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HYBRID MEAN SQUARE OF L-FUNCTIONS

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§1. Introduction. Let $k\geq 1$ be an integer and χ run through the set of characters mod k. Let $T\geq 0$ and $T_0 = T+2$. Then R. Balasubramanian and K. Ramachandra proved the following theorem. For the history of the theorem see [1].

Theorem 1. We have

$$\frac{1}{2\pi\phi(k)} \sum_{\chi \bmod k} \int_0^T |L(\tfrac{1}{2}+it,\chi)|^2 dt = \frac{\phi(k)}{k^2} A(k,T) + O\left(\frac{\phi(k)}{k} T^{1/2} \log T_0\right)$$

where $A(k,T) = \frac{kT}{2\pi} \log \frac{kT}{2\pi} + \left(2\gamma - 1 + \sum_{p|k} \frac{\log p}{p-1}\right) \frac{kT}{2\pi}$, γ being the Euler's constant.

In this paper I prove the following

Theorem 2. For $T\geq 2$ and $k\geq 3$,

$$\frac{1}{2\pi\phi(k)} \sum_{\chi \bmod k} \int_0^T |L(\tfrac{1}{2}+it,\chi)|^2 dt = \frac{\phi(k)}{k^2} A(k,T) +$$

$$+ B(k)T^{1/2} + O\left(\frac{\phi(k)}{k} T^{5/12} (\log T)^2\right)$$

where $B(k)$ is a negative constant and depends only on k. Moreover

$$|B(k)| \gg \frac{\phi(k)}{k} \quad \text{and} \quad |B(k)| \ll \frac{\phi(k)}{k}.$$

§2. Proof of the Theorem 2.

Lemma 1. We have

$$\frac{1}{\phi(k)} \sum_{\chi \bmod k} |L(\tfrac{1}{2}+it,\chi)|^2 = \frac{1}{k} \sum_{\substack{(a,k)=1\\ 1\le a\le k}} |\zeta(\tfrac{1}{2}+it,\tfrac{a}{k})|^2$$

where $\zeta(s,\alpha)$ for $0<\alpha<1$ is the Hurwitz Zeta-function defined by $\zeta(s,\alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s}$ for $\mathrm{Re}\ s>1$.

Proof. Follows from the fact that if $F(a)$ is a complex number for every residue class a mod k with $(a,k) = 1$, then

$$\frac{1}{\phi(k)} \sum_{\chi \bmod k} \Big| \sum_{\substack{(a,k)=1\\ 1\le a\le k}} \chi(a)F(a)\Big|^2 = \sum_{\substack{(a,k)=1\\ 1\le a\le k}} |F(a)|^2 .$$

Lemma 2. Theorem 1 gives Theorem 2 for $1\le T\le 2$ and hence it is sufficient to prove that for $T\ge 2$

$$\frac{1}{2\pi\phi(k)} \sum_{\chi \bmod k} \int_1^T |L(\tfrac{1}{2}+it,\chi)|^2 dt = \frac{\phi(k)}{k^2} \{A(k,T)-A(k,1)\} + B(k)T^{1/2}+O(\frac{\phi(k)}{k} T^{5/12}(\log T)^2).$$

where $B(k)$ is as in Theorem 2.

Lemma 3. For $T\ge 2$, $0<\alpha<1$,

$$\frac{1}{2\pi} \int_1^T |\zeta(\tfrac{1}{2}+it,\alpha)|^2 dt = \frac{T}{2\pi} \log \frac{T}{2\pi} + \frac{T}{2\pi} [C(\alpha)+\gamma-1] - \frac{1}{2\pi\alpha}+A_1(\alpha)T^{1/2} + O(T^{5/12}(\log T)^2)+O\left(\frac{\log T}{\|2\alpha\|}\right).$$

where $\|2\alpha\|$ denotes the distance of 2α from the nearest integer and

$$C(\alpha) = \lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{(n+\alpha)} - \log N\right] \quad \text{and}$$

$$A_1(\alpha) = \frac{1}{\sqrt{2\pi}}\left[-2H(\alpha)+\int_{1-\alpha}^{1}\frac{\cos\{\pi(u^2-u-1/4)\}}{\cos \pi u}+2\sum_{r=1}^{\infty}\int_{r}^{\infty}\cos\{\pi(u^2-1/4)\}du\right],$$

$$H(\alpha)=\int_{\frac{1-\alpha}{2}}^{\frac{1+\alpha}{2}} G^2(u)du + \int_{\alpha/2}^{1-\alpha/2} G^2(u)du$$

where $G(u) = \dfrac{\cos\{2\pi(u^2-u-1/16)\}}{\cos(2\pi u)}$.

