On Waring's Problem:

$g(4) < 20$

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§ 1. In [1], I proved that every integer is a sum of atmost 21 biquadrates. The object of this paper is to prove a refinement namely

Theorem : Every positive integer is a sum of not more than twenty biquadrates.

Remarks :

(1) Since 79 is not expressible as a sum of eighteen biquadrates, nineteen, if true, is best possible ; but we are unable to improve the theorem further at present.

(2) All numbers upto 10^{310} are sums of 19 biquadrates. This is proved by Henry Thomas Jr. using extensive numerical calculation ([7], Th. 3.3). Our method yields that all numbers bigger than 10^{700} are sums of 19 biquadrates.

(3) For the history of the problem, we refer the reader to [1]

Our work is based upon the papers of Chen Jing run [2] and Davenport [4]. The extensive computer work necessary was done by Thomas ([6] and [7]) and we have freely borrowed these results.

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§ 2. Notation :

The following notation will be used throughout this paper.

$$e(x) = e^{2\pi i x}$$

$$S_{a,q} = \sum_{x=1}^{q} e\left(\frac{ax^4}{q}\right).$$

Let N ($> 10^{400}$) be a given integer to be represented as a sum of twenty biquadrates.

$$P = [N^{1/4}] ;$$

$$T(\alpha) = \sum_{1 < x < P} e(\alpha x^4)$$

Let m be an integer. (In § 9, we shall choose m = 10 ; In other sections, m can either be 9 or 10).

The singular series S (n)

$$= \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right)$$

The truncated singular series

$$S_1(n) = \sum_{q < P^{1/2}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{S_{a,q}}{q} \right)^m e\left(-\frac{an}{q}\right)$$

$$\Psi(\alpha) = \int_0^P e(\alpha x^4)\, dx$$

The major arc $\mathfrak{m}$ is defined by

$$\left\{ \alpha : \left| \alpha - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q < P^{1/2} \text{ and } (h,q) = 1 \right\}$$

The minor arc m is defined by

$$\left\{ \alpha : \left| \alpha - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q,\ P^{1/2} < q < 8P^3 \text{ and } (h,q) = 1 \right\}$$

$$W_0(N_0) = \int_{\mathfrak{m}} (T(\alpha))^m e(-\alpha N_0)\, d\alpha$$

$$W(N_0) = \int_0^1 (T(\alpha))^m e(-\alpha N_0)\, d\alpha$$

$$R(N_0) = \int_{-\infty}^{\infty} (\psi(\alpha))^m \, e(-\alpha N_0)\, d\alpha$$

Each section contains a proposition, which is the main result of the section. Lemmas are subsidiary results needed to prove the proposition.

§ 3. An upper bound for $S_{a,q}$

We recall that $S_{a,q} = \sum_{x=1}^{q} e\left(\frac{ax^4}{q}\right)$. We define

$$S(q) = \max_{\substack{a \\ (a,q)=1}} (S_{a,q}).$$

In this section, we prove

Prop. 1 : *There holds* $|S(q)| < (4.3)q^{3/4}$

In order to prove fhe proposition, we need

Lemma 1 (a) $S(q)$ *is a multiplicative function of q*

(b) *If* $p \neq 2$, *and* $a \not\equiv 0 \pmod p$,

$$|S_{a,p}| < (\delta - 1)\, p^{1/2}$$

where $\delta = (4, p-1)$

(c) If $p \neq 2$, $\left| S_{a,p^\nu} \right| = p^{\nu - 1}$

if $2 < \nu < 4$

(d) If $p > 2$, $\left| S_{a,p^\nu} \right| = p^3 \left| S_{a,p^{\nu-4}} \right|$

if $\nu > 4$

Proof :

A proof can be found in Davenport [3] (Lemma 6(page 31) Lemma 12 (page 42), Lemma 13 (page 43) and Lemma 14 (page 44))

From Lemma 1, it follows that

$$\frac{S(q)}{q^{3/4}} = \prod_{\substack{\alpha \\ p \| q}} \frac{S(p^{\alpha})}{p^{\frac{3}{4}\alpha}}$$

$$< \prod_{p} \max \left(1, \max_{\alpha > 0} \frac{S(p^{\alpha})}{p^{\frac{3}{4}\alpha}} \right)$$

which being a finite product can be evaluated and this gives the proposition. For further details, we refer the reader to Theorem 2.1 in page 38 of Thomas [6].

§ 4. A lower bound for $R(N_0)$

Let us recall that

$$\Psi(\alpha) = \int_0^P e(\alpha x^4)\, dx$$

$$R(N_0) = \int_{-\infty}^{\infty} (\Psi(\alpha))^m\, e(-\alpha N_0)\, d\alpha$$

$$W(N_0) = \int_0^1 (T(\alpha))^m\, e(-\alpha N_0)\, d\alpha$$

Define $B = B(\alpha) = \begin{cases} P & \text{if } |\alpha| < P^{-4} \\ 2|\alpha|^{-1/4} & \text{if } |\alpha| > P^{-4} \end{cases}$

Lemma 2 :

Let f(x) *be a real function which is twice differentiable in* $A \leq x \leq B$ *; suppose that, in the interval* $A \leq x \leq B$, *we have* $0 \leq f'(x) \leq \frac{1}{2}$ *and* $f''(x) > 0$. *Then*

$$\sum_{A < n \leq B} e(f(n)) = \int_A^B e(f(x))dx + 4\theta$$

Proof :

This is Lemma 13 (page 34) in Vinogradov [9]

Lemma 3

We have $|\Psi(\alpha)| \leq B(\alpha)$

Proof :

Clearly $|\Psi(\alpha)| \leq P$.

