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PROOF OF SOME CONJECTURES ON THE MEAN-VALUE OF TITCHMARSH SERIES-I

BY

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§ 1. **INTRODUCTION.** When we are integrating a function related to a series which we call TITCHMARSH SERIES, (a function of a real variable t) $\mid F(it) \mid$, or $\mid F(it) \mid^2$ from $t = 0$ to $t = H$ $(H \geq 10)$ we encounter the following situation. Let $a_1 = \lambda_1 = 1$ and $\{a_n\}$ $(n = 1,2,3,\cdots)$ be a sequence of complex numbers and $\{\lambda_n\}$ $(n = 1,2,3,\cdots)$ an increasing sequence of real numbers with $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$ for $n \geq 1$, where C is a positive constant. We suppose a_n to depend on n and H such that $\mid a_n \mid \leq (nH)^A$ for $n \geq 2$ and more generally we suppose that $\mid a_n \mid n^{-A}$ is bounded above by a suitable big function (of A and) H, where A is a positive integer constant. (Also in the paper that follows K. Ramachandra $[R]_2$ considers the case where instead of these conditions $\sum_{n \leq X} \mid a_n \mid$ is bounded above by suitable functions of X and H for all $X \geq 2$). We refer to all such series $(F(s) = \sum_{n=1}^{\infty}(a_n\lambda_n^{-s}), s = \sigma + it, \sigma \geq A+2)$ as TITCHMARSH SERIES. Trivially $F(s)$ is analytic in $\sigma \geq A + 2$ and we suppose that $F(s)$ can be continued analytically in $(\sigma \geq 0, 0 \leq t \leq H)$ some times with some "growth conditions on certain lines". We put $B = A + 2$. We prove in all five main theorems (the last two are in $[R]_2$, the paper that follows) on Titchmarsh series. Theorems 2 and 3 are sharper versions of two conjectures (stated by K. Ramachandra $[R]_1$ in Durham Conference held in 1979). The last two main theorems essentially due to K. Ramachandra $[R]_2$ are published in the next paper. The first three are jointly due to R. Balasubramanian and K. Ramachandra and are published in the present paper. We begin by stating a main lemma.

§ 2. **MAIN LEMMA.** *Let r be a positive integer $H \geq (r+5)U, U \geq 2^{70}(16B)^2$ and N and M positive integers subject to $N > M \geq 1$. Let $b_m(m \leq M)$ and $c_n(n \geq N)$ be complex numbers and $A(s) = \sum_{m \leq M} b_m \lambda_m^{-s}$. Let $B(s) = \sum_{n \geq N} c_n \lambda_n^{-s}$ be absolutely convergent in $\sigma \geq A+2$ and continuable analytically in $\sigma \geq 0$. Write $g(s) = A(-s)B(s)$,*

$$G(s) = U^{-r} \int_0^U du_r \cdots \int_0^U du_1 (g(s+i\lambda))$$

(here and elsewhere $\lambda = u_1 + u_2 + \cdots + u_r$). Assume that there exist real numbers T_1 and T_2 with $0 \leq T_1 \leq U, H-U \leq T_2 \leq H$, such that

$$| g(\sigma + iT_1) | + | g(\sigma + iT_2) | \leq ExpExp(\frac{U}{16B})$$

uniformly in $0 \leq \sigma \leq B$. (As stated already $B = A+2$). Let

$$S_1 = \sum_{m \leq M, n \geq N} | b_m c_n | (\frac{\lambda_m}{\lambda_n})^B 2^r (U log \frac{\lambda_n}{\lambda_m})^{-r},$$

and

$$S_2 = \sum_{m \leq M, n \geq N} | b_m c_n | (\frac{\lambda_m}{\lambda_n})^B.$$

Then

$$\begin{aligned} | \int_{2U}^{H-(r+3)U} G(it)dt | &\leq | U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} g(it)dt | \\ &\leq 2B^2U^{-10} + 54BU^{-1} \int_0^H | g(it) | dt \\ &\quad +(H+64B^2)S_1 + 16B^2 Exp(-\frac{U}{8B})S_2. \end{aligned}$$

To prove this main lemma we need five lemmas. After proving these we complete the proof of the main lemma.

