On Waring’s Problem: \( g(4) \leq 20 \).
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On Waring's Problem:

g(4) < 20

R. BALASUBRAMANIAN

§ 1. In [1], I proved that every integer is a sum of atmost 21 biquadrates. The object of this paper is to prove a refinement namely

Theorem: Every positive integer is a sum of not more than twenty biquadrates.

Remarks:

(1) Since 79 is not expressible as a sum of eighteen biquadrates, nineteen, if true, is best possible; but we are unable to improve the theorem further at present.

(2) All numbers upto $10^{310}$ are sums of 19 biquadrates. This is proved by Henry Thomas Jr. using extensive numerical calculation ([7], Th. 3.3). Our method yields that all numbers bigger than $10^{700}$ are sums of 19 biquadrates.

(3) For the history of the problem, we refer the reader to [1]

Our work is based upon the papers of Chen Jing run [2] and Davenport [4]. The extensive computer work necessary was done by Thomas ([6] and [7]) and we have freely borrowed these results.
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§ 2. Notation:

The following notation will be used throughout this paper.

\[ e(x) = e^{2\pi i x} \]

\[ S_{a, q} = \sum_{x=1}^{q} e\left(\frac{ax^4}{q}\right) \]

Let \( N (> 10^{400}) \) be a given integer to be represented as a sum of twenty biquadrates.

\[ P = \lfloor N^{1/4} \rfloor \]

\[ T(a) = \sum_{1 \leq x < P} e(ax^4) \]

Let \( m \) be an integer. (In § 9, we shall choose \( m = 10 \); in other sections, \( m \) can either be 9 or 10).
The singular series \( S(n) \)

\[
{\sum_{q = 1}^{\infty} \sum_{a = 1}^{q} (S_{a, q})^m \left( -\frac{an}{q} \right)}
\]

\((a, q) = 1\)

The truncated singular series

\[
S_1(n) = {\sum_{q < P^{1/2}} \sum_{a = 1}^{q} (S_{a, q})^m \left( -\frac{an}{q} \right)}
\]

\((a, q) = 1\)

\(\Psi(d) = \int_0^P e(d x^4) \, dx\)

The major arc \( m \) is defined by

\[
\{ d : \left| d - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q < P^{1/2} \text{ and } (h, q) = 1 \}
\]

The minor arc \( m \) is defined by

\[
\{ d : \left| d - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q, P^{1/2} < q < 8P^3 \\
\text{and } (h, q) = 1 \}
\]

\[
W_0(N_0) = \int_m (T(d))^m e(-dN_0) \, dd
\]

\[
W(N_0) = \int_0^1 (T(d))^m e(-dN_0) \, dd
\]
Each section contains a proposition, which is the main result of the section. Lemmas are subsidiary results needed to prove the proposition.

§ 3. An upper bound for $S_{a, q}$

We recall that $S_{a, q} = \sum_{x=1}^{q} e\left(\frac{ax}{q}\right)$. We define

$$S(q) = \max_{a, q} \left( S_{a, q} \right).$$

$(a, q) = 1$

In this section, we prove

**Prop. 1**: There holds $|S(q)| \leq (4.3) q^{3/4}$

In order to prove the proposition, we need

**Lemma 1**

(a) $S(q)$ is a multiplicative function of $q$

(b) If $p \neq 2$, and $a \neq 0 \pmod{p}$,

$$|S_{a, p}| \leq (\delta - 1) p^{1/2}$$

where $\delta = (4, p-1)$

(c) If $p \neq 2$,

$$|S_{a, p}^\nu| = p^{\nu - 1}$$

if $2 \leq \nu < 4$

(d) If $p > 2$,

$$|S_{a, p}^\nu| = p^3 |S_{a, p}^\nu - 4|$$

if $\nu > 4$
Proof:

A proof can be found in Davenport [3] (Lemma 6 (page 31), Lemma 12 (page 42), Lemma 13 (page 43) and Lemma 14 (page 44)).

From Lemma 1, it follows that

\[
\frac{S(q)}{q^{3/4}} = \frac{\pi}{p \parallel q} \frac{S(p^d)}{p^3 d} < \pi_{\max} \left( 1, \frac{\max}{d > 0} \frac{S(p^d)}{p^3 d} \right)
\]

which being a finite product can be evaluated and this gives the proposition. For further details, we refer the reader to Theorem 2.1 in page 38 of Thomas [6].

§ 4. A lower bound for \( R(N_0) \)

Let us recall that

\[
\Psi (d) = \int_0^p e (dx^4) dx
\]

\[
R(N_0) = \int_0^\infty (\Psi(d))^m e(-d N_0) d\lambda
\]

\[
W(N_0) = \int_0^1 (T(d))^m e(-d N_0) d\lambda
\]

Define \( B = B(d) = \begin{cases} P & \text{if } |d| < P^{-4} \\ 2 |d|^{-1/4} & \text{if } |d| > P^{-4} \end{cases} \)
Lemma 2:

Let \( f(x) \) be a real function which is twice differentiable in \( A < x < B \); suppose that, in the interval \( A < x < B \), we have \( 0 < f'(x) < \frac{1}{4} \) and \( f''(x) > 0 \). Then

\[
\int_{A}^{B} e(f(n)) = \int_{A}^{B} e(f(x)) dx + 4\theta
\]

Proof:

This is Lemma 13 (page 34) in Vinogradov [9]

Lemma 3

We have \( |\Psi(\alpha)| < B(\alpha) \)

Proof:

Clearly \( |\Psi(\alpha)| < P. \)

To prove that \( |\Psi(\alpha)| < 2 |\alpha|^{-1/4} \), it suffices to prove the result for \( \alpha > 0 \). Now a change of variable \( dx^4 = y \) transforms the integral to

\[
\Psi(\alpha) = \frac{1}{4\alpha^{1/4}} \int_{0}^{\alpha^{1/4}} \frac{e(y) dy}{y^{3/4}}
\]

and the result is immediate.