To prove that $|\Psi(\alpha)| \leq 2|\alpha|^{-1/4}$, it suffices to prove the result for $\alpha > 0$. Now a change of variable $\alpha x^4 = y$ transforms the integral to

$$\Psi(\alpha) = \frac{1}{4\alpha^{1/4}} \int_0^{\alpha P^4} \frac{e(y)\,dy}{y^{3/4}}$$

and the result is immediate.

Lemma 4 :

If $N_o - p^{3\frac{3}{4}} \leq N_1 \leq N_0$, *then*

$$W(N_1) = R(N_0) + \int_{\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} (T(\alpha))^m e(\alpha N_1) d\alpha$$

$$+ \theta\, 10^6\, p^{m-5+3/4}$$

Proof :

By Lemma 2, if $|\alpha| < \frac{1}{8P^3}$, we have

$$T(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^4) = \int_o^P e(\alpha x^4)\, dx + 4\theta$$

$$= \psi(\alpha) + 4\theta$$

Hence by Lemma 3,

$$|(T(\alpha))^m - (\psi(\alpha))^m| \leq 4m(\max(T(\alpha), \psi(\alpha)))^{m-1}$$

$$\leq 4m(B+4)^{m-1}$$

$$W(N_1) = \int_o^1 (T(\alpha))^m e(-\alpha N_1) d\alpha$$

$$= \int_{-\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_1) d\alpha$$

$$= \int_{|\alpha| \leq \frac{1}{8P^3}} \cdots\cdots + \int_{-\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} \cdots\cdots$$

In the first integral, we replace $(T(\alpha))^m$ by $(\Psi(\alpha))^m$ with an error E_1. Then we replace $e(-\alpha N_1)$ by $e(-\alpha N_o)$ with an error E_2. Now we extend the range of integration to $[-\infty, \infty]$ with an error E_3. Hence

$$W(N_1) = \int_{-\infty}^{\infty} (\Psi(\alpha))^m e(-\alpha N_o)\, d\alpha$$

$$= \int_{-\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_1)\, d\alpha + E_1 + E_2 + E_3$$

$$= R(N_o) + \int_{\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_1) d\alpha$$

$$+ E_1 + E_2 + E_3.$$

Now $\left| E_1 \right| \leq$

$$\left| \int_{|\alpha| \leq \frac{1}{8P^3}} \left((T(\alpha))^m - (\Psi(\alpha))^m\right) e(-\alpha N_1)\, d\alpha \right|$$

$$\leq \left| \int_{|\alpha| \leq \frac{1}{8P^3}} (4m) \quad (B + 4)^{m-1} \, d\alpha \right|$$

$$\leq \left| \int_{|\alpha| \leq P^{-4}} (4m) \quad (P + 4)^{m-1} \, d\alpha \right|$$

$$+ \left| \int_{\frac{1}{8P^3} \geq |\alpha| \geq P^{-4}} (4m) \, (2 \, |\alpha|^{-\frac{1}{4}} + 4)^{m-1} \, d\alpha \right.$$

$\leq 10^5 \; P^{m-5}$, since m is either 9 or 10

Similarly,

$$\left| E_2 \right| \leq \left| \int_{|\alpha| < \frac{1}{8P^3}} (\psi(\alpha))^m \left| e(-\alpha N_1) - e(-\alpha N_o) \right| d\alpha \right|$$

$$\leq \int_{|\alpha| \leq \frac{1}{8P^3}} B^m (2\pi) \, |\alpha| \, |N_1 - N_0| \; d\alpha$$

$$\leq P^{3\frac{3}{4}} \; (2\pi) \int_{|\alpha| \leq P^{-4}} P^m \, |\alpha| \, d\alpha \; +$$

$$+ P^{3\frac{3}{4}} (2\pi) \int_{|\alpha| > P^{-4}} (2 \; |\alpha|^{-\frac{1}{4}})^{m} |\alpha| \, d\alpha$$

$$< 10^{5} P^{m-5+\frac{3}{4}}$$

$$\left| E_3 \right| < \int_{|\alpha| > \frac{1}{8P^3}} \left| \psi(\alpha) \right|^{m} d\alpha$$

$$< \int_{|\alpha| > \frac{1}{8P^3}} B^{m} \, d\alpha$$

$$\leq 10^{5} P^{\frac{3}{4} m - 3}$$

This proves the result.

Lemma 5 :

With $M = \left[\frac{P^{3\frac{3}{4}}}{2} \right]$

$$\sum_{1 \leq A \leq M} \; \sum_{1 \leq B \leq M} W(N_0 - A - B) = M^{2} R(N_0)$$

$$+ \theta \, 10^{7} M^{2} P^{m-5+\frac{3}{4}}$$

Proof :

In lemma 4, we take $N_1 = N_o - A - B$ and sum over $1 \leq A \leq M$ and $1 \leq B \leq M$. This gives

$$\sum_A \sum_B W(N_o - A - B) = M^2 R(N_o)$$

$$+ \int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} (T(\alpha))^m e(-\alpha N_0) \left(\sum_A e(\alpha A)\right)^2 d\alpha$$

$$+ \theta\, 10^6 M^2 P^{m - 5 + \frac{3}{4}}$$

Since $\left| \sum_A e(\alpha A) \right| \leq \frac{1}{\|\alpha\|}$, the integral appearing on the right, is bounded, in absolute value, by

$$\int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} P^m \left(\frac{1}{\|\alpha\|} \right)^2 d\alpha \leq 10^4 P^{m+3}$$

Lemma 6

We have

$$\sum_{1 \leq A \leq M} \sum_{1 \leq B \leq M} W(N_0 - A - B)$$

$$> T_m \left(\frac{m-1}{4} \right) M^2 N_0^{\frac{m}{4} - 1}$$

where $$T_m = \frac{(\Gamma(5/4))^m}{\Gamma(1+\frac{m}{4})}$$

Proof :