LEMMA 2.1. *Let $z = x+iy$ be a complex variable with $| x | \leq \frac{1}{4}$. Then,* *we have,*

(a) $| Exp((Sin\ z)^2) | \leq e^{\frac{1}{2}} < 2$ *for all* y

and

(b) If $| y | \geq 2$,

$$| Exp((Sin\ z)^2) | \leq e^{\frac{1}{2}}(ExpExp\ | y |)^{-1} < 2(ExpExp\ | y |)^{-1}.$$

PROOF. We have

$$\begin{aligned} Re(Sin\ z)^2 &= -\tfrac{1}{4}\ Re\left\{\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)^2\right\} \\ &= -\tfrac{1}{4}\ Re\left\{e^{2ix-2y} + e^{-2ix+2y} - 2\right\} \\ &= \tfrac{1}{2} - \tfrac{1}{4}\left\{(e^{-2y} + e^{2y})\ cos(2x)\right\}. \end{aligned}$$

But in $| x | \leq \frac{1}{4}$, we have $cos(2x) = cos(| 2x |) \geq cos\ \frac{1}{2} \geq cos\ \frac{\pi}{6} \geq \frac{\sqrt{3}}{2}$. The rest of the proof is trivial since (i) $cosh\ y$ is an increasing function of $| y |$ and (ii) for $| y | \geq 2$

$$Exp(-\frac{\sqrt{3}}{8}e^{2|y|}) \leq (ExpExp\ | y |)^{-1}$$

since $e^2 > (2.7)^2$ and $\frac{8}{\sqrt{3}} < \frac{8\times1.8}{3} = 4 \cdot 8$ and so $e^2 > \frac{8}{\sqrt{3}}$. The lemma is completely proved.

LEMMA 2.2. *For any two real numbers* k *and* σ *with* $0 < | \sigma | \leq 2B$, *we have,*

$$\int_{-\infty}^{\infty} | Exp\left(Sin^2\left(\frac{ik - \sigma - iu_1}{8B}\right)\right) \frac{du_1}{ik - \sigma - iu_1} | \leq 12 + 4\ log\ | \frac{2B}{\sigma} |.$$

PROOF. Split the integral into three parts J_1, J_2 and J_3 corresponding to $| u_1 - k | \geq 2B, | \sigma | \leq | u_1 - k | \leq 2B$ and $| u_1 - k | \leq \sigma$. The contribution to J_1 from $| u_1 - k | \geq 16B$ is (by (b) of Lemma 2.1)

$$\begin{aligned} &\leq \tfrac{2e^{\frac{1}{2}}}{16B} \int_{16B}^{\infty} Exp\left(-\tfrac{u_1}{8B}\right) du_1 \\ &= e^{\frac{1}{2}} \int_{2}^{\infty} Exp(-u_1) du_1 = Exp(-\tfrac{3}{2}). \end{aligned}$$

The contribution to J_1 from $2B \leq | u_1 - k | \leq 16B$ is (by (a) of Lemma 2.1)

$$\leq e^{\frac{1}{2}} \int_{2B \leq |u_1 - k| \leq 16B} | u_1 - k |^{-1} du_1 = 2e^{\frac{1}{2}} \log 8 = 6e^{\frac{1}{2}} \log 2.$$

Now

$$6e^{\frac{1}{2}} \log 2 + Exp\left(-\frac{3}{2}\right) < 6\left(1 + \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{6 \cdot 2^2}\right)\left(\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^2}\right)$$
$$+ \left(\frac{1}{2 \cdot 7}\right)^{3/2} < 8.$$

Thus $| J_1 | \leq 8$. Using (a) of Lemma 2.1 we have $| J_2 | \leq 4 \log \left| \frac{2B}{\sigma} \right|$. In J_3 the integrand is at most $e^{\frac{1}{2}} \sigma^{-1}$ in absolute value and so $| J_3 | \leq 2e^{\frac{1}{2}} \leq 4$. Hence the lemma is completely proved.