Lemma 4:

If \( N_0 - p^{3/4} \leq N_1 < N_0 \), then
\[ W(N_1) = R(N_0) + \int \frac{1}{8p^3} (T(\alpha))^m e^{-\alpha N_1} d\alpha \]

\[ = \frac{1}{8p^3} + \theta 10^6 p^{m-5+3/4} \]

**Proof:**

By Lemma 2, if \( |\alpha| \leq \frac{1}{8p^3} \), we have

\[ T(\alpha) = \sum_{1 \leq x \leq p} e(\alpha x^4) = \int_0^p e(\alpha x^4) \, dx + 4\theta \]

\[ = \psi(\alpha) + 4\theta \]

Hence by Lemma 3,

\[ |(T(\alpha))^m - (\psi(\alpha))^m| \leq 4m (\max(T(\alpha), \psi(\alpha)))^{m-1} \]

\[ \leq 4m (B + 4)^{m-1} \]

\[ W(N) = \int_0^1 (T(\alpha))^m e^{-\alpha N_1} d\alpha \]

\[ = \int_0^1 \frac{1}{8p^3} (T(\alpha))^m e^{-\alpha N_1} d\alpha \]

\[ = \int_0^1 \frac{1}{8p^3} (T(\alpha))^m e^{-\alpha N_1} d\alpha \]
In the first integral, we replace \((T(\alpha)) \cdot m\) by \((\psi(\alpha)) \cdot m\) with an error \(E_1\). Then we replace \(e(-dN_1)\) by \(e(-dN_0)\) with an error \(E_2\). Now we extend the range of integration to \([-\infty, \infty]\) with an error \(E_3\). Hence

\[
W(N_1) = \int_{-\infty}^{\infty} \left(\psi(\alpha)\right)^m e(-dN_0) \, d\alpha
\]

\[
= \int_{-\infty}^{1} \frac{1 - \frac{1}{8p^3}}{8p^3} (T(\alpha))^m e(-dN_1) \, d\alpha + \frac{1}{8p^3} + E_1 + E_2 + E_3
\]

\[
= R(N_0) + \int_{1}^{\infty} \frac{1 - \frac{1}{8p^3}}{8p^3} (T(\alpha))^m e(-dN_1) \, d\alpha + \frac{1}{8p^3} + E_1 + E_2 + E_3
\]

Now

\[
\left| E_1 \right| \leq \left| \int_{1}^{\infty} (T(\alpha))^m - (\psi(\alpha))^m e(-dN_1) \, d\alpha \right| \leq \frac{1}{8p^3}
\]
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\[ \left| \int (4m) (B + 4)^{m-1} d\lambda \right| \leq \frac{1}{8P^3} \]

\[ \left| \int (4m) (P + 4)^{m-1} d\lambda \right| \leq P^{-4} \]

\[ + \left| \int \frac{1}{8P^3} \geq |\lambda| \geq P^{-4} \right| (4m) (2 |\lambda|^{-\frac{1}{4}} + 4)^{m-1} d\lambda \]

\[ \leq 10^5 P^{m-5}, \text{ since } m \text{ is either } 9 \text{ or } 10 \]

Similarly,

\[ \left| E_2 \right| \leq \left| \int (\psi(\lambda))^m \left| e(-\lambda N_1) - e(-\lambda N_0) \right| d\lambda \right| \leq \frac{1}{8P^3} \]

\[ \leq \int B^m (2\pi) |\lambda| |N_1 - N_0| d\lambda \leq \frac{1}{8P^3} \]

\[ \leq P^{3\frac{2}{3}} (2\pi) \int P^m |\lambda| d\lambda + \left| \lambda \right| \leq P^{-4} \]
+ P^{3 \frac{3}{4}} (2 \pi) \int_{|\omega| > P^{-4}} (2 |\omega|^{- \frac{1}{m}})^m |\omega| d\omega \\
< 10^5 P^{m-5 + \frac{3}{4}} \\
|E_3| < \int \left| \Psi(\omega) \right|^m d\omega \\
|\omega| > \frac{1}{8P^3} \\
< \int B^m d\omega \\
|\omega| > \frac{1}{8P^3} \\
< 10^5 P^{\frac{3}{4}} m^{-3}

This proves the result.

Lemma 5:

With \( M = \left[ \frac{P^{3 \frac{3}{4}}}{2} \right] \)

\[
\sum_{1 \leq A \leq M} \sum_{1 \leq B \leq M} W(N_0 - A - B) = M^2 R(N_0)
\]

+ \( \theta 10^7 M^2 P^{m-5 + \frac{3}{4}} \)

Proof:

In lemma 4, we take \( N_1 = N_0 - A - B \) and sum over \( 1 \leq A \leq M \) and \( 1 \leq B \leq M \). This gives
\begin{equation}
\frac{x}{A} \quad \frac{y}{B} \quad W(N_0 - A - B) = M^2 R(N_0)
\end{equation}

\begin{equation}
1 - \frac{1}{8p^3} \int \frac{1}{8p^3} \left( (T(x))^m e(-\mathcal{A} N_0) \left( \frac{\mathcal{A}}{\mathcal{A}} e(\mathcal{A}) \right)^2 \right) d \mathcal{A}
\end{equation}

\begin{equation}
+ \theta 10^6 m^2 p^m - 5 + \frac{3}{2}
\end{equation}

Since \( \sum_A e(\mathcal{A} A) \leq \frac{1}{\| \mathcal{A} \|} \), the integral appearing on the right, is bounded, in absolute value, by

\begin{equation}
1 - \frac{1}{8p^3} \int \frac{1}{8p^3} p^m \left( \frac{1}{\| \mathcal{A} \|} \right)^2 d \mathcal{A} \leq 10^4 p^{m+3}
\end{equation}

