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THE NUMBER OF FINITE NON-ISOMORPHIC ABELIAN GROUPS IN MEAN SQUARE

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ABSTRACT. Let $\Delta(x) = \sum_{n\le x} a(n) - \sum_{j=1}^{6} c_j x^{1/j}$ denote the error term in the abelian group problem. Using zeta-function methods it is proved that

$$\int_1^X \Delta^2(x)dx \ll X^{39/29} \log^2 X,$$

where the exponent $39/29 = 1.344827\ldots$ is close to the best possible exponent $4/3$ in this problem.

Let as usual $a(n)$ denote the number of non-isomorphic abelian groups with n elements. This is a well-known multiplicative function, whose generating Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\zeta(4s)\ldots \qquad (\mathrm{Re}\ s>1)$$

shows that methods from zeta-function theory may be used in various problems involving $a(n)$. A classical problem, investigated first by P. Erdös and G. Szekeres [2], is to determine the order of the error term in the asymptotic formula for $A(x) = \sum_{n\le x} a(n)$. Since $F(s)$ has only simple poles for $s = 1/j$ $(j = 1,2,\ldots)$, it is natural to expect that $A(x)$ is well approximated by a finite sum of residues of $F(s)x^s s^{-1}$ at $s=1/j$. With this in mind we define

$$\Delta(x) = \sum_{n\le x} a(n) - \sum_{j=1}^{6} \operatorname*{Res}_{s=1/j} F(s)x^s s^{-1} = \sum_{n\le x} a(n) - \sum_{j=1}^{6} c_j x^{1/j},$$

where $c_j = \prod_{k=1, k\ne j} \zeta(k/j)$. Upper bound estimates for $\Delta(x)$ have a long history (see [6], [7], and Ch. 14 of [3]), and the best published result

$$\Delta(x) \ll x^{97/381}\log^{35}x$$

is due to G. Kolesnik [5]. This is far from the bound $\Delta(x) \ll x^{1/6+\varepsilon}$, which one may conjecture in this problem. In the other direction, W. Schwarz [6] proved $\Delta(x) = \Omega(x^{1/6-\delta})$ for any $\delta>0$ under the truth of the Riemann hypothesis. Here as usual, $f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \longrightarrow \infty$. A simple unconditional proof of this result is given in Ch. 14 of [3], but a sharper result, namely $\Delta(x) = \Omega(x^{1/6}\log^{1/2}x)$, may be derived from the general method of R. Balasubramanian and K. Ramachandra [1]. It was kindly pointed to the author by K. Ramachandra that their methods lead to the slightly sharper

$$\int_1^X \Delta^2(x)dx = \Omega(X^{4/3}\log X), \tag{1}$$

which was independently proved in [4], where it was conjectured that the integral in (1) is asymptotic to $CX^{4/3}\log X$ $(C>0, X \longrightarrow\infty)$. It was also stated in [1] without proof that

$$\int_1^X \Delta^2(x)dx \ll X^{4/3}\log^2 X \tag{2}$$

holds. However, in correspondence with Professor K. Ramachandra it transpired that this claim is, unfortunately, unsubstantiated. It seems that, at present, it is not possible to obtain (2) even if one assumes the Riemann hypothesis. If one assumes the Lindelöf hypothesis ($\zeta(\frac{1}{2}+it) \ll t^{\varepsilon}$; a well-known consequence of the Riemann hypothesis), then it is possible to obtain a result only a little weaker than (2), namely $X^{4/3}\log^3 X$ in place of $X^{4/3}\log^2 X$. The author is very grateful to Professor K. Ramachandra for this and other remarks concerning this topic.

The aim of this note is to prove a good, unconditional bound for the mean square of $\Delta(x)$. This is contained in the following

Theorem.

(3)
$$\int_1^X \Delta^2(x)dx \ll X^{39/29} \log^2 X.$$

The exponent $\frac{39}{29} = \frac{40-1}{30-1} = 1.34482758\ldots$ is close to the best possible exponent $\frac{4}{3}$, and it could be still slightly decreased, but not beyond $\frac{67}{50} = 1.34$ by existing results concerning power moments of $\zeta(s)$.

