Hardy-Ramanujan Journal
Vol. 9 (1986) 1-10

ON THE FREQUENCY OF TITCHMARSH'S PHENOMENON FOR $\zeta(s)$-IV

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§ 1. INTRODUCTION. The object of this paper is to prove the following

Theorem 1 : The following lower bound holds :

$$\max_{T\le t\le T+H} \left|\zeta\left(\frac{1}{2}+it\right)\right| \ge \operatorname{Exp}\left(\frac{3}{4}\left(\frac{\log H}{\log\log H}\right)^{1/2}\right)$$

where $100 \log\log T \le H \le T$, and $T \ge T_0$.

Remark 1. In [BR1] we proved a weaker version of this Theorem where $\frac{3}{4}$ was replaced by a small constant. The constant $\frac{3}{4}$ is a substantial improvement. It can be improved slightly and this seems to be the limit of our method (see Theorem 4 below).

Remark 2. The result $\max\limits_{0<t<T} \left|\zeta\left(\frac{1}{2}+it\right)\right| > \operatorname{Exp}\left(\frac{1}{20}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right)$ was obtained by Montgomery [M] on the assumption of Riemann Hypothesis. Our result is independent of any hypothesis.

Remark 3. A similar result can be proved for zeta function of number fields etc. one can refer, for example to Remark 3 of [BR1].

§ 2. TITCHMARSH SERIES. In this section, we state some results on the mean square of Titchmarsh series and these results are essentially due to Ramachandra.

Let $A(>1)$ be a constant. Let $\{a_n\}$ be a sequence of complex numbers, possibly depending on a parameter $H(>10)$ such that $a_1=1$; and $|a_n| < (nH)^A$. Let $F(s)=\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be analytically continuable in the rectangle $R(T,H)=\{\sigma\geq 0; T\leq t\leq T+H\}$ and maximum of $|F(s)|$ within the rectangle be bounded by $\exp\exp\left(\frac{H}{100}\right)$. Also assume that $T\geq H\geq H_0$, a large constant. Then we have

<u>Theorem 2</u>. <u>There exists a constant</u> $c=c(A)>0$ <u>such that</u>

$$\frac{1}{H}\int_T^{T+H} |F(it)|^2 dt \geq c_A \sum_{n<\frac{H}{100}} |a_n|^2\left(1-\frac{\log n}{\log H}+\frac{1}{\log\log H}\right).$$

§ 3. <u>MAIN THEOREM</u>. In this section, we state the main theorem:

For $l>0$, let $I(l)=\int_{2l}^{\infty} \frac{e^{-x}}{x}\, dx$

$$D(l) = \frac{1}{2}(I(l)+e^{-2l})\sqrt{2l}\, e^{l} \text{ and let}$$

$B = \max D(l)$, the maximum being over all real $l>0$. Then we have, $\mathcal{L} >1$

<u>Theorem 3</u>. $\max\limits_{k>1} \frac{1}{2k}\left(\frac{\log X}{\log\log X}\right)^{-1/2} \log\left(\sum_{n<X} \frac{d_k^2(n)}{n}\right)$ <u>tends to</u> B <u>as</u> $X\longrightarrow\infty$ <u>where the maximum is taken over all natural numbers</u> k.

Having defined B, we can slightly strengthen Theorem 1 as

<u>Theorem 4</u>. $\max\limits_{T<t<T+H} |\zeta\left(\frac{1}{2}+it\right)|>\operatorname{Exp}\left(B_1\left(\frac{\log H}{\log\log H}\right)^{1/2}\right)$, <u>where</u> $100\log\log T<H<T$ <u>provided</u> $B_1<B$ <u>and T is sufficiently large.</u>

§ 4. <u>PROOFS OF THEOREMS</u> 1,2 <u>AND</u> 4.