LEMMA 2.3. *If $n > m$, we have, for all real k,*

$$\left| \int_0^U du_r \cdots \int_0^U du_1 \left(\frac{\lambda_m}{\lambda_n}\right)^{i(k+\lambda)} \right| \leq 2^r \left(\log \frac{\lambda_n}{\lambda_m}\right)^{-}$$

PROOF. Trivial.

LEMMA 2.4. *For all real t and all $D \geq B$, we have,*

$$| G(D + it) | \leq S_1 \text{ and } | g(D + it) | \leq S_2.$$

PROOF. We have, trivially,

$$| g(D + it) | \leq \sum_{m \leq M, n \geq N} | b_m c_n | \left(\frac{\lambda_m}{\lambda_n}\right)^D$$

and the second result follows on observing that $\frac{\lambda_m}{\lambda_n} < 1$ and so $\left(\frac{\lambda_m}{\lambda_n}\right)^D \leq \left(\frac{\lambda_m}{\lambda_n}\right)^B$.

Next

$$G(D + it) = U^{-r} \int_0^U du_r \cdots \int_0^U du_1 (g(D + it + i\lambda))$$

$$= U^{-r} \sum_{m \leq M, n \geq N} b_m c_n \left(\frac{\lambda_m}{\lambda_n}\right)^D \int_0^U du_r \cdots \int_0^U du_1 \left(\frac{\lambda_m}{\lambda_n}\right)^{i(t+\lambda)}.$$

Using Lemma 2.3 and observing $\left(\frac{\lambda_m}{\lambda_n}\right)^D \leq \left(\frac{\lambda_m}{\lambda_n}\right)^B$ the first result follows.

LEMMA 2.5 *Let* $0 < \sigma \leq B$ *and* $2U \leq t \leq H - (r+3)U$. *Then, for* $H \geq (r+5)U$ *and* $U \geq (20)!(16B)^2$, *we have,*

$$| G(\sigma + it) | \leq BU^{-10} + U^{-1}(2 + 4log\frac{2B}{\sigma}) \int_0^H | g(it) | \, dt$$

$$+16 \; S_1 log(2B) + 8BS_2 Exp\left(-\frac{U}{8B}\right).$$

REMARK. $(20)! < 2^{70}$.

PROOF. We note, by Cauchy's theorem, that

$$2\pi i g(\sigma + it + i\lambda) = \int_{iT_1}^{B+1+iT_1} + \int_{B+1+iT_1}^{B+1+iT_2} - \int_{iT_2}^{B+1+iT_2} - \int_{iT_1}^{iT_2}$$

$$\left\{g(w) Exp\left(Sin^2\left(\tfrac{w-\sigma-it-i\lambda}{8B}\right)\right)\right\} \tfrac{dw}{w-\sigma-it-i\lambda}$$

$J_1 + J_2 - J_3 - J_4$ say.

We write

$$\begin{aligned} 2\pi i G(\sigma + it) &= 2\pi i U^{-r} \textstyle\int_0^U du_r \cdots \int_0^U du_1 (g(\sigma + it + i\lambda)) \\ &= U^{-r} \textstyle\int_0^U du_r \cdots \int_0^U du_1 (J_1 + J_2 - J_3 - J_4) \\ &= J_5 + J_6 - J_7 - J_8 \text{ say.} \end{aligned}$$

Let us look at J_5. In J_1 (also in J_3) $| g(w) | \leq Exp Exp(\frac{U}{16B})$ (by the definition of T_1 and T_2). Also by using Lemma 2.1 (b) (since $| Re\, w - \sigma | \leq B+1 \leq 2B$, and $| Im(w - it - i\lambda) | \geq U \geq (20)!(16B)^2$), we have,

$$\left| Exp\left(\left(\frac{w - \sigma - it - i\lambda}{8B}\right)\right) \right| \leq 2 \; Exp\left(-\frac{U}{8B}\right).$$