By lemma 3 (page 22) of Vinogradov [9], the number of integer solutions $K_r(N)$ of $x_1^4 + x_2^4 + \dots\dots + x_r^4 \leq N$ is given by

$$K_r(N) = T_r N^{\frac{r}{4}} - \theta r N^{\frac{r-1}{4}}$$ where

$$T_r = \frac{(\Gamma(5/4))^r}{\Gamma(1+\frac{r}{4})}$$

Now,

$$\sum_{1 \leq B \leq M} W(N_0 - A - B)$$

$$= \sum_B (K_m(N_0 - A - B) - K_m(N_0 - A - B - 1))$$

$$= K_m(N_0 - A - 1) - K_m(N_0 - A - M - 1)$$

$$= T_m [(N_0 - A - 1)^{\frac{m}{4}} - (N_0 - A - M - 1)^{\frac{m}{4}}] + 2m\theta N_0^{\frac{m-1}{4}}$$

$$> T_m \frac{m-1}{4} M N_0^{\frac{m}{4}-1}$$

and hence the lemma.

Prop. 2

We have, for m = 9 or 10,

$$R(N_0) > 0.25 \; N_0^{\frac{m}{4} - 1}$$

Proof :

From lemmas 5 and 6, it follows that

$$R\,(N_0) > \left(\frac{T_m\,(m-1)}{4} - 10^{-5} \right) N_0^{\frac{m}{4} - 1}$$

and hence the result.

§ 5 : *A lower bound for $S_1\,(N_0)$:*

Let us recall that S(n) = S(n, m) is given by

$$S\,(n) = \sum_{q=1}^{\infty} \sum_{a=1}^{q} \left(\frac{S_{a,q}}{q} \right)^m e\left(- \frac{an}{q} \right)$$

$$= \sum_q A\,(n;\,q) \quad \text{say}$$

Write $\chi_p\,(n) = \chi_p\,(n;\,m) = \sum_{i=1}^{\infty} A\,(n;\,p^i)$.

Lemma 7 :

A (n ; q) *is a multiplicative function of q and hence*

$$S\,(n) = \prod_p \left(1 + \chi_p\,(n) \right)$$

Proof :

We refer the reader to Lemma 2.11 (page 20) of Vaughan [8]

Lemma 8 :

(a) *For any prime* p, *and natural number* λ, *let* $N_m(p^{\lambda}, n)$ *denote the number of solutions of*

$$x_1^4 + x_2^4 + \cdots\cdots + x_m^4 \equiv n \pmod{p^{\lambda}},$$

$$1 \leq x_i \leq p^{\lambda} \quad \textit{and not all } x_i \equiv 0 \pmod{p}$$

Further set $\gamma = 1$ if p *is odd and* $\gamma = 4$ if $p = 2$. Let p^{4r+s} *exactly divide* n, $0 \leq s \leq 3$.

$$k_0 = \max(4r + s + 1, 4r + \gamma). \quad \text{Then}$$

$A(n, p^{\lambda}) = 0$ *for* $\lambda > k_0$ *and*

$$1 + X_p(n, m) = p^{-(m-1)\gamma} N_m(p^{\gamma}, 0) \left\{ \sum_{\tau=0}^{r-1} p^{\tau(4-m)} \right\}$$

$$+ p^{(4-m)r - (m-1)\gamma} N_m(p^{\gamma}, np^{-4r})$$

where the empty sum is understood to be zero.

(b) *Let* d = g.c.d. of $(4, p-1)$. *Then for* $p \neq 2$,

$$\left| N_m(p, n) - p^{m-1} \right| \leq \left(1 - \frac{1}{p}\right)(d-1)^m \, p^{\frac{m}{2}}$$

Proof :

For the proof of Lemma 8 (a), we refer the reader to Hilfsatz 293 of LANDAU [5] or Prop. 4. 3 of Thomas [6]. For the proof of lemma 8 (b), we refer to lemma 4 1 (page 91) of Thomas [6].

From Lemma 8, we have

Lemma 9 :

Let $K > 100$; $S_2 = \{ p > K;\ p \equiv 3 \pmod 4 \}$

$S_4 = \{ p > K;\ p \equiv 1 \pmod 4 \}$. *Then for* $d = 2$ *or* 4

$$\prod_{p \in S_d} (1 + \chi_p(n))$$

$$> \exp\left(-2\,(d-1)^m\, K^{1-\frac{m}{2}} \left(1 + \frac{K}{2m-8}\right)\right)$$

Lemma 10 :

We have $\prod_{p > 114} (1 + \chi_p(n)) > 0.97$

Since $\prod_{p > 114} (1 + \chi_p(n))$

$$= \prod_{\substack{p > 127 \\ p \equiv 3 \pmod 4}} (1 + \chi_p(n)) \prod_{\substack{p > 137 \\ p \equiv 1 \pmod 4}} (1 + \chi_p(n))$$

the result follows from Lemma 9.

Lemma 11 :

We have $\prod_{2 < p < 113} (1 + \chi_p(n)) > 0.297.$

Proof :

This is easily verified using Lemma 8 (a). For details, we refer the reader to Thomas [6]

Lemma 12 :

If $n \equiv \tilde{n} \pmod{16}$, $2 < \tilde{n} < m-2$, *then*

$$1 + X_2(n) = \binom{m}{\tilde{n}} 2^{-m+4}$$ *and hence*

$$1 + X_2(n) > \frac{45}{64}$$

Here $\binom{m}{\tilde{n}}$ *is the binomial coefficient*

Proof :

This is straight forward. For details, we refer the reader to Theorem 4. 2 of Thomas [6].

Lemma 13 :

We have $S(n) > 0.2025$

provided $n \equiv 2, 3, 4, 5, 6$ or $7 \pmod{16}$

Proof :

This follows from lemmas 7, 10, 11 and 12.