Proof of the Theorem. We suppose $X/2 \le x \le X$, $1/6 < \sigma < 1/5$, $X^{\varepsilon} \ll G \ll X^{1-\varepsilon}$, $F(s) = \zeta(s)\zeta(2s)\zeta(3s)\ldots$. An application of Perron's formula (see the Appendix of [3]) gives

(4)
$$\Delta(x) - c_6 x^{1/6} = O(GX^{\varepsilon}) + O(G\int_{\sigma}^{1+\varepsilon} |F(\alpha+iXG^{-1})| x^{\alpha-1} d\alpha) + (2\pi)^{-1}\int_{-X/G}^{X/G} F(\sigma+it)\frac{x^{\sigma+it}}{\sigma+it}\,dt.$$

The first step in the proof is to show that

(5)
$$X^{\varepsilon}G + G\int_{\sigma}^{1+\varepsilon} |F(\alpha+iXG^{-1})| X^{\alpha-1} d\alpha \ll X^{1/6}$$

for a suitable G, and the choice will be $G = X^{\varepsilon}$. To bound F it will be sufficient to use

$$\zeta(\sigma+it) \ll (1+t^{(1-\sigma)/3}\log t) \quad (\sigma\ge 1/2,\, t\ge t_0) \tag{6}$$

and the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where $t^{\frac{1}{2}-\sigma} \ll \chi(\sigma+it) \ll t^{\frac{1}{2}-\sigma}$ (see [3]). Since the exponent of t in (6) is a linear function of σ it is easily checked that $|F(\alpha+iXG^{-1})|$ is majorized by bounds valid for $\alpha=\sigma$ and $\alpha=1+\varepsilon$, respectively. In the latter case the bound in question is trivially $\ll X^{\varepsilon}$, while for $\alpha = \sigma$ we have

$$F(\sigma+iXG^{-1}) \ll \prod_{j=1}^{5} |\zeta(j\sigma+ijXG^{-1})|$$

$$\ll (XG^{-1})^{1-3\sigma}|\zeta(1-\sigma+iXG^{-1})\zeta(1-2\sigma+2iXG^{-1})| \prod_{j=3}^{5} |\zeta(j\sigma+ijXG^{-1})|$$

$$\ll (XG^{-1})^{1-3\sigma+\varepsilon}(XG^{-1})^{(\sigma+2\sigma+1-3\sigma+1-4\sigma+1-5\sigma)/3} \ll X^{2-6\sigma+\varepsilon}.$$

Therefore the left-hand side of (5) is $\ll X^{1-5\sigma+\varepsilon} \ll X^{1/6}$ for ε sufficiently small, since $\sigma > 1/6$. Thus from (4) with $G=X^{\varepsilon}$ we obtain

$$\int_{X/2}^{X} \Delta^2(x)dx \ll X^{4/3} + \int_{-X^{1+\varepsilon}}^{X^{1-\varepsilon}} \int_{-X^{1-\varepsilon}}^{X^{1-\varepsilon}} \frac{F(\sigma+it)F(\sigma-iu)}{(\sigma+it)(\sigma-iu)} \int_{X/2}^{X} x^{2\sigma+it-iu}dxdtdu$$

$$\ll X^{4/3} + X^{2\sigma+1} \int_{-X^{1-\varepsilon}}^{X^{1-\varepsilon}} \int_{-X^{1-\varepsilon}}^{X^{1-\varepsilon}} \left|\frac{F(\sigma+it)F(\sigma+iu)}{(\sigma+it)(\sigma+iu)}\right| \frac{dt\,du}{1+|t-u|}.$$

In view of $|ab|\le \frac{1}{2}(|a|^2+|b|^2)$ the double integral does not exceed

$$\int_{-X^{1-\varepsilon}}^{X^{1-\varepsilon}} |F(\sigma+it)|^2(\sigma^2+t^2)^{-1}\left(\int_{-X^{1-\varepsilon}}^{X^{1-\varepsilon}} \frac{du}{1+|t-u|}\right)dt$$

$$\ll \log X \int_{-X^{1-\varepsilon}}^{X^{1-\varepsilon}} |F(\sigma+it)|^2(\sigma^2+t^2)^{-1}dt.$$