Hence

$$\begin{aligned} | J_1 | &\leq \tfrac{2(B+1)}{U} Exp\left(Exp \tfrac{U}{16B} - Exp \tfrac{U}{8B}\right) \\ &\leq \tfrac{2(B+1)}{U} Exp\left(-\left(Exp \tfrac{U}{16B}\right)\left(Exp \tfrac{U}{16B} - 1\right)\right) \\ &\leq \tfrac{B}{2} U^{-10}, \end{aligned}$$

since $U \geq (20)!(16B)^2$ and so $Exp\ \frac{U}{16B} - 1 \geq 1$ and $Exp\left(-Exp\ \frac{U}{16B}\right) \leq Exp\left(-Exp\ U^{\frac{1}{2}}\right) \leq Exp\left(-U^{\frac{1}{2}}\right) \leq (20)!U^{-10}$. Thus $| J_5 | \leq \frac{1}{2}BU^{-10}$.

Similarly, $| J_7 | \leq \frac{1}{2}BU^{-10}$. Next

$$J_8 = U^{-r}\int_{iT_1}^{iT_2} g(w)dw \int_0^U du_r \cdots \int_0^U du_2 \int_0^U$$

$$Exp\left(sin^2\left(\frac{w-\sigma-it-i\lambda}{8B}\right)\right)\frac{du_1}{w-\sigma-it-i\lambda}.$$

We note that $w-\sigma-it-i\lambda = ik-\sigma-iu_1$ where $k = Im\ w - t - u_2 \cdots - u_r$. Hence the u_1-integral is in absolute value (by Lemma 2.2)

$$\leq 12 + 4\ log\ \frac{2B}{\sigma}.$$

This shows that

$$\begin{aligned} | J_8 | &\leq U^{-r}\textstyle\int_{iT_1}^{iT_2} | g(w)dw | \left\{U^{r-1}\left(12+4\ log\ \frac{2B}{\sigma}\right)\right\} \\ &\leq U^{-1}\left(12+4\ log\ \frac{2B}{\sigma}\right)\textstyle\int_0^H | g(it) | \,dt \end{aligned}$$

Finally we consider J_6.

$$\begin{aligned} J_6 &= U^{-r}\textstyle\int_0^U du_r \cdots \int_0^U du_1 \int_{B+1+iT_1}^{B+1+iT_2} g(w)Exp\left(Sin^2\left(\frac{w-\sigma-it-i\lambda}{8B}\right)\right)\frac{dw}{w-\sigma-it-} \\ &= U^{-r}\textstyle\int_0^U du_r \cdots \int_0^U du_1 \displaystyle\int_{B+1-\sigma+iT_1-it-i\lambda}^{B+1-\sigma+iT_2-it-i\lambda} g(w+\sigma+it+i\lambda) \\ &\quad Exp\left(Sin^2\left(\tfrac{w}{8B}\right)\right)\tfrac{dw}{w}. \end{aligned}$$

Using Lemma 2.1 (b) we extend the range of integration of w to $(B+1-\sigma-i\infty, B+1-\sigma+i\infty)$ and this gives an error which is at most

$$U^{-r}\int_0^U du_r \cdots \int_0^U du_1 \int_{|Im\ w|\geq U, Re\ w=B+1-\sigma} | g(w+\sigma+it+i\lambda)$$

$$Exp\left(Sin^2\left(\frac{w}{8B}\right)\right)\frac{dw}{w} |.$$

By Lemma 2.4 this is

$$\leq S_2U^{-r}\int_0^U du_r \cdots \int_0^U du_1 \int_{|Im\ w|\geq U, Re\ w=B+1-\sigma} | Exp\left(Sin^2\ \frac{w}{8B}\right)\frac{dw}{w} |.$$

Here the innermost integral is (by Lemma 2.1(b))

$$\leq \frac{4}{U}\int_U^\infty Exp\left(-\frac{u}{8B}\right)du \leq \int_U^\infty Exp\left(-\frac{u}{8B}\right)du = 8B\ Exp\left(-\frac{U}{8B}\right).$$