The proof of Theorem 2, either in slightly weaker form (but sufficient for our purpose) or slightly stronger form can be found in [B] (Theorem 4), [BR1] [BR2] [R1] and [R2]. The proof of Theorem 3 will be given in the next few sections. To deduce

Theorem 4 from Theorems 2 and 3, we choose $F(s)=\left(\zeta\left(\frac{1}{2}+s\right)\right)^k$. Then by Theorems 2 and 3, we have, for a suitable choice of k, and for any $\varepsilon>0$,

$$\max_{T<t<T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|\geq\left(\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt\right)^{1/2k}$$

$$>\left(c_A\sum_{n<H/100}\frac{d_k^2(n)}{n}\left(1-\frac{\log n}{\log H}+\frac{1}{\log\log H}\right)\right)^{1/2k}$$

$$>\operatorname{Exp}\left\{(B-\varepsilon)\left(\frac{\log H}{\log\log H}\right)^{1/2}\right\}$$ and hence Theorem 4.

Even though the expression for B is unwieldy, it is possible to get an approximate value numerically and this gives that $B>\frac{3}{4}$ and this proves Theorem 1. (In fact $B=0.75\ldots$.)

§ 5. <u>SOME PRELIMINARY LEMMAS.</u> In this section, we prove some preliminary lemmas.

$$\text{Let } \theta=\theta(a)=\frac{1}{2}+\frac{5a}{\log k}$$

$$A_p=A_p(a)=\sum_{r=0}^{\infty}\frac{d_k^2(p^r)}{p^{2r\theta}}$$

$$G(a)=\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\theta}}$$

$$M=10k^2e^{-10a};$$

We assume that a is less than $\frac{\log k}{100}$ so that

$$\frac{1}{2}\leq\theta\leq\frac{3}{4}\text{ and }M\geq k^{1.9}.$$

<u>Lemma 1</u>. <u>We have</u>

$$\text{(a)}\quad\sum_{p<M}\frac{k}{p^{\theta}}=O\left(\frac{k^2}{\log k}e^{-15a}\right)$$

$$\text{(b)} \quad \sum_{p\geq M} \frac{k^4}{p^{4\Theta}} = O\left(\frac{k^2}{\log k}\, e^{-15a}\right)$$

$$\text{(c)} \quad \sum_{p\geq M} \frac{k^2}{p^{2\Theta}} = I(10a)k^2 + O\left(\frac{k^2}{\log k}\, e^{-15a}\right)$$

Proof. Since $\sum_{p<M} \frac{k}{p^{\Theta}} = O\left(\frac{kM^{1-\Theta}}{\log M}\right)$, (a) follows.

$\sum_{p\geq M} \frac{k^4}{p^{4\Theta}} = O\left(\frac{k^4\ M^{1-4\Theta}}{\log M}\right)$ and hence (b) follows.

To prove (c), we use prime number theorem in the form

$$\vartheta(u) = \sum_{p\leq u} \log p = u+O\left(\frac{u}{(\log u)^{10}}\right).$$

Hence

$$\sum_{p\geq M} \frac{k^2}{p^{2\Theta}} = k^2 \int_M^{\infty} \frac{1}{u^{2\Theta}\log u}\, d(\vartheta(u))$$

$$= k^2 \int_M^{\infty} \frac{1}{u^{2\Theta}\log u}\, du + k^2 \int_M^{\infty} \frac{d(\vartheta(u)-u)}{u^{2\Theta}\log u}$$

$$= S_1 + S_2, \text{ say.}$$

In S_1, we make a change of variable

$$v = \frac{10a}{\log k} \log u \text{ to get}$$

$$S_1 \doteq k^2 \int_A^{\infty} \frac{e^{-v}dv}{v} \text{ where } A = 20a - \frac{100a^2}{\log k} + \frac{10a \log 10}{\log k}.$$

Thus $S_1 = k^2 \int_{20a}^{\infty} \frac{e^{-v}dv}{v} + O\left(k^2 \int_A^{20a} \frac{e^{-v}dv}{v}\right)$.