Lemma 6

We have:

\[
\sum_{1 \leq A \leq M} \sum_{1 \leq B \leq M} W(N_0 - A - B) \geq T_m \left( \frac{m - 1}{4} \right) M^2 N_0^{\frac{m}{4} - 1}
\]
where \[ T_m = \frac{\left( \Gamma \left( \frac{5}{4} \right) \right)^m}{\Gamma \left( 1 + \frac{m}{4} \right)} \]

**Proof:**

By lemma 3 (page 22) of Vinogradov [9], the number of integer solutions \( K_r(N) \) of \( x_1^4 + x_2^4 + \ldots + x_r^4 \leq N \) is given by

\[
K_r(N) = T_r N - \theta r N
\]

where

\[
T_r = \frac{\left( \Gamma \left( \frac{5}{4} \right) \right)^r}{\Gamma \left( 1 + \frac{r}{4} \right)}
\]

Now,

\[
\sum_{1 \leq B \leq M} W(N_0 - A - B)
\]

\[
= \sum_B (K_m(N_0 - A - B) - K_m(N_0 - A - B - 1))
\]

\[
= K_m(N_0 - A - 1) - K_m(N_0 - A - M - 1)
\]

\[
= T_m \left[ (N_0 - A - 1)^4 - (N_0 - A - M - 1)^4 \right] + 2m\theta N_0^4
\]

\[
\geq T_m \frac{m - 1}{4} M N_0
\]

and hence the lemma.
Prop. 2

We have, for \( m = 9 \) or \( 10 \),

\[
R(N_0) > 0.25 \cdot \frac{m}{4^4} - 1
\]

Proof:

From lemmas 5 and 6, it follows that

\[
R(N_0) > \left( \frac{T_m (m-1)}{4} - 10^{-5} \right) \frac{m}{4^4} - 1
\]

and hence the result.

§ 5: A lower bound for \( S_1(N_0) \):

Let us recall that \( S(n) = S(n, m) \) is given by

\[
S(n) = \sum_{q=1}^{\infty} \sum_{a=1}^{q} \left( \frac{S_a, q}{q} \right)^m e \left( - \frac{an}{q} \right)
\]

Write \( \chi_p(n) = \chi_p(n; m) = \sum_{i=1}^{\infty} A(n; p^i) \).

Lemma 7:

\( A(n; q) \) is a multiplicative function of \( q \) and hence

\[
S(n) = \prod_p (1 + \chi_p(n))
\]
Proof:

We refer the reader to Lemma 2.11 (page 20) of Vaughan [8]

Lemma 8:

(a) For any prime $p$, and natural number $\lambda$, let $N_m(p^\lambda, n)$ denote the number of solutions of

$$x_1^4 + x_2^4 + \ldots + x_m^4 \equiv n \pmod{p^\lambda},$$

$$1 \leq x_i \leq p^\lambda \text{ and not all } x_i \equiv 0 \pmod{p}$$

Further set $\gamma = 1$ if $p$ is odd and $\gamma = 4$ if $p = 2$. Let $p^4r+s$ exactly divide $n$, $0 \leq s \leq 3$.

$$k_0 = \max(4r + s + 1, 4r + \gamma).$$

Then

$$A(n, p^\lambda) = 0 \text{ for } \lambda > k_0 \text{ and}$$

$$1 + X_p(n, m) = p^{-(m-1)\gamma} N_m(p^\gamma, 0) \left\{ \sum_{\tau=0}^{r-1} p^\tau (4-m) \right\}$$

$$+ p^{(4-m)r - (m-1)\gamma} N_m(p^\gamma, np^{-4r})$$

where the empty sum is understood to be zero.

(b) Let $d = \text{g.c.d. of } (4, p-1)$. Then for $p \neq 2$,

$$\left| N_m(p, n) - p^{m-1} \right| \leq (1 - \frac{1}{p}) (d - 1)^{\frac{m}{2}}$$
Proof:
For the proof of Lemma 8 (a), we refer the reader to Hilfsatz 293 of LANDAU [5] or Prop. 4.3 of Thomas [6]. For the proof of lemma 8 (b), we refer to lemma 41 (page 91) of Thomas [6].

From Lemma 8, we have

Lemma 9:

Let $K > 100$; $S_2 = \{ p > K; \ p \equiv 3 \pmod{4} \}$

$S_4 = \{ p > K\; ; \; p \equiv 1 \pmod{4} \}$. Then for $d = 2$ or 4

$$\prod_{p \in S_d} (1 + X_p(n)) > \exp\left(-2(d-1)^m K \frac{1-m}{2} \left(1 + \frac{K}{2m-8}\right)\right)$$

Lemma 10:

We have $\prod_{p \geq 114} (1 + X_p(n)) > 0.97$

Since $\prod_{p \geq 114} (1 + X_p(n))$

$$= \prod_{p \geq 127} (1 + X_p(n)) \prod_{p \geq 137} (1 + X_p(n))$$

$p \equiv 3 \pmod{4}$ $p \equiv 1 \pmod{4}$

the result follows from Lemma 9.

Lemma 11:

We have $\prod_{2 < p \leq 113} (1 + X_p(n)) > 0.297$. 
Proof:

This is easily verified using Lemma 8 (a). For details, we refer the reader to Thomas [6].

Lemma 12:

If \( n \equiv \tilde{n} \pmod{16}, \ 2 < \tilde{n} < m - 2, \) then

\[
1 + \chi_2(n) = \binom{\tilde{m}}{\tilde{n}} 2^{-\tilde{m} + 4}
\]

and hence

\[
1 + \chi_2(n) \geq \frac{45}{64}
\]

Here \( \binom{\tilde{m}}{\tilde{n}} \) is the binomial coefficient.