Thus the error does not exceed $8BS_2\ Exp\left(-\frac{U}{8B}\right)$ and so

$$\mid J_6 \mid \leq \mid U^{-r}\int_0^U du_r \cdots \int_0^U du_1 \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} g(w+\sigma+it+i\lambda)Exp\left(Sin^2\left(\tfrac{w}{8B}\right)\right)\tfrac{dw}{w} \mid$$
$$+8BS_2\ Exp\left(-\tfrac{U}{8B}\right)$$
$$=\mid U^{-r}\int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} Exp\left(Sin^2\left(\tfrac{w}{8B}\right)\right)\tfrac{dw}{w}\int_0^U du_r \cdots \int_0^U du_1 g(w+\sigma+it+$$
$$i\lambda) \mid +8BS_2\ Exp\left(-\tfrac{U}{8B}\right)$$
$$=\mid \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} G(w+\sigma+it)Exp\left(Sin^2\left(\tfrac{w}{8B}\right)\right)\tfrac{dw}{w} \mid +8BS_2\ Exp\left(-\tfrac{U}{8B}\right).$$

Using the first part of Lemma 2.4 we obtain

$$\begin{aligned} \mid J_6 \mid &\leq S_1\int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} \mid Exp\left(Sin^2\left(\tfrac{w}{8B}\right)\right)\tfrac{dw}{w} \mid +8BS_2Exp\left(-\tfrac{U}{8B}\right) \\ &\leq S_1\left(12+4\ log\ \tfrac{2B}{B+1-\sigma}\right)+8BS_2\ Exp\left(-\tfrac{U}{8B}\right) \end{aligned}$$

by using Lemma 2.2. Thus

$$\mid J_6 \mid \leq 16S_1 log(2B) + 8BS_2 ExpExp\left(-\frac{U}{8B}\right).$$

This completes the proof of the lemma.

We are now in a position to complete the proof of the main lemma. We first remark that

$$4\int_0^B log\ \tfrac{2B}{\sigma}d\sigma = 4B\ log\ 2 + 4\sqrt{2}\int_0^B\left(\tfrac{B}{\sigma}\right)^{\frac{1}{2}}d\sigma$$
$$< 4\left(\tfrac{1}{2}+\tfrac{1}{2\cdot 2^2}+\tfrac{1}{3\cdot 2^3}\right)B + (8\times 1.415)B < 15B.$$

By Cauchy's theorem, we have,

$$\begin{aligned} \int_{2U}^{H-(r+3)U} G(it)idt &= \int_{i(2U)}^{i(H-(r+3)U)} G(s)ds \\ &= \int_{i(2U)}^{B+i(2U)} G(s)ds + \int_{B+i(2U)}^{B+i(H-(r+3)U)} G(s)ds- \\ &\quad -\int_{i(H-(r+3)U)}^{B+i(H-(r+3)U)} G(s)ds \\ &= J_1 + J_2 - J_3 \text{ say.} \end{aligned}$$

Using the estimate given in Lemma 2.5, we see that

$$
\begin{aligned}
|J_1| &\leq \int_0^B (BU^{-10} + \frac{(12+4\log\frac{2B}{\sigma})}{U}\int_0^H |g(it)|\,dt \\
&\quad +16(\log(2B))S_1 + 8BS_2\, Exp(-\tfrac{U}{8B}))d\sigma \\
&\leq B^2U^{-10} + \frac{12B+15B}{U}\int_0^H |(g(it)|\,dt + 16BS_1\log(2B) \\
&\quad +8B^2S_2\, Exp\left(-\tfrac{U}{8B}\right).
\end{aligned}
$$

The same estimate holds for $|J_3|$ also. For $|J_2|$ we use the estimate given in Lemma 2.4 to get

$$|J_2| \leq HS_1.$$

This completes the proof of the main lemma.

