrt{\log x}}$, we use trivial estimate and get

$$\sum_{d \leq xe^{-c\sqrt{\log x}}} \mu(d) \left\{ \frac{x}{d} \right\} \ll xe^{-c\sqrt{\log x}}$$
For $z \geq d \geq ze^{-c\sqrt{\log z}}$, we notice that $\{\frac{z}{d}\}$ is monotonic in $\frac{z}{k+1} < d \leq \frac{z}{k}$ for any $k$ such that $1 \leq k \leq e^{c\sqrt{\log z}}$. Therefore we get
\[
\sum_{\frac{z}{k+1} < d \leq \frac{z}{k}} \mu(d)\{\frac{z}{d}\} \ll \frac{z}{k} \exp(-c\sqrt{\log \frac{z}{k}}) \\
\ll \frac{z}{k} \exp(-c_1 \sqrt{\log z})
\]

Now summing up over all $k$'s in $1 \leq k \leq \exp(c\sqrt{\log z})$, we get
\[
\sum_{1 \leq k \leq \exp(c\sqrt{\log z})} \mu(d)\{\frac{z}{d}\} \ll \exp(-c_2 \sqrt{\log z}).
\]

The sum $\sum_{d \leq z} \mu(d)\{\frac{z}{d}\}^2$ can be treated similarly. Other sums are easy to deal with. Hence the Theorem 2 follows.
REFERENCES


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