Lemma 14 :

We have $S(n) - S_1(n) < 0.002$

Proof :

Since $|S(n) - S_1(n)| =$

$$\left| \sum_{q > p^{\frac{1}{3}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{S_{a,q}}{q} \right)^m e\left(- \frac{an}{q} \right) \right|$$

$$< \sum_{q > P^{\frac{1}{3}}} \sum_{a=1}^{q} \left(4.3\, q^{-\frac{1}{4}} \right)^m,$$

we are through.

Prop 3 :

There holds the inequality

$$S_1(n) > 0.2$$

if $n \equiv l \pmod{16}$, and $l \in (2, 3, 4, 5, 6, 7)$.

§ 6 : *A lower bound for* $W_0(N_0)$:

Let us recall that $W_0(N_0) =$

$$\int_{m} (T(\alpha))^m \, e(-\alpha N_0)\, d\alpha.$$

Lemma 15 :

If $|\beta| < \dfrac{1}{8q\,P^3}$ *, then*

$$(T(\frac{a}{q} + \beta))^m = (\psi(\beta)\,\frac{S_{a,q}}{q})^m$$

$$+ \theta.\ 4qm\,(4.3\ B\ (\beta) q^{-\frac{1}{4}} + 4q)^{m-1}$$

Proof :

We have,

$$T\left(\frac{a}{q} + \beta\right) = \sum_{x=1}^{P} e\left(\left(\frac{a}{q} + \beta\right) x^4\right)$$

$$= \sum_{y=0}^{q-1} \sum_{-\frac{y}{q} < t < \frac{P-y}{q}} e\left(\left(\frac{a}{q} + \beta\right)(qt + y)^4\right)$$

$$= \sum_{y} \sum_{t} e\left(\frac{ay^4}{q} + \beta\ (qt + y)^4\right)$$

$$= \sum_{y} e\left(\frac{ay^4}{q}\right)\ D_y\ (z), \text{ say}$$

Now, using Lemma 2,

$$D_y(z) = \int_{-\frac{y}{q}}^{\frac{P-y}{q}} e(\beta(qt+y)^4)\,dt + 4\theta$$

$$= \frac{1}{q}\int_0^P e(\beta x^4)\,dx + 4\theta$$

$$= \frac{1}{q}\,\Psi(\beta) + 4\theta.$$

Hence $T\left(\frac{a}{q}+\beta\right) = \frac{S_{a,q}}{q}\,\Psi(\beta) + 4\theta q$

The result follows from the following inequalities.

$$\left|\frac{S_{a,q}}{q}\right| < (4.3)\, q^{-\frac{1}{4}}$$

$|\Psi(\beta)| < B(\beta)$ and

$$a^m - b^m < m(a-b)\max(|a|^{m-1}, |b|^{m-1})$$

Lemma 16 :

We have

$$W_0(N_0) = S_1(N_0)\,R(N_0) + \theta\, 10^{13}\, P^{m-\frac{9}{2}}$$

Proof :

Since the proof is similar to that of Lemma 4, we give only the sketch of the proof.

$$W_0(N_0) = \int_{m} (T(\alpha))^m e(-\alpha N_0)\, d\alpha$$

$$= \sum_{q \leq P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\frac{a}{q} - \frac{1}{8qP^3}}^{\frac{a}{q} + \frac{1}{8qP^3}} (T(\alpha))^m e(-N_0 \alpha)\, d\alpha$$

$$= \sum_{q \leq P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-\frac{1}{8qP^3}}^{\frac{1}{8qP^3}} \left(T\left(\frac{a}{q} + \beta\right)\right)^m e\left(-N_0\left(\frac{a}{q} + \beta\right)\right) d\beta$$

First we replace $\left(T\left(\frac{a}{q} + \beta\right)\right)^m$ by $\left(\psi(\beta)\ \frac{S_{a,q}}{q}\right)^m$ and the error is, by Lemma 15, at most $10^{12}\ P^{m-9/2}$. Now we extend the range of integration of the integral to $[-\infty, \infty]$ which gives an error at most $10^{12}\ P^{m-9/2}$. Hence

$$W_0(N_0) = \sum_{q < P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-\infty}^{\infty} (\psi(\beta))^m \left(\frac{S_{a,q}}{q}\right)^m e\left(-N_0\left(\frac{a}{q} + \beta\right)\right) d\beta + \theta\, 10^{13}\ P^{m-9/2}$$

$$= S_1(N_0)\, R(N_0) + \theta\, 10^{13}\ P^{m-9/2}$$

Prop 4 : *We have*

$$W_0 (N_0) > 0.05 \; N_0^{\frac{m}{4} - 1}$$

provided $N_0 \equiv 2, 3, 4, 5, 6$ or $7 \pmod{16}$

Proof :

This follows from Prop. 2, Prop. 3 and Lemma 16.

§ 7 : *The estimate on the minor arc :*

We define $f(n) = n^{-a} \prod_{p/n} (1 - p^{-a})^{-1}$ where a (>0) is a constant. Ultimately we shall choose a = 0.1

$$g(n) = \sqrt{f(n)}; \quad h(n) = \sum_{\substack{l/n \\ l \leq P; \; \frac{n}{l} \leq P}} \frac{1}{g(l)}$$

$$C_a = \prod_p \left(1 + \frac{1}{P(P^a - 1)} \right)$$

$$D_a = \prod_p \left(1 + \frac{(1 - P^{-a})^{-\frac{1}{2}} - 1}{P} \right)$$

$$k(m) = \sum_{d/m} \frac{1}{h(d)}$$

Lemma 17

Let $\lambda(n)$ *be a non negative multiplicative function with* $\prod_p \left(1 + \frac{\lambda(p)}{p} \right)$ *convergent. Then*

$$\sum_{n \leq X} \left(n^{-\beta} \prod_{p/n} (1 + \lambda (p))\right) < 2 \prod_{p} \left(1 + \frac{\lambda (p)}{p} \right) X^{1-\beta}$$