Using again the functional equation for $\zeta(s)$ we obtain then

$$(7) \qquad \int_{X/2}^{X} \Delta^2(x)dx \ll X^{4/3} + X^{2\sigma+1}\log^2 X\Big(1 + \sum_{1\ll M\le X^{1-\varepsilon}} M^{-6\sigma}I_M\Big),$$

where for brevity we have set

$$(8) \quad I_M = \int_{M/2}^{M} |\zeta(1-\sigma+it)\zeta(1-2\sigma+2it)\zeta(3\sigma+3it)\sigma(4\sigma+4it)\zeta(5\sigma+5it)|^2dt.$$

To bound I_M we shall use Hölder's inequality and power moments for $\zeta(s)$. Namely, for $1/2<\sigma<1$ fixed we define $m(\sigma)(\ge 4)$ as the supremum of all numbers m such that

$$\int_1^T |\zeta(\sigma+it)|^m dt \ll T^{1+\varepsilon}$$

holds for any $\varepsilon>0$. We shall use the following bounds for $m(\sigma)$ (Th.8.4 of [3]), which are hitherto the sharpest ones:

$m(\sigma) \ge 4/(3-4\sigma)$	for $1/2 < \sigma \le 5/8$,
$m(\sigma) \ge 10/(5-6\sigma)$	for $5/8 \le \sigma \le 35/54$,
$m(\sigma) \ge 19/(6-6\sigma)$	for $35/54 \le \sigma \le 41/60$,
$m(\sigma) \ge 2112/(859-948\sigma)$	for $41/60 \le \sigma \le 3/4$,
$m(\sigma) \ge 12408/(4537-4890\sigma)$	for $3/4 \le \sigma \le 5/6$,
$m(\sigma) \ge 4324/(1031-1044\sigma)$	for $5/6 \le \sigma \le 7/8$.

Now we choose $\sigma = 5/29 = 0.172413793...$ and we obtain

$\sigma_1 = 3\sigma = 0.51724137...$	$m(\sigma_1) \ge 4.29629629... = p_1$
$\sigma_2 = 1-2\sigma = 0.65517241...$	$m(\sigma_2) \ge 9.18333333... = p_2$,
$\sigma_3 = 4\sigma = 0.68965517...$	$m(\sigma_3) \ge 10.29205176... = p_3$,

$\sigma_4 = 1-\sigma = 0.82758620\ldots \qquad m(\sigma_4) \geq 25.31710406\ldots = p_4,$

$\sigma_5 = 5\sigma = 0.86206896\ldots \qquad m(\sigma_5) \geq 33.00763359\ldots = p_5.$

It follows then by Hölder's inequality for integrals that

$$I_M \leq \prod_{j=1}^{4} \left(\int_{M/2}^{M} |\zeta(\sigma_j\ldots)|^{p_j} dt \right)^{2/p_j} \left(\int_{M/2}^{M} |\zeta(\sigma_5+5it)|^{2A} dt \right)^{1/A},$$

where

$$\frac{1}{A} + \frac{2}{p_1} + \frac{2}{p_2} + \frac{2}{p_3} + \frac{2}{p_4} = 1,$$

hence $A = 23.05516\ldots$, $2A = 46.11033\ldots$, $2A-p_5 = 13.10270\ldots$.

Therefore by the choice of $p_1,\ldots,p_5$ we have

$$I_M \ll M^{1+\varepsilon} \max_{M/2\leq t\leq M} |\zeta(\sigma_5+5it)|^{(2A-p_5)/A}$$

$$\ll M^{1+(2A-p_5)(1-\sigma_5)/(5A)+\varepsilon} = M^{1.01567\ldots+\varepsilon},$$

since $\zeta(\sigma+it) \ll t^{(1-\sigma)/5}\log t$ for $5/6\leq\sigma\leq1$, $t\geq t_0$. But $-6\sigma = -1.03448\ldots$, hence (7) and (8) yield

$$\int_{X/2}^{X} \Delta^2(x)dx \ll X^{4/3}+X^{2\sigma+1}\log^2 X \ll X^{2\sigma+1}\log^2 X$$

for $\sigma = 5/29$. Replacing X by $X/2$, $X/2^2,\ldots$ etc. and adding one easily obtains (3).

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