Finally, $A_1(\alpha) < -\frac{1}{\sqrt{2\pi}}H(\alpha)$

Proof. See [2].

Lemma 4. We have

$$B(k) = \frac{1}{\sqrt{2\pi}\,k}\sum_{\substack{(a,k)=1\\1\le a\le k}}\left[-2H(\tfrac{a}{k})+\int_{1-\frac{a}{k}}^{1}\frac{\cos\{\pi(u^2-u-1/4)\}}{\cos \pi u}du\right.$$

$$\left.+\ 2\sum_{r=1}^{\infty}\int_{r}^{\infty}\cos\{\pi(u^2-1/4)\}du\right]$$

and $|B(k)| \ll$ and $\gg \frac{\phi(k)}{k}$.

Proof. Follows from the expression for $A_1(\alpha)$ and the last part of

Lemma 3.

Lemma 5. In Lemma 3, the error term $O\left(\frac{\log T}{\|2\alpha\|}\right)$ can be replaced by $O\left[\min\left\{\frac{\log T}{\|2\alpha\|},\ T^{1/2}\log T\right\}\right]$.

Proof. The sums in Lemmas 11 to 14 of [2] which contribute to this O-term add up to

$$2\ \mathrm{Re}\sum_{m=1}^{[\sqrt{T/2\pi}\,]}\ \sum_{0\le r\le m^{1/4}} \mathrm{Exp}\left[2\pi i(2m\alpha+2r\alpha+\alpha^2)\right]\int_{r+\alpha}^{\infty}\mathrm{Exp}\left[i\pi\left(\Theta^2-\frac{1}{4}\right)\right]d\Theta.$$

Now it is easy to see that this sum is $O(T^{1/2}\log T)$. We only have to integrate by parts and obtain an estimate $O(\frac{1}{r+\alpha})$ for $r\ge 1$ and $O(1)$ for $r=0$, for the expression inside the summation sign.

Lemma 6. We have

$$\frac{1}{k}\sum_{\substack{(a,k)=1\\1\le a<k}}\min\left[\frac{\log T}{\|2a/k\|},T^{1/2}\log T\right] \ll_{\varepsilon} \frac{\phi(k)}{k}\,T^{\varepsilon}\ .$$

Proof. Put $S=\frac{1}{\phi(k)}\sum_{\substack{(a,k)=1\\1\le a<k}}\min\left[\frac{\log T}{\|2a/k\|},T^{1/2}\log T\right]$

and $S_U=\frac{1}{\varphi(k)}\sum_{\substack{1/2U\le\|2a/k\|<1/U\\(a,k)=1,\ 1\le a<k}}\min\left[\frac{\log T}{\|2a/k\|},T^{1/2}\log T\right]$

where U runs through powers 2^r of 2, $2^r<k$. We divide the totality of S_U into 3 parts according as the integer nearest to $\frac{2a}{k}$ is 0, 1 or 2. In each case the number of integers a such that $\frac{1}{2U}\le\|\frac{2a}{k}\|\le\frac{1}{U}$ is $\ll \frac{\varphi(k)}{U}+2^{\omega(k)}$, where $\omega(k)$ is the number of different prime factors of k. Hence, if $U\le X$, taking the first term in the

minimum,

$$S_U \ll \left[1+U\,\frac{2^{\varphi(k)}}{\varphi(k)}\right]\log T.$$

For $U>X$, we take the second term in the minimum, so that

$$S_U \ll T^{1/2}\log T\left[\frac{1}{U}+\frac{2^{\omega(k)}}{\varphi(k)}\right].$$

Now $2^{\omega(k)} \ll k^{\varepsilon}$, hence

$$\frac{S}{\log T} \ll_{\varepsilon} \log X + \frac{X}{X^{1-\varepsilon}} + \frac{T^{1/2}}{k^{1-\varepsilon}}\,.$$

If $k<T^{10}$ we take $X=k$ and we do not get the last two terms as the case $U>X$ does not arise. If $k\geq T^{10}$, take $X=T^{1/2}$ so that

$$\frac{S}{\log T} \ll T^{\varepsilon} \text{ in either case.}$$

This proves the Lemma and Theorem 2 is now completely proved.

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References

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[2] M.J. Narlikar, "On the mean square value of Hurwitz zeta-function", Proc. Indian Acad. Sci. 90 (1981), 195-212.

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ADDED IN THE END. Since the submission of this paper the following paper has appeared :- Tom Meurman, A generalisation of Atkinson's formula to L-functions, Acta Arith., XLVII (1986), p.351-370.