if $|\beta| < \frac{1}{2}$

Proof:

$$\sum_{n \leq X} \left(n^{-\beta} \prod_{p/n} (1 + \lambda (p))\right) = \sum_{n \leq X} n^{-\beta} \sum_{d/n} \mu^2 (d) \lambda (d)$$

$$= \sum_{d} \mu^2 (d) \lambda (d) \sum_{\substack{n \leq X \\ n \equiv 0 \pmod d}} n^{-\beta}$$

$$\leq 2 X^{1-\beta} \sum_{d} \frac{\mu^2 (d) \lambda (d)}{d}$$

$$< 2 X^{1-\beta} \prod_{p} \left(1 + \frac{\lambda (p)}{p} \right)$$

Lemma 18

We have

(a) $\sum_{n \leq X} f(n) \leq 2 C_a X^{1-a}$

(b) $\sum_{n \leq X} g(n) \leq 2 D_a X^{1-\frac{a}{2}}$

(c) $\sum_{n \leq X} g(n) n^{a/2} \leq 2 D_a X$

(d) $\sum_{n \leq X} \frac{1}{g (n)} < X^{1 + \frac{a}{2}}$

(d) $\sum_{n \leq X} \frac{1}{f (n)} < X^{1+a}$

Proof:

(a), (b), and (c) follow from Lemma 17 by the proper choice of $\lambda(n)$ and β; since $\frac{1}{g(n)} \leq n^{a/2}$ and $\frac{1}{f(n)} \leq n^{a}$, (d) and (e) follow.

Lemma 19: *We have*

$$\sum_{n \leq P^2} f(n)\,(h(n))^2 \leq 20P^{2-a}\, C_a\, D_a^2 \,(\log P)$$

Proof :

$$\sum_{n \leq P^2} f(n)\, h^2(n) = \sum_{n \leq P^2} f(n) \Big(\sum_{\substack{l/n \\ l \leq P,\ \frac{n}{l} \leq P}} \frac{1}{g(l)} \Big)^2$$

$$< 2 \sum_{l_1 \leq P} \frac{1}{g(l_1)} \sum_{l_2 \leq l_1} \frac{1}{g(l_2)} \sum_{\substack{n \leq P^2 \\ n \equiv 0 \,(\mathrm{mod}\, [l_1, l_2])}} f(n)$$

$$< 2 \sum_{l_1} \frac{1}{g(l_1)} \sum_{l_2 \leq l_1} \frac{1}{g(l_2)} \sum_{m \leq \frac{P^2}{[l_1, l_2]}} f(m)\, f([l_1, l_2])$$

$$< 4 C_a \sum_{l_1} \frac{1}{g(l_1)} \sum_{l_2} \frac{f([l_1, l_2])}{g(l_2)} \left(\frac{P l_2}{[l_1, l_2]} \right)^{1-a}$$

$$< 4 C_a P^{1-a} \sum_{d} \sum_{(l_1, l_2) = d} \frac{1}{g(l_1)} \frac{g^2 \left(\frac{l_1 l_2}{d} \right)}{g(l_2)} \left(\frac{d}{l_1} \right)^{1-a}$$

$$< 4 C_a P^{1-a} \sum_{d} \sum_{(l_1, l_2) = d} \frac{1}{g(l_1) g(l_2)} \left(\frac{g(l_1) g(l_2)}{g(d)} \right)^2 \frac{d^{1-a}}{l_1^{1-a}}$$

$$< 4 C_a P^{1-a} \sum_{d} \frac{d^{1-a}}{g^2(d)} \sum_{l_1 \equiv 0 \pmod d} \frac{g(l_1)}{l_1^{1-a}} \sum_{\substack{l_2 \equiv 0 \pmod d \\ l_2 < l_1}} g(l_2)$$

$$< 8 D_a P^{1-a} \sum_{d} \frac{1}{d^{a/2} g(d)} \sum_{\substack{l_1 \leq P \\ l_1 \equiv 0 \pmod d}} g(l_1) l_1^{a/2}$$

$$< 16 C_a D_a^2 P^{2-a} \sum_{d} \frac{1}{d}$$

$$< 20\, C_a\, D_a^2\, P^{2-a} (\log P).$$

Lemma 20 :

If $f(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \dots \dots \alpha_0$ *and*

$g_l(x) = f(x+l) - f(x) = lk\, \alpha_k x^{k-1} + \dots,$

then
$$\left| \sum_{x=1}^{P} e(f(x)) \right|^2 <$$
$$2 \sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e(g_l(x)) \right| + P$$

Proof :

$$\left| \sum_{x=1}^{P} e(f(x)) \right|^2 = \sum_x \sum_y e(f(y) - f(x))$$

$$< \left| \sum_{x} \sum_{\substack{y \\ x=y}} e(f(y) - f(x)) \right|$$
$$+ 2 \left| \sum_{x} \sum_{\substack{y \\ y > x}} e(f(y) - f(x)) \right|$$

We write $y = x + l$ in the second sum and simplify. This gives the result.