Proof:

This is straight forward. For details, we refer the reader to Theorem 4.2 of Thomas [6].

Lemma 13:

We have \( S(n) \geq 0.2025 \)

provided \( n \equiv 2, 3, 4, 5, 6 \) or 7 (mod 16)

Proof:

This follows from lemmas 7, 10, 11 and 12.
Lemma 14:

We have $S(n) - S_1(n) \leq 0.002$

Proof:

Since $|S(n) - S_1(n)| = \left| \sum_{q > \frac{1}{p}} \sum_{a = 1}^{q} \left( \frac{S_{a,q}}{q} \right)^m e \left( -\frac{an}{q} \right) \right|$

we are through.

Prop 3:

There holds the inequality

$S_1(n) > 0.2$

if $n \equiv l \pmod{16}$, and $l \in \{2, 3, 4, 5, 6, 7\}$.

§ 6: A lower bound for $W_0(N_0)$:

Let us recall that $W_0(N_0) = \int (T(\alpha))^m e (-\alpha N_0) \, d\alpha$. 


Lemma 15:

If \(|\beta| < \frac{1}{8q^3}\), then

\[(T\left( \frac{a}{q} + \beta \right))^m = (\Psi(\beta) \frac{S_a, q}{q})^m + \theta \cdot 4qm \left((4.3 B(\beta)q^{-1} + 4q)^m\right)\]

Proof:

We have,

\[
T\left( \frac{a}{q} + \beta \right) = \sum_{x=1}^{P} \sum_{y=0}^{q-1} \sum_{-\frac{y}{q} < t < \frac{P-y}{q}} \exp\left(\left( \frac{a}{q} + \beta \right)(qt + y)^4\right)
\]

\[
= \sum_{y} \sum_{t} \exp\left(\left( \frac{ay^4}{q} + \beta (qt + y)^4\right)\right)
\]

\[
= \sum_{y} \exp\left(\frac{ay^4}{q}\right) D_y(z), \text{ say}
\]

Now, using Lemma 2,
\[ D_y(z) = \int \frac{e^{(\beta (qt + y)^4)}}{q} dt + 4\theta \]

\[ = \frac{1}{q} \int_0^P e^{(\beta x^4)} dx + 4\theta \]

\[ = \frac{1}{q} \Psi(\beta) + 4\theta. \]

Hence \[ T \left( \frac{a}{q} + \beta \right) \leq \frac{S_{a/q}}{q} \Psi(\beta) + 4\theta q \]

The result follows from the following inequalities.

\[ \left| \frac{S_{a/q}}{q} \right| \leq (4.3) q^{-\frac{1}{2}} \]

\[ |\Psi(\beta)| \leq B(\beta) \]

\[ a^m - b^m \leq m (a-b) \max(|a|^{m-1}, |b|^{m-1}) \]

**Lemma 16:**

We have

\[ W_0(N_0) = S_1(N_0) R(N_0) + \Theta 10^{13} p^{m-\frac{9}{2}} \]

**Proof:**

Since the proof is similar to that of Lemma 4, we give only the sketch of the proof.
\[ \mathcal{W}_0(N_0) = \int (T(\mathcal{A}))^m e(-\mathcal{A}N_0) \, d\mathcal{A} \]

\[ \frac{a}{q} + \frac{1}{8qP^3} \]

\[ \sum_{q < P^{1/2}} \sum_{a=1}^{q} \int (T(\mathcal{A}))^m e(-\mathcal{A}N_0) \, d\mathcal{A} \]

\[ \frac{1}{8qP^3} \]

\[ \sum_{q < P^{1/2}} \sum_{a=1}^{q} \int \left( T\left( \frac{a}{q} + \beta \right) \right)^m e\left( -N_0\left( \frac{a}{q} + \beta \right) \right) d\beta \]

\[ \frac{1}{8qP^3} \]

First we replace \( \left( T\left( \frac{a}{q} + \beta \right) \right)^m \) by \( \left( \psi(\beta) \frac{S_a}{q} \right)^m \)

and the error is, by Lemma 15, at most \( 10^{12} p^{m-9/2} \). Now we extend the range of integration of the integral to \([-\infty, \infty]\) which gives an error at most \( 10^{12} p^{m-9/2} \). Hence

\[ \mathcal{W}_0(N_0) = \sum_{q < P^{1/2}} \sum_{a=1}^{q} \int_{-\infty}^{\infty} (\psi(\beta))^m \left( \frac{S_a}{q} \right)^m e\left( -N_0\left( \frac{a}{q} + \beta \right) \right) d\beta + \theta 10^{13} p^{m-9/2} \]

\[ = S_1(N_0) R(N_0) + \theta 10^{13} p^{m-9/2} \]
Prop 4: We have

\[ W_0 (N_0) > 0.05 \ N_0 \]

provided \( N_0 \equiv 2, 3, 4, 5, 6 \) or 7 (mod 16)

Proof:

This follows from Prop. 2, Prop. 3 and Lemma 16.