§ 3. **FIRST MAIN THEOREM.** *Let A, B, C be as before $0 < \varepsilon \leq \frac{1}{2}, r \geq [(200A + 200)\varepsilon^{-1}]$, $|a_n| \leq n^A H^{\frac{r\varepsilon}{8}}$. Then $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ is analytic in $\sigma \geq A + 2$. Let $K \geq 30, U = H^{1-\frac{\varepsilon}{2}} + 50B \log\log K_1$. Assume that*

$$H \geq \left(120B^2C^{2A+4}(4rC^2)^r\right)^{\frac{400}{\varepsilon}} + (100rB)^{20}\log\log K_1,$$

and that there exist T_1, T_2 with $0 \leq T_1 \leq U, H - U \leq T_2 \leq H$ such that

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K$$

uniformly in $0 \leq \sigma \leq B$ where $F(s)$ is assumed to be analytically continuable in $\sigma \geq 0$. Then

$$\int_0^H |F(it)|^2\,dt \geq (H - 10rC^2H^{1-\frac{\varepsilon}{4}} - 100rB\log\log K_1) \sum_{n \leq H^{1-\varepsilon}} |a_n|^2,$$

where

$$K_1 = \left(\sum_{n \leq H^{1-\varepsilon}} |a_n|\lambda_n^B\right) K + \left(\sum_{n \leq H^{1-\varepsilon}} |a_n|\lambda_n^B\right)^2.$$

REMARK 1. We need the conditions $H \geq (r+5)U, U \geq 2^{70}(16B)^2$ in the application of the main lemma. All such conditions are satisfied by our lower bound choice for H. We have not attempted to obtain economical lower bounds.

REMARK 2. Taking $F(s) = (\zeta(\frac{1}{2} + it + iT))^k$ in the first main theorem we obtain the following as an immediate corollary. Let $C(\varepsilon, k) loglog\ T \leq H \leq T$. Then for all integers $k \geq 1$

$$\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k}\,dt \geq (1-\varepsilon)\sum_{n\leq H^{1-\varepsilon}} (d_k(n))^2 n^{-1} \geq (C'_k - 2\varepsilon)(log\ H)^{k^2},$$

where

$$C'_k = (\Gamma(k^2+1))^{-1}\prod_p\left\{(1-p^{-1})^{k^2}\sum_{m=0}^{\infty}\left(\frac{\Gamma(k+m)}{\Gamma(k)m!}\right)^2 p^{-m}\right\}.$$

(This is because it is well-known that

$$\sum_{n\leq X}(d_k(n))^2 n^{-1} = \left\{C'_k + O\left(\frac{1}{log\ X}\right)\right\}(log\ X)^{k^2}).$$

Our third main theorem gives a sharpening of this. The third main theorem is sharper than the conjecture (stated by K. Ramachandra $[R]_1$ in Durham conference 1979). The conjecture (as also the weaker form of the conjecture proved by him in the conference) would only give

$$\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k}\,dt \gg_k (log\ H)^{k^2} \text{ in } C(k) loglog\ T \leq H \leq T.$$

But the third main Theorem gives

$$\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k}\,dt \geq C'_k(log\ H)^{k^2} + O\left(\frac{loglog\ T}{H}(log\ H)^{k^2}\right) + O\left((log\ H)^{k^2-1}\right)$$

where the O-constants depend only on k.

REMARK 3. The first main theorem gives a lower bound for $\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k}\,dt$ uniformly in $1 \leq k \leq log\ H, T \geq H \geq 30$ and $C\ loglog\ T \leq H \leq T$. From this it follows (as was shown in $[B]_1$) that for $C\ loglog\ T \leq H \leq T$ we have uniformly

$$\max_{T\leq t\leq T+H} |\zeta(\frac{1}{2}+it) > Exp\left(\frac{3}{4}\sqrt{\frac{log\ H}{loglog\ H}}\right)$$

if C is choosen to be a large positive constant. On Riemann hypothesis we can deduce from the first main theorem the following more general result. Let θ be fixed and $0 \le \theta < 2\pi$. Put $z = e^{i\theta}$. Then (on Riemann hypothesis), we have,

$$\max_{T \le t \le T+H} |\, ((\zeta(\frac{1}{2} + it))^{z} \,| > Exp\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right)$$

where the LHS is interpreted as $\lim_{\sigma \to \frac{1}{2}+0}$ of the same expression with $\frac{1}{2} + it$ replaced by $\sigma + it$. This result with $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ gives a quantitative improvement of some results of J.H. Mueller [M].