Lemma 21 :

If $\left| \alpha - \frac{h}{q} \right| < \frac{1}{8qP^3}$ *and* $P < q < 8P^3$, *then*

$$|T(\alpha)| < 4\, C_a^{1/8}\, D_a^{1/2}\, P^{\frac{7+a}{8}}\, (\log P)^{\frac{1}{4}}$$

Proof :

By Lemma 20,

$$|T(\alpha)|^2 < 2 \sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e(4\alpha l x^3 + \ldots) \right| + P$$

Hence

$$T_1(\alpha) = |T(\alpha)|^2 - P$$

$$< 2 \sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e(4\alpha l x^3 + \ldots) \right|$$

$$|T_1(\alpha)|^2 < 4 \left(\sum_{l=1}^{P} g(l) \right) \left(\sum_{l=1}^{P} \frac{1}{g(l)} \left| \sum_{x} e(4\alpha l x^3 + \ldots) \right|^2 \right)$$

$$< 8\, D_a\, P^{1-\frac{a}{2}} \left(\sum_{l=1}^{P} \frac{1}{g(l)} \left| \sum_{x} e(4\alpha l x^3 + \ldots) \right|^2 \right)$$

By Lemma 20 and 19 (d),

$$T_2(\alpha) = \frac{|T_1(\alpha)|^2}{8D_a P^{1-\frac{a}{2}}} \leqslant \sum_{l=1}^{P} \frac{1}{g(l)}$$

$$\left\{ 2 \sum_{l_1=1}^{P-l} \left| \sum_{x=1}^{P-l-l_1} e(12\alpha l l_1 x^2 + \dots) \right| + P \right\}$$

$$\leqslant 2 \sum_{l=1}^{P} \frac{1}{g(l)} \sum_{l_1=1}^{P-l}$$

$$\left| \sum_{x=1}^{P-l-l_1} e(12\alpha l l_1 x^2 + \dots) \right| + P^{2+a}$$

$$T_3(\alpha) = T_2(\alpha) - P^{2+a} \leqslant 2 \sum_{l, l_1} \frac{1}{g(l)}$$

$$\left| \sum_{x=1}^{P-l-l_1} e(12\alpha l l_1 x^2 + \dots\dots) \right|$$

$$\leqslant 2 \sum_{n \leqslant \frac{P^2}{4}} h(n)$$

$$\max_{l l_1 = n} \left| \sum_{x=1}^{P-l-l_1} e(12\alpha l l_1 x^2 + \dots\dots) \right|$$

$$|T_3(\alpha)|^2 < 4\left(\sum_n f(n)\, h^2(n)\right)\left(\sum_n \frac{1}{f(n)} \max_{\ell\ell_1 = n} \left|\sum_x e(12\alpha\ell\ell_1 x^2 + \ldots\ldots)\right|^2\right)$$

$$< \frac{(80 P^{2-a}}{2^{2-a}} C_a D_a^2 \log P)\, (\sum_n \frac{1}{f(n)} \max_{\ell\ell_1 = n} \left|\sum_x e(12\alpha\ell\ell_1 x^2 + \ldots\ldots)\right|^2$$

$$T_4(\alpha) = \frac{2^{2-a} T_3(\alpha)^2}{80 P^{2-a} C_a D_a^2 \log P} < \sum_n \frac{1}{f(n)} \max_{\ell\ell_1 = n} \left|\sum_x e(12\alpha\ell\ell_1 x^2 + \ldots\ldots)\right|^2$$

$$< \sum_n \frac{1}{f(n)} \max_{\ell\ell_1 = n} \left\{ 2 \sum_{\ell_2 = 1}^{P-\ell-\ell_1} \left| \sum_{x=1}^{P-\ell-\ell_1-\ell_2} e(24\alpha\ell\ell_1\ell_2 x + \ldots\ldots) \right| + P \right\}$$

$$< 2 \sum_n \frac{1}{f(n)} \max_{\ell\ell_1 = n} \sum_{\ell_2 = 1}^{P-\ell-\ell_1} \min\left(P, \frac{1}{2\|24\alpha\ell\ell_1\ell_2\|}\right) + P^{3+2a}$$

$$< 2 \sum_{m} k(m) \cdot \min\left(P, \frac{1}{2\|24\alpha m\|}\right) + P^{3+2a}$$

One easily checks thaf $k(m) < m^a$ and $m = l_1 l_2 < \frac{P^3}{27}$;

We also have $a = 0.1$. Hence

$$T_4(\alpha) < 2 \sum_{m < \frac{P^3}{27}} m^a \min\left(P, \frac{1}{2\|24\alpha m\|}\right) + P^{3+2a}$$

$$< 2\left(\frac{P^3}{27}\right)^a \sum_{m < \frac{P^3}{27}} \min\left(P, \frac{1}{2\|24\alpha m\|}\right) + P^{3+2a}$$

For a given α, we know that there exist h and q, $(h, q) = 1$ such that $\left|\alpha - \frac{h}{q}\right| < \frac{1}{8qP^3}$, $q < 8P^3$.

Let $q' = \frac{q}{(24, q)}$. The m-sum is divided into at most $\left(\frac{P^3}{27q'} + 1\right)$ subintervals, each of length q'. In each subinterval, we check that the sum is atmost $2P + q' \log q' + q'$.

Hence

$$|T_4(\alpha)| < 2\frac{P^{0.3}}{(27)^{0.3}}\left(\frac{P^3}{27q'} + 1\right)$$

$$(2P + q' \log q' + q') + P^{3+2a}$$

$$< 25 P^{3.3} \log P$$

Hence $\left| T_3(\alpha) \right|^2 < 10^3 P^{5.2} C_a D_a^2 (\log P)^2$

Hence $\left| T_2(\alpha) \right| < (31.63) P^{2.6} C_a^{\frac{1}{2}} D_a (\log P)$

Hence $\left| T_1(\alpha) \right| < (15.91) P^{\frac{7.1}{4}} C_a^{\frac{1}{4}} D_a (\log P)^{\frac{1}{2}}$

which proves the result.

Lemma 22 :

If $\left| \alpha - \frac{h}{q} \right| < \frac{1}{8\,qP^3}$ *and* $P^{\frac{1}{2}} < q < P,$

then $\left| T(\alpha) \right| < 100\; P^{7/8}.$

Proof :

Arguing as in Lemma 9.5 of Thomas [6] or Lemma 9 of Davenport [3], we have the result. (Lemma 9.5 of Thomas [6] is proved under the restriction $\left| \alpha - \frac{h}{q} \right| < \frac{1}{64qP^3}$ but the same proof holds in our case).