§ 7: The estimate on the minor arc:

We define \( f (n) = n^{-a} \pi (1 - p^{-a})^{-1} \) where \( a > 0 \)

is a constant. Ultimately we shall choose \( a = 0.1 \)

\[
g(n) = \sqrt{f(n)}; \quad h(n) = \sum_{l/n} \frac{1}{g(l)} \quad l \leq P; \quad \frac{n}{l} \leq P
\]

\[
C_a = \frac{\pi}{p} \left( 1 + \frac{1}{P(P^a - 1)} \right)
\]

\[
D_a = \frac{\pi}{p} \left( 1 + \frac{(1 - p^{-a})^{-1} - 1}{P} \right)
\]

\[
k(m) = \sum_{d|m} \frac{1}{h(d)}
\]

Lemma 17

Let \( \lambda(n) \) be a non negative multiplicative function with

\[
\frac{\pi}{p} \left( 1 + \frac{\lambda(p)}{p} \right) \text{ convergent. Then}
\]
\[
\sum_{n \leq X} \left( n^{-\beta} \pi \frac{1 + \lambda(p)}{p} \right) < 2 \pi \left( 1 + \frac{\lambda(p)}{p} \right)^{1 - \beta} X^{1 - \beta}.
\]

**Proof:**

\[
\sum_{n \leq X} \left( n^{-\beta} \pi \frac{1 + \lambda(p)}{p} \right) = \sum_{n \leq X} n^{-\beta} \sum_{d \mid n} \mu^2(d) \lambda(d)
\]
\[
= \sum_{d} \mu^2(d) \lambda(d) \sum_{n \leq X} n^{-\beta} \quad \text{if } n \equiv 0 \pmod{d}
\]
\[
< 2 X^{1 - \beta} \sum_{d} \frac{\mu^2(d) \lambda(d)}{d}
\]
\[
< 2 X^{1 - \beta} \pi \left( 1 + \frac{\lambda(p)}{p} \right).
\]

**Lemma 18**

We have

(a) \( \sum_{n \leq X} f(n) \leq 2 C_a X^{1 - a} \)

(b) \( \sum_{n \leq X} g(n) \leq 2 D_a X^{1 - \frac{a}{2}} \)

(c) \( \sum_{n \leq X} g(n) n^{a/2} \leq 2 D_a X \)

(d) \( \sum_{n \leq X} \frac{1}{g(n)} \leq X^{1 + \frac{a}{2}} \)

(d) \( \sum_{n \leq X} \frac{1}{f(n)} \leq X^{1 + a} \)
Proof:

(a), (b), and (c) follow from Lemma 17 by the proper choice of \( \lambda(n) \) and \( \beta \); since \( \frac{1}{g(n)} \leq n^{a/2} \) and \( \frac{1}{f(n)} \leq n^a \), (d) and (e) follow.

Lemma 19: We have

\[
\sum_{n \leq P^2} f(n) (h(n))^2 \leq 20P^{2-a} C_a D_a^2 (\log P)
\]

Proof:

\[
\sum_{n \leq P^2} f(n) h^2(n) = \sum_{n \leq P^2} f(n) \left( \sum_{l \leq n \leq n/P} \frac{1}{g(l)} \right)^2
\]

\[
\leq 2 \left( \sum_{l_1 \leq P} \frac{1}{g(l_1)} \right) \left( \sum_{l_2 \leq l_1} \frac{1}{g(l_2)} \right) \sum_{n \leq P/2} f(n) \quad n \equiv 0 (\mod \lfloor l_1, l_2 \rfloor)
\]

\[
\leq 2 \left( \sum_{l_1} \frac{1}{g(l_1)} \right) \left( \sum_{l_2 \leq l_1} \frac{1}{g(l_2)} \right) \sum_{m \leq \frac{P/2}{\lfloor l_1, l_2 \rfloor}} f(m) f(\lfloor l_1, l_2 \rfloor)
\]
\[
< 4C_a \sum_{l_1} \frac{1}{g(l_1)} \sum_{l_2} \frac{f([l_1', l_2'])}{g(l_2')} \left( \frac{P_l}{[l_1', l_2']} \right)^{1-a}
\]

\[
< 4C_a p^{1-a} \sum_d \sum_{(l_1', l_2')} = d \frac{1}{g(l_1')}
\]

\[
g^2 \frac{(l_1/l_2)}{d} \left( \frac{d}{l_1'} \right)^{1-a}
\]

\[
< 4C_a p^{1-a} \sum_d \left( \sum_{(l_1', l_2')} = d \frac{1}{g(l_1') g(l_2')}ight)
\]

\[
\left( \frac{g(l_1') g(l_2')}{g(d)} \right)^2 \frac{d^{1-a}}{l_1'^{1-a}}
\]

\[
< 4C_a p^{1-a} \sum_d \frac{d^{1-a}}{g^2(d)} \sum_{l_1 \equiv 0 \pmod{d}} \frac{g(l_1)}{l_1^{1-a}}
\]

\[
\sum_{l_2 \equiv 0 \pmod{d}} g(l_2')
\]

\[
l_2 < l_1
\]

\[
< 8D_a p^{1-a} \sum_d \frac{1}{d^{a/2}g(d)} \sum_{l_1 \equiv 0 \pmod{d}} \frac{g(l_1)}{l_1^{a/2}}
\]

\[
< 16C_a D_a p^{2-a} \sum_d \frac{1}{d}
\]
< 20 C D \sqrt{a} P^{2-a} (\log P).

**Lemma 20:**

If \( f(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0 \) and

\( g_1(x) = f(x+1) - f(x) = l_k a_k x^{k-1} + \ldots, \)

then

\[
\sum_{x=1}^{P} \left| e(f(x)) \right|^2 < 2 \sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e(g_1(x)) \right| + P
\]

**Proof:**

\[
\sum_{x=1}^{P} \left| e(f(x)) \right|^2 = \sum_{x} \sum_{y} e(f(y) - f(x))
\]

\[
\leq \left| \sum_{x} \sum_{y \neq x} e(f(y) - f(x)) \right| + 2 \left| \sum_{x} \sum_{y > x} e(f(y) - f(x)) \right|
\]

We write \( y = x + l \) in the second sum and simplify. This gives the result.