Since $A\ge 19a$, the O-term is $O\left(k^2\int_A^{20a} e^{-19a}dv\right) = O\left(\frac{a^2k^2}{\log k} e^{-19a}\right)$.

Thus $S_1 = I(10a)k^2+O\left(e^{-15a}\frac{k^2}{\log k}\right)$. By integrating by parts, we check that $S_2 = O\left(\frac{e^{-15a}k^2}{\log k}\right)$ and this completes the proof.

<u>Lemma 2.</u> <u>We have</u> $\log G(a) = I(10a)k^2+O\left(\frac{k^2}{\log k} e^{-15a}\right)$.

<u>Proof</u>. Since $G(a) = \Pi_p A_p(a)$, it follows that

$$\log G(a) = \sum_p \log A_p(a).$$

Since $A_p \le \left(\sum_{r=0}^{\infty} \frac{d_k(p^r)}{p^{r\theta}}\right)^2 = (1-p^{-\theta})^{-2k}$, $\log A_p = O\left(\frac{k}{p^{\theta}}\right)$ and by

Lemma 1(a), $\sum_{p\le M} \log A_p = O\left(\frac{k^2}{\log k} e^{-15a}\right)$. Since $\frac{k}{p^{2\theta}} \le \frac{1}{10}$ for $p\ge M$,

$$A_p = 1 + \frac{k^2}{p^{2\theta}} + O\left(\frac{k^4}{p^{4\theta}}\right) \quad \text{for } p\ge M.$$

$$\text{Hence } \log A_p = \frac{k^2}{p^{2\theta}} + O\left(\frac{k^4}{p^{4\theta}}\right)$$

$\sum_{p\ge M} \log A_p = \sum_{p\ge M} \frac{k^2}{p^{2\theta}} + O\left(\sum_{p\ge M} \frac{k^4}{p^{4\theta}}\right)$ and the result follows from Lemmas 1(b) and 1(c).

<u>Lemma 3.</u> <u>If</u> a <u>is bounded above and below and</u>

$$\frac{\log y}{k^2\log k} \ge \frac{e^{-20a}}{10a} + (\log k)^{-1/8}, \text{ \underline{then}}$$

$$\log\left(\sum_{n\geq Y}\frac{d_k^2(n)}{n^{2\Theta(a)}}\right)\leq I(10a)k^2-\frac{k^2}{\sqrt{\log k}}.$$

<u>Proof</u>. Let $b=2(\log k)^{-1/8}$. Then

$$\sum_{n\geq Y}\frac{d_k^2(n)}{n^{2\Theta(a)}}\leq Y^{-\frac{10b}{\log k}}\sum_{n\geq Y}\frac{d_k^2(n)}{n^{2\Theta(a-b)}}$$

$$\leq Y^{-\frac{10b}{\log k}}G(a-b)$$

$$\log\left(\sum_{n\geq Y}\frac{d_k^2(n)}{n^{2\Theta(a)}}\right)\leq \log G(a-b)-\frac{10b}{\log k}\log Y$$

$$=I(10a-10b)\,k^2+O\left(\frac{k^2}{\log k}\right)-\frac{10b}{\log k}\log Y$$

$$=(I(10a)-10bI'(10a)+O(b^2))k^2-\frac{10b}{\log k}\log Y$$

$$=\left\{I(10a)+(10b)\frac{e^{-20a}}{10a}+O(b^2)-\frac{10b}{k^2\log k}\log Y\right\}k^2$$

and hence the result.