PROOF. Write $M = [H^{1-\epsilon}]\,, N = M + 1, A(s) = \sum_{m \le M} \overline{a}_m \lambda_m^{-s}, \overline{A}(s) = \sum_{m \le M} a_m \lambda_m^{-s}, B(s) = \sum_{n \ge N} a_n \lambda_n^{-s}$. Then we have, in $\sigma \ge A + 2$,

$$F(s) = \overline{A}(s) + B(s).$$

Also,

$$\begin{aligned} |\, F(it) \,|^2 &= |\, \overline{A}(it) \,|^2 + 2\, Re(A(-it)B(it)) + |\, B(it) \,|^2 \\ &\ge |\, \overline{A}(it) \,|^2 + 2\, Re(g(it)) \end{aligned}$$

where $g(s) = A(-s)B(s)$. Hence

$$\begin{aligned} \int_0^H |\, F(it) \,|^2\, dt &\ge U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} |\, F(it) \,|^2\, dt \\ &\ge U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} (|\, \overline{A}(it) \,|^2 + 2\, Re\; g(it))dt \\ &= J_1 + 2J_2 \text{ say.} \end{aligned}$$

Now $\log\left(\frac{\lambda_{n+1}}{\lambda_n}\right) = -\log\left(1 - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)\right) \ge \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \ge (2nC^2)^{-1}$. Hence by Montgomery-Vaughan theorem,

$$\begin{aligned} J_1 &\ge \int_{2U}^{H-(r+3)U} |\, \overline{A}(it) \,|^2\, dt \\ &\ge \sum_{n \le M} (H - (r+5)U - 100C^2 n) \,|\, a_n \,|^2 . \end{aligned}$$

We have

$$| g(s) | = | A(-s)B(s) | = | A(-s)(F(s) - A(s)) |$$

$$\leq \left(\sum_{n \leq H^{1-\epsilon}} | a_n | \lambda_n^B \right) K + \left(\sum_{n \leq H^{1-\epsilon}} | a_n | \lambda_n^B \right)^2$$

$$= K_1.$$

By the main lemma, we have,

$$\begin{aligned} | J_2 | &\leq | U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} g(it)dt | \\ &\leq \tfrac{2B^2}{U^{10}} + \tfrac{54B}{U} \int_0^H | g(it) | dt + (H + 64B^2)S_1 + 16B^2 S_2 Exp\left(-\tfrac{U}{8B}\right). \quad (3.1) \end{aligned}$$

We simplify the last expression in (3.1). We can assume that $\int_0^H | F(it) |^2 dt \leq H \sum_{n \leq H^{1-\epsilon}} | a_n |^2$ (otherwise the result is trivially true). Hence

$$\int_0^H | g(it) | dt = \int_0^H | A(-it)B(it) | dt$$

$$\leq \int_0^H | A(-it) |^2 dt + \int_0^H | B(it) |^2 dt$$

$$\leq \int_0^H | A(-it) |^2 dt + \int_0^H | F(it) - \overline{A}(it) |^2 dt$$

$$\leq 3 \int_0^H | A(-it) |^2 dt + 2 \int_0^H | F(it) |^2 dt$$

$$\leq 3 \sum_{n \leq M} (H + 100C^2 n) | a_n |^2 + 2H \sum_{n \leq M} | a_n |^2$$

$$\leq (300C^2 + 5)H \sum_{n \leq M} | a_n |^2 .$$

$$\begin{aligned} S_2 &\leq \sum_{m \leq M, n \geq N} | b_m c_n | \left(\tfrac{\lambda_m}{\lambda_n}\right)^{A+2} \\ &\leq \sum_{m \leq M, n \geq N} | a_m a_n | \left(\tfrac{\lambda_m}{\lambda_n}\right)^{A+2} \\ &\leq \sum_{m \leq M, n \geq N} m^A H^{\frac{r\epsilon}{8}} n^A H^{\frac{r\epsilon}{8}} (C^2 m n^{-1})^{A+2} \\ &\leq H^{\frac{r\epsilon}{4}} C^{2A+4} \sum_{m \leq M} m^{2A+2} \sum_{n \geq N} n^{-2} \\ &\leq H^{\frac{r\epsilon}{4}+2A+3} C^{2A+4} \text{ since } \tfrac{\pi^2}{6} - 1 < 1. \end{aligned}$$