**Lemma 21:**

If \( \left| \mathcal{A} - \frac{h}{q} \right| < \frac{1}{8qP^3} \) and \( P < q < 8P^3 \), then
Proof:
By Lemma 20,

\[
|T_1(d)|^2 < 2 \sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e\left(4d/l^3 + \ldots\right) \right| + P
\]

Hence

\[
T_1(d) = |T(d)|^2 - P
\]

\[
< 2 \sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e\left(4d/l^3 + \ldots\right) \right|
\]

\[
|T_1(d)|^2 < 4 \left( \sum_{l=1}^{P} g(l) \right) \left( \sum_{l=1}^{P} \frac{1}{g(l)} \right)
\]

\[
\left| \sum_{x} e\left(4d/x^3 + \ldots\right) \right|^2
\]

\[
< 8D_a P \left( \sum_{l=1}^{P} \frac{1}{g(l)} \right)
\]

By Lemma 20 and 19 (d),
\[ T_2(\alpha) = \frac{|T_1(\alpha)|^2}{8D_a P} \leq \sum_{l=1}^{P} \frac{1}{g(l)} \left\{ \begin{array}{c} \sum_{l_1=1}^{P-l} \sum_{x=1}^{P-l-l_1} e^{(12 \alpha / l_1 x^2 + \ldots)} \end{array} \right\} + P \]

\[ \leq 2 \sum_{l=1}^{P} \frac{1}{g(l)} \sum_{l_1=1}^{P-l} \left\{ \sum_{x=1}^{P-l-l_1} e^{(12 \alpha / l_1 x^2 + \ldots)} \right\} + P^2 + a \]

\[ T_3(\alpha) = T_2(\alpha) - P^2 + a \leq 2 \sum_{l, l_1} \frac{1}{g(l)} \left\{ \sum_{x=1}^{P-l-l_1} e^{(12 \alpha / l_1 x^2 + \ldots)} \right\} \]

\[ \leq 2 \sum_{n} h(n) \]

\[ n \leq \frac{P^2}{4} \]

\[ \max \left\{ \sum_{l, l_1} \frac{1}{g(l)} \sum_{x=1}^{P-l-l_1} e^{(12 \alpha / l_1 x^2 + \ldots)} \right\} \]
\[
\left| T_3 (d) \right|^2 < 4 \left( \sum_{n} f(n) h^2(n) \right) \left( \sum_{n} \frac{1}{f(n)} \right)^{\max_{L/1 = n} \sum_{x} e \left( 12 \frac{d}{L} x^2 + \ldots \right) }^2
\]

\[
< \frac{(80 P^{2-a} C_a D_a^2 \log P)}{2^{2-a}} \left( \sum_{n} \frac{1}{f(n)} \right)^{\max_{L/1 = n} \sum_{x} e \left( 12 \frac{d}{L} x^2 + \ldots \right) }^2
\]

\[
T_4 (d) = \frac{2^{2-a} T_3 (d)^2}{80 P^{2-a} C_a D_a^2 \log P} < \sum_{n} \frac{1}{f(n)}
\]

\[
< \sum_{n} \frac{1}{f(n)} \max_{L/1 = n} \left\{ 2 \sum_{l_2 = 1}^{P - l_1} \sum_{x = 1}^{P - l_1 - l_2} e \left( 24 \frac{d}{L} l_1 l_2 x + \ldots \right) + P \right\}
\]

\[
< 2 \sum_{n} \frac{1}{f(n)} \max_{L/1 = n} \sum_{l_2 = 1}^{P - l_1} \min \left( P, \frac{1}{2 \left| 24 \frac{d}{L} l_1 l_2 \right|} \right) + P^3 + 2a
\]
One easily checks that \( k(m) \leq m^a \) and \( m = \frac{1}{2} \). Hence

\[
T_4(\alpha) < 2 \sum_{m \leq \frac{P^3}{27}} m^a \min \left( P, \frac{1}{24 \alpha m \|} \right) + P^3 + 2a
\]

For a given \( \alpha \), we know that there exist \( h \) and \( q \), \((h, q) = 1\) such that

\[
\left| \alpha - \frac{h}{q} \right| < \frac{1}{8qP^3}, \quad q < 8P^3.
\]

Let \( q' = \frac{q}{(24, q)} \). The \( m \)-sum is divided into at most

\[
\left( \frac{P^3}{27q'} + 1 \right)
\]

subintervals, each of length \( q' \). In each subinterval, we check that the sum is at most \( 2P + q' \log q' + q' \).

Hence

\[
| T_4(\alpha) | < 2 \frac{P^{0.3}}{(27)^{0.3}} \left( \frac{P^3}{27q'} + 1 \right)
\]

\[
\left( 2P + q' \log q' + q' \right) + P^3 + 2a
\]

\[
< 25 P^{3.3} \log P
\]
Hence $\left| T_3(\mathcal{A}) \right|^2 < 10^3 P^{5.2} C_a D_a^2 (\log P)^2$

Hence $\left| T_2(\mathcal{A}) \right| < (31.63) P^{2.6} C_a D_a (\log P)\frac{7.1}{4}$

Hence $\left| T_1(\mathcal{A}) \right| < (15.91) P^{\frac{7}{4}} C_a D_a (\log P)\frac{7}{4}$

which proves the result.

Lemma 22:

If $\left| \mathcal{A} - \frac{h}{q} \right| < \frac{1}{8qP^3}$ and $P^{\frac{7}{4}} < q < P$,

then $\left| \mathcal{T}(\mathcal{A}) \right| < 100 P^{7/8}$.

Proof:

Arguing as in Lemma 9.5 of Thomas [6] or Lemma 9 of Davenport [3], we have the result. (Lemma 9.5 of Thomas [6] is proved under the restriction $\left| \mathcal{A} - \frac{h}{q} \right| < \frac{1}{64qP^3}$ but the same proof holds in our case).