Proof :

On the minor are $\left| T(\alpha) \right| < 40\; P^{\frac{71}{8}} (\log P)^{\frac{1}{4}}$

Proof :

The proposition follows from Lemmas 21 and 22

§ 8 : A lower bound for the number of integers upto X representable as a sum of five fourth powers

Lemma 23 :

Let $P \geq 10^{10}$. *Let* M *be the number of solutions of the equation* $x^4 + u_h = y^4 + u_j$, *subject to* (i) $P < x, y < 2P$ (ii) $x \equiv y \pmod 2$ (iii) $u_h, u_j < P^{3+\mu}$ *and* (iv) $u_h, u_j \in \mathcal{U}$. *Then*

$$M \leq PU + P^{\frac{1+3\mu}{4}} U^{5/4} B.$$

Here $0 < \mu \leq \frac{1}{4}$; $\mathcal{U}$ *is a set of integers in* $[0, P^{3+\mu}]$. U *is the number of elements of* $\mathcal{U}$.

$$B^4 = \tfrac{1}{4} \quad \max \quad d_4(m)$$

$$M \leq \frac{P^{3+\mu}}{192}$$

Proof :

Even though this has not been explicitly stated anywhere, this is essentially contained in Lemma 1 of Davenport [4] and Lemma 7.1 of Thomas [6].

Lemma 24 : *The number of solutions of the equation*

$$a^4 + b^4 = x^4 + y^4, \text{ with } a, b, x, y \leq A;$$

$a \equiv x \pmod 2$; $b \equiv y \pmod 2$ *is atmost* $A^2 (\log A)^4$ *provided* $A > 10^{10}$.

Proof :

We write $x + a = 2k$; $x - a = 2l$; $y + b = 2m$ $y - b = 2n$.

Then we have to find the number of solutions of the equation $kl(k^2 + l^2) = m(m^2 + n^2)$, with $k, m \leq A$; $l, n \leq A/2$. The number of solutions are

$$= \sum_{k, l} \sum_{\substack{m, n \\ mn(m^2 + n^2) = kl(k^2 + l^2)}} 1$$

$$= \sum_{k, l} d(kl(k^2 + l^2))$$

$$\leq \sum \frac{d^2(k) + d^2(l)}{2} d(k^2 + l^2)$$

$$\leq \sum_{k \leq A} d^2(k) \ d(k^2 + l^2)$$

$$\leq 2 \sum_{k \leq A} d^2(k) \sum_{l} \sum_{\substack{j / k^2 + l^2 \\ j \leq 2A}} 1$$

$$\leq 2 \sum_{k \leq A} d^2(k) \sum_{j \leq 2A} \sum_{\substack{k^2 + l^2 \equiv 0 \pmod{j} \\ l \leq A}} 1$$

$$\leq 2 \sum_{k \leq A} d^2(k) \sum_{j \leq 2A} P(j)\left(\frac{A}{j} + 1\right)$$

where $\rho(j)$ is the number of solutions of $k^2 + l^2 \equiv 0 \pmod{j}$, $1 \leq l \leq j$, for a fixed k. We observe that $\rho(j) \leq \sum_{d/j} \chi(d)$,

where χ is the Dirichlet character (mod 4) and hence we easily check that

$$\sum_{j \leq 2A} \frac{\rho(j)}{j} \leq \log A$$

$$\sum_{j \leq 2A} \rho(j) \leq 4A$$

and $$\sum_{k \leq A} d^2(k) \leq \frac{A}{3} \log^3 (A+2)$$

This proves the result.

Lemma 25 :

Let l be 0, 1 or 2. *The number of integers upto X, representable as a sum of two biquadrates and belonging to* l (mod 16) *is atleast* $X^{0.45}$ *provided* $X \geq 10^{188}$.

Proof :

Choose f_0 and $f_1 \in (0, 1)$ such that $f_0 + f_1 = l$.

Let r(m) be the number of solutions of $m = x^4 + y^4$,

$$x, y \leq \left(\frac{X}{2}\right)^{\frac{1}{4}} ; \; x \equiv f_0 \pmod{2} ; \; y \equiv f_1 \pmod{2}$$

Then $$\sum r(m) \geq \left(\left(\frac{X}{2}\right)^{\frac{1}{4}} \tfrac{1}{2} - 1\right)^2$$

By lemma 24, $\sum r^2(m) < \left(\frac{X}{2}\right)^{\frac{1}{2}} (\log X^{\frac{1}{4}})^4$

$$< X^{0.5424}.$$

Hence the required estimate

$$> \sum_{r(m) \neq 0} 1 > \frac{(\sum r(m))^2}{\sum r^2(m)}$$

and we are through.

Lemma 26 :

Let $X > 0$; $P = \left[\frac{X}{17}\right]^{\frac{1}{4}} + 1$; *Let* μ, B *and* U_0 *satisfy the following conditions*

(i) $0 < \mu < \frac{1}{2}$

(ii) $B^4 = \frac{1}{4} \cdot \max_{m < \frac{1}{192} P^{3+\mu}} d_4(m)$

(iii) *The number of integers upto* $P^{3+\mu}$ *which are sums of* l *biquadrates and belonging to* f_0 (mod 16) *is atleast* U_0 *for every* $f_0 \in (0, 1, 2, \ldots\ldots l)$

(iv) $B^4 < U_0 P^{3-3\mu}$.

Then the number of integers upto X which are sum of $(l + 1)$ *biquadrates and belonging to* f (mod 16) *is atleast* $\frac{PU_0}{8}$ for every f $\in (0, 1, 2, \ldots\ldots (l + 1))$.