§6. <u>The upper bound in Theorem 3.</u>

We now prove that $\frac{1}{2k}\log\left(\sum_{n\leq X}\frac{d_k^2(n)}{n}\right)\left(\frac{\log\log X}{\log X}\right)^{1/2}\leq B+\varepsilon$ for any k; X sufficiently large. Suppose $k\geq\sqrt{\log X}\,(\log\log X)^2$. Then

$$d_k(n)\leq k^{\Omega(n)}\leq k^{2\log n}\text{ so that}$$

$$\sum_{n\le X} \frac{d_k^2(n)}{n} \le \sum_{n\le X} \frac{k^{4\log n}}{n} \le k^{4\log X}(\log X+O(1)).$$ Hence

$$\frac{1}{2k}\log\left(\sum_{n\le X}\frac{d_k^2(n)}{n}\right) \le \frac{5(\log X)(\log k)+\log\log X}{2k}$$

which is $O\left(\frac{\sqrt{\log X}}{\log\log X}\right)$.

Hence the result is true in this case. If $k\le\sqrt{\log X}\,(\log\log X)^2$, define a by

$$\frac{\log X}{k^2\log k} = \frac{e^{-20a}}{10a}. \text{ Then } a \le \frac{\log k}{100}.$$

Further $k \le \sqrt{20a}\, e^{10a}\sqrt{\frac{\log X}{\log(10a\log X)}}$.

$$\sum_{n\le X}\frac{d_k^2(n)}{n} \le X^{\frac{10a}{\log k}}\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\Theta(a)}} = X^{\frac{10a}{\log k}}G(a),$$

$$\frac{1}{2k}\log\left(\sum_{n\le X}\frac{d_k^2(n)}{n}\right) \le \frac{10a}{2k\log k}\log X + \frac{\log G(a)}{2k}$$

$$= \frac{e^{-20a}}{2}k + \frac{I(10a)k^2}{2k} + O\left(\frac{k\,e^{-15a}}{\log k}\right)$$

$$\le \frac{1}{2}k(e^{-20a}+I(10a))+O\left(\frac{k\,e^{-15a}}{\log k}\right).$$

Substituting the values of k, the O-term is easily seen to be

$O\left(\sqrt{\frac{\log X}{\log\log X}}\right)$ and the main term is atmost $(B+\varepsilon)\sqrt{\frac{\log X}{\log\log X}}$.

§7. The lower bound in Theorem 3.

In this section, we prove that for a suitable choice of k,

$$\frac{1}{2k} \log\left[\sum_{n\le X} \frac{d_k^2(n)}{n} \right] \ge (B-\varepsilon) \sqrt{\frac{\log X}{\log \log X}} .$$

Choose a to be a real number such that $D(l)$ takes its maximum at $l = 10a$; then a is bounded above and below. Choose the largest integer k such that

$$\frac{\log X}{k^2 \log k} \ge \frac{e^{-20a}}{10a} + (\log k)^{-1/8}$$

Now $G(a) = \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\Theta(a)}} = \mathrm{Exp}\left[I(10a)k^2 + O\left(\frac{k^2}{\log k}\right)\right]$. By Lemma 3,

$\sum_{n\le X} \frac{d_k^2(n)}{n^{2\Theta(a)}}$ is small. Hence

$$\sum_{n\le X} \frac{d_k^2(n)}{n^{2\Theta(a)}} = \mathrm{Exp}\left[I(10a)k^2 + O\left(\frac{k^2}{\log k}\right)\right].$$

Hence $\sum_{n\le X} \frac{d_k^2(n)}{n} \ge X^{(2\Theta-1)} \mathrm{Exp}\left[I(10a)k^2 + O\left(\frac{k^2}{\log k}\right)\right]$.

$$= \mathrm{Exp}\left[\frac{10a}{\log k} \log X + I(10a)k^2 + O\left(\frac{k^2}{\log k}\right)\right].$$

Substituting the value of k, the result follows.

§8. Acknowledgements. The author is grateful to Professor K. Ramachandra for his interest in this work and encouragement. He is also indebited to Professor H.E. Richert for his interest in this work and in helping the author in simplifying the unwiedldy

expression for B and getting the lower bound $\frac{3}{4}$ for B.

A REMARK. The author had proved this result immediately after the results of [BR1] were discovered. But due to various reasons the result of the present paper could not be published earlier.

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