Proof:

$\left| \mathcal{T}(\mathcal{A}) \right| < 40 P (\log P)^{\frac{7}{4}}$

Proof:

The proposition follows from Lemmas 21 and 22.
§ 8: A lower bound for the number of integers up to $X$ representable as a sum of five fourth powers

Lemma 23:

Let $P > 10^{10}$. Let $M$ be the number of solutions of the equation $x^4 + u_h = y^4 + u_j$, subject to (i) $P < x, y < 2P$ (ii) $x \equiv y \pmod{2}$ (iii) $u_h, u_j < P^3 + \mu$ and (iv) $u_h, u_j \in \mathcal{U}$. Then

$$M \leq PU \sqrt{P} \frac{1 + 3\mu}{192}.$$

Here $0 < \mu < 1$; $\mathcal{U}$ is a set of integers in $[0, P^3 + \mu]$. $U$ is the number of elements of $\mathcal{U}$.

Proof:

Even though this has not been explicitly stated anywhere, this is essentially contained in Lemma 1 of Davenport [4] and Lemma 7.1 of Thomas [6].

Lemma 24: The number of solutions of the equation

$$a^4 + b^4 = x^4 + y^4$$

with $a, b, x, y \leq A$; $a \equiv x \pmod{2}$; $b \equiv y \pmod{2}$ is at most $A^2 (\log A)^4$ provided $A > 10^{10}$. 
Proof:

We write \( x + a = 2k \); \( x - a = 2l \); \( y + b = 2m \); \( y - b = 2n \).

Then we have to find the number of solutions of the equation

\[
k / (k^2 + l^2) = m (m^2 + n^2),
\]

with \( k, m \leq A \); \( l, n \leq A/2 \). The number of solutions are

\[
= \sum_{k, l} \sum_{m, n} 1
\]

\[
m n (m^2 + n^2) = k / (k^2 + l^2)
\]

\[
= \sum_{k, l} d \left( k / (k^2 + l^2) \right)
\]

\[
\leq \sum \frac{d^2(k) + d^2(l)}{2} d \left( k^2 + l^2 \right)
\]

\[
\leq \sum_{k \leq A} d^2(k) d \left( k^2 + l^2 \right)
\]

\[
\leq 2 \sum_{k \leq A} d^2(k) \sum_{l} \sum_{j / k^2 + l^2} 1
\]

\[
\leq 2 \sum_{k \leq A} d^2(k) \sum_{j \leq 2A} \sum_{k^2 + l^2 \equiv 0 \pmod{j}} 1
\]

\[
\leq 2 \sum_{k \leq A} d^2(k) \sum_{j \leq 2A} p(j) \left( A - j + 1 \right)
\]
where \( P(j) \) is the number of solutions of \( k^2 + l^2 \equiv 0 \pmod{j} \), \( 1 < l < j \), for a fixed \( k \). We observe that \( P(j) \leq \sum_{d \mid j} \chi(d) \),

where \( \chi \) is the Dirichlet character \( \pmod{4} \) and hence we easily check that

\[
\sum_{j \leq 2A} \frac{P(j)}{j} \leq \log A
\]

\[
\sum_{j \leq 2A} P(j) \leq 4A
\]

and

\[
\sum_{k \leq A} d^2(k) \leq \frac{A}{3} \log^3 (A + 2)
\]

This proves the result.

**Lemma 25:**

Let \( I \) be 0, 1 or 2. The number of integers up to \( X \), representable as a sum of two biquadrates and belonging to \( I \pmod{16} \) is at least \( X^{0.45} \) provided \( X \geq 10^{188} \).

**Proof:**

Choose \( f_0 \) and \( f_1 \in (0, 1) \) such that \( f_0 + f_1 = I \).

Let \( r(m) \) be the number of solutions of \( m = x^4 + y^4 \), \( x, y \leq \left( \frac{X}{2} \right)^{\frac{1}{4}} \); \( x \equiv f_0 \pmod{2} \); \( y \equiv f_1 \pmod{2} \)

Then

\[
\sum r(m) \geq \left( \left( \frac{X}{2} \right)^{\frac{1}{4}} - 1 \right)^2
\]
By lemma 24, \( \sum_{r(m) \neq 0} \frac{2}{r(m)} < \left( \frac{X}{2} \right)^{\frac{1}{4}} \left( \log X \right)^{\frac{1}{2}} \)
\(< X^{0.5424}. \)

Hence the required estimate
\[
\sum_{r(m) \neq 0} 1 > \frac{\left( \sum r(m) \right)^2}{\sum r^2(m)}
\]
and we are through.

Lemma 26:

Let \( X > 0 \); \( P = \left[ \frac{X}{17} \right] + 1 \); Let \( \mu, B \) and \( U_0 \)
satisfy the following conditions

(i) \( 0 < \mu < \frac{1}{3} \)

(ii) \( B^4 = \frac{1}{2}, \max_{m < \frac{1}{192}} P^3 + \mu d_4(m) \)

(iii) The number of integers upto \( P^3 + \mu \) which are
sums of \( I \) biquadrates and belonging to \( f_0 \) (mod 16) is at least
\( U_0 \) for every \( f_0 \epsilon (0, 1, 2, \ldots) \)

(iv) \( B^4 < U_0 P^3 - 3 \mu. \)

Then the number of integers upto \( X \) which are sum of \( (l + 1) \)
biquadrates and belonging to \( f (\text{mod 16}) \) is at least \( \frac{PU_0}{8} \) for
every \( f \epsilon (0, 1, 2, \ldots (l + 1)) \).
Proof:

Let \( f_0 \in (0, 1, 2, \ldots) \) and \( f_1 \in (0, 1) \) such that \( f_0 + f_1 = f \). Let \( U = \{ n < P^{3+\mu} : n \text{ is sum of biquadrates and } n \equiv f_0 \pmod{16} \} \); \( U \) be the number of elements of \( U \).

By (iii), \( U > U_0 \).