Proof :

Let $f_0 \in (0, 1, 2, \ldots\ldots l)$ and $f_1 \in (0, 1)$ such that $f_0 + f_1 = f$. Let $\mathcal{U} = \{ n < P^{3+\mu}$: n is sum of l biquadrates and $n \equiv f_0 \pmod{16} \}$; U be the number of elements of $\mathcal{U}$; By (iii), $U > U_0$.

Let r (m) be the number of solutions of $m = x^4 + u_h$, $P < x < 2P$; $x \equiv f_1 \pmod 2$ and $u_h \in \mathcal{U}$. The number N of integers upto X, which are representable as a sum of $(l + 1)$ biquadrates and $\equiv f \pmod{16}$ is atleast

$$\sum_{\substack{m \\ r(m) \neq 0}} 1$$

$$> \left(\sum r(m) \right)^2 / \left(\sum r^2(m) \right)$$

Now, using lemma 23

$$N > \left(\frac{PU}{2} \right)^2 / PU + P^{\frac{1+3\mu}{4}} U^{5/4} B$$

$$> \left(\frac{PU_0}{2} \right)^2 / PU_0 + P^{\frac{1+3\mu}{4}} U_0^{5/4} B.$$

By (iv), $P^{\frac{1+3\mu}{4}} U_0^{5/4} B < PU_0$.

Hence $N > \left(\frac{PU_0}{2} \right)^2 / 2PU_0 > \frac{PU_0}{8}$.

Lemma 27 :

Let f *be* 0, 1, 2, or 3. *Then the number of integers upto* X, *representable as a sum of three biquadrates belonging to* f(mod 16) *is atleast* $X^{0.603}$ *provided* $X > 10^{236}$.

Proof :

We choose $l = 2$; $\mu = 0.20096$ and $U_0 = P^{(3+\mu)(0.45)}$ in lemma 26 ; since $B^4 < 10^{12} P^{(3+\mu)(0.235)}$, we verify (iv) in lemma 26. This gives that the required estimate

$$> PU_0 > \frac{1}{8}\left(\frac{X}{17}\right)^{0.610108} > X^{0.603} .$$

Lemma 28 :

Let f *be* 0, 1, 2, 3 or 4. *Then the number of integers upto* X, *representable as a sum of* 4 *biquadrates and belonging to* f (mod 16) *is atleast* $X^{0.7095}$ *provided* $X > 10^{307}$

Proof :

We choose $l = 3$; $\mu = 0.0869$; $U_0 = P^{(3+\mu)(0.603)}$ in lemma 26. We have to use $B^4 < 10^{14} P^{(3+\mu)(0.225)}$.

This yields the result.

Prop 6 :

Let f *be* 0, 1, 2, 3, 4 or 5. *Then the number of integers upto* X *representable as a sum of* 5 *biquadrates and belonging to* f (mod 16) *is atleast* $X^{0.7795}$ *provided* $X > 10^{410}$

Proof :

We take $l = 4$; $\mu = 0.01133$; and $U_0 = P^{(3+\mu)(0.7095)}$.

We use $B^4 \leq (1.5) \times 10^{23}$. This gives the result.

§ 9. Conclusion of the proof :

Lemma 30 :

If $N > 10^{412}$ *, then* N *can be written as a sum of at most twenty biquadrates.*

Proof :

Choose $f \in (0, 1, 2, 3, 4, 5)$ and

$l \in (2, 3, 4, 5, 6, 7)$ such that

$N - 2f \equiv l \pmod{16}$

$P = [N^{\frac{1}{4}}]$

Let $\mathcal{U} = \Big\{ n \leq \frac{N}{4}$; n is representable as a sum of five biquadrates : $n \equiv f \pmod{16} \Big\}$

Let $U(\alpha) = \sum_{u \in \mathcal{U}} e(u\alpha)$

$$r(N) = \int_0^1 (T(\alpha))^{10} . (u(\alpha))^2 e(-N\alpha)\, d\alpha$$

It suffices to prove that $r(N) > 0$

Now $r(N) = \int_{m} + \int_{\mathfrak{m}}$

Now, using Proposition 4,

$$\int_{m} = \sum_{u_1, u_2} \int_{m} (T(\alpha))^{10} e(-(N - u_1 - u_2)\alpha) d\alpha$$

$$\asymp \sum_{u_1, u_2} W_0(N - u_1 - u_2)$$

$$> 0.01\, U^2 P^6$$

n the other hand, using Proposition 5 and 6,

$$\left| \int_{\mathfrak{m}} (T(\alpha))^{10} (U(\alpha))^2 e(-N\alpha))) d\alpha \right|$$

$$< \max_{\alpha \in \mathfrak{m}} \left| T(\alpha)) \right|^{10} \int_0^1 \left| u(\alpha) \right|^2 d\alpha$$

$$< (40)^{10} P^{\frac{71}{8}} . (\log P)^{2.5} U$$

$$< (40)^{10} P^{\frac{71}{8}} . (\log P)^{2.5} U^2 / \left(\frac{N}{4}\right)^{0\ 7795}$$

$$\leq 4^{11}\ 10^{10}\ P^{5.757}\ U^2 (\log P)^{2.5}$$

It suffices to to verify that, for $P \leqslant 10^{102}$,

$$4^{11} \; 10^{10} \; P^{5.757} \, (\log P)^{2.5} < 0.01 \, P^{6}$$

and this is true.

Lemma : 31

If $N < 10^{412}$, *then* N *can be written as a sum of atmost 20 biquadrates.*

Proof :

If $N \leqslant 10^{412}$, then $N - [N^{\frac{1}{4}}]^{4} < 10^{310}$ and hence representable as a sum of atmost 19 biquadrates by Th. 3.3 of Thomas [7]

Thus the proof of the theorem is complete.

Note :

Some doubts have been raised about the correctness of lemma 31 and the author has not verified the calculations given in [7]. But even a weaker result, compared to lemma 31, will suffice for our purpose and we shall return to this subject at a later date.

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