Let \( r(m) \) be the number of solutions of \( m = x^4 + u_h \), \( P < x < 2P \); \( x \equiv f_1 \pmod{2} \) and \( u_h \in U \). The number \( N \) of integers up to \( X \), which are representable as a sum of \( (l + 1) \) biquadrates and \( \equiv f \pmod{16} \) is at least

\[
\sum_{m} \frac{1}{r(m)}
\]

\( r(m) \neq 0 \)

\[
> \left( \sum r(m) \right)^2 / \left( \sum r^2(m) \right)
\]

Now, using lemma 23

\[
N > \left( \frac{PU_0}{2} \right)^2 / PU_0 + P \left( \frac{1 + 3\mu}{4} \right) U_0^{5/4} B
\]

\[
> \left( \frac{PU_0}{2} \right)^2 / PU_0 + P \left( \frac{1 + 3\mu}{4} \right) U_0^{5/4} B.
\]

By (iv), \( P \left( \frac{1 + 3\mu}{4} \right) U_0^{5/4} B < PU_0 \).

Hence \( N > \left( \frac{PU_0}{2} \right)^2 / 2PU_0 > \frac{PU_0}{8} \).
Lemma 27:

Let \( f \) be 0, 1, 2, or 3. Then the number of integers upto \( X \), representable as a sum of three biquadrates belonging to \( f(\text{mod } 16) \) is at least \( X^{0.603} \) provided \( X > 10^{236} \).

Proof:

We choose \( l = 2; \mu = 0.20096 \) and \( U_0 = p^{(3+\mu)} (0.45) \) in lemma 26; since \( B^4 < 10^{12} p^{(3+\mu)} (0.235) \), we verify (iv) in lemma 26. This gives that the required estimate

\[
> PU_0 > \frac{1}{8} \left( \frac{X}{17} \right)^{0.610108} > X^{0.603}.
\]

Lemma 28:

Let \( f \) be 0, 1, 2, 3 or 4. Then the number of integers upto \( X \), representable as a sum of 4 biquadrates and belonging to \( f(\text{mod } 16) \) is at least \( X^{0.7095} \) provided \( X > 10^{307} \).

Proof:

We choose \( l = 3; \mu = 0.0869 \); \( U_0 = p^{(3+\mu)} (0.603) \) in lemma 26. We have to use \( B^4 < 10^{14} p^{(3+\mu)} (0.225) \).

This yields the result.

Prop 6:

Let \( f \) be 0, 1, 2, 3, 4 or 5. Then the number of integers upto \( X \) representable as a sum of 5 biquadrates and belonging to \( f(\text{mod } 16) \) is at least \( X^{0.7795} \) provided \( X > 10^{410} \).
Proof:

We take \( l = 4 \); \( \mu = 0.01133 \); and \( U_0 = (3 + \mu)(0.7095) \).

We use \( B^4 \lesssim (1.5) \times 10^{23} \). This gives the result.

§ 9. Conclusion of the proof:

Lemma 30:

If \( N > 10^{412} \), then \( N \) can be written as a sum of at most twenty biquadrates.

Proof:

Choose \( f \in \{ 0, 1, 2, 3, 4, 5 \} \) and \( l \in \{ 2, 3, 4, 5, 6, 7 \} \) such that

\[ N - 2f \equiv l \pmod{16} \]

\[ P = \lfloor \frac{N}{4} \rfloor \]

Let \( \mathcal{U} = \{ n \leq \frac{N}{4}; n \text{ is representable as a sum of five biquadrates: } n \equiv f \pmod{16} \} \)

Let \( U(\lambda) = \sum_{u \in \mathcal{U}} e(u \lambda) \)

\[ r(N) = \int_0^1 (T(\lambda))^{10} \cdot (U(\lambda))^2 e(-N \lambda) \, d\lambda \]
It suffices to prove that \( r(N) > 0 \).

Now \( r(N) = \int_{m} + \int_{m} \).

Now, using Proposition 4,

\[
\int_{m} = \sum_{u_1, u_2} \int_{m} (T(\alpha))^{10} e^{-(N-u_1-u_2)\alpha) d\alpha
\]

\[
= I W_0(N-u_1-u_2) u_1, u_2 > 0.01 U^2 P^6
\]

On the other hand, using Proposition 5 and 6,

\[
\left| \int_{m} (T(\alpha))^{10} (U(\alpha))^{2} e^{(-N \alpha)) d\alpha \right|
\]

\[
< \max_{\alpha \in m} \left| T(\alpha) \right|^{10} \int_{0}^{1} \left| u(\alpha) \right|^{2} d\alpha
\]

\[
< \left( \frac{40}{8} \right)^{10} \left( \frac{71}{8} \right) \left( \log P \right)^{2.5} U
\]

\[
< \left( \frac{40}{8} \right)^{10} \left( \frac{71}{8} \right) \left( \log P \right)^{2.5} U^2 \left( \frac{N}{4} \right)^{0.7795}
\]

\[
< 4^{11} 10^2 \left( \log P \right)^{2.5}
\]

\[
< 10^10 P^{5.757} U^2 (\log P)^{2.5}
\]
It suffices to verify that, for $P < 10^{102}$,
\[
4^{11} 10^{10} P^{5.757} (\log P)^{2.5} < 0.01 P^6
\]
and this is true.

**Lemma:** 31

If $N < 10^{412}$, then $N$ can be written as a sum of at most $20$ biquadrates.

**Proof:**

If $N < 10^{412}$, then $N - \lfloor N^{\frac{4}{3}} \rfloor < 10^{310}$ and hence representable as a sum of at most $19$ biquadrates by Th. 3:3 of Thomas [7].

Thus the proof of the theorem is complete.

**Note:**

Some doubts have been raised about the correctness of lemma 31 and the author has not verified the calculations given in [7]. But even a weaker result, compared to lemma 31, will suffice for our purpose and we shall return to this subject at a later date.
References


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