Now

$$S_1 \leq \left(U\ log\ \frac{\lambda_N}{\lambda_M}\right)^{-r} 2^r S_2$$

and

$$log\ \frac{\lambda_N}{\lambda_M} \geq \frac{1}{2}\frac{\lambda_N - \lambda_M}{\lambda_M} \geq (2C^2M)^{-1},$$

$$U\ log\left(\frac{\lambda_N}{\lambda_M}\right) \geq (2C^2)^{-1}H^{\frac{\varepsilon}{2}}.$$

Thus

$$\begin{aligned} |\ J_2\ | \ \leq\ & \tfrac{2B^2}{U^{10}} + 54B(300C^2+5)HU^{-1}\sum_{n\leq M} |\ a_n\ |^2 \\ & +(H+64B^2)H^{-\frac{r\varepsilon}{4}+2A+3}2^r(2C^2)^rC^{2A+4} \\ & +16B^2Exp\left(-\tfrac{U}{8B}\right)H^{\frac{r\varepsilon}{4}+2A+3}C^{2A+4}.\text{(Note } a_1=\lambda_1=1). \end{aligned}$$

So

$$\begin{aligned} & (r+5)U + 100C^2H^{1-\varepsilon} + 2\ |\ J_2\ |\ (\sum_{n\leq M} |\ a_n\ |^2)^{-1} \\ & \leq (r+5)H^{1-\frac{\varepsilon}{2}} + 100C^2H^{1-\varepsilon} + 100Br\,loglog\ K_1 \\ & +\tfrac{4B^2}{H^5} + 108B(300C^2+5)H^{\frac{\varepsilon}{2}} + 128(2^r)(2C^2)^rB^2H^{2A+4-50A}C^{2A+4} \\ & +32B^2C^{2A+4}r!(8B)^rH^{2A+3+\frac{r\varepsilon}{2}-\frac{r}{2}} \\ & \leq 100\ Br\ loglog\ K_1 + rC^2H^{1-\frac{\varepsilon}{4}}\{\tfrac{r+5}{rC^2H^{\frac{\varepsilon}{4}}} + \tfrac{100C^2}{H^{\frac{3\varepsilon}{4}}} \\ & +\tfrac{4B^2}{H^5} + \tfrac{108B(300C^2+5)}{H^{1-\frac{3\varepsilon}{4}}} + 128(2^r)(2C^2)^rB^2H^{-1}C^{2A+4} + 32B^2C^{2A+4}r!(8B)^rH^{-1}\} \\ & \leq 100Br\ loglog\ K_1 + 10C^2rH^{1-\frac{\varepsilon}{4}}. \end{aligned}$$

This completes the proof of the theorem.

§ 4. **SECOND MAIN THEOREM.** *We assume the same conditions as in the first main theorem except that we change the definition of U to $U = H^{\frac{7}{8}} + 50B\ loglog\ K_2$. Then there holds*

$$\int_0^H |\ F(it)\ |\ dt \geq H - 10\ rH^{\frac{7}{8}} - 100r\ B\ loglog\ K_2,$$

where $K_2 = K + 1$.

REMARK. Conditions like $H \geq (r+5)U, U \geq 2^{70}(16B)^2$ are taken care of by the inequality for H.

PROOF. We have,

$$\int_0^H \mid F(it) \mid dt \geq U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} \mid F(it) \mid dt$$

$$\geq U^{-r} Re \left(\int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} F(it) dt \right)$$

$$= U^{-r} Re \left\{ \int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} (1 + A(-it)B(it)) dt \right\}$$

(where $A(s) \equiv 1$ (i.e. $a_1 = 1 = M$) and $B(s) = F(s) - 1) = J_1 + Re\ J_2$ say. Clearly $J_1 \geq H - (r+5)U$. For J_2 we use the main lemma.

$$\mid J_2 \mid \leq \frac{2B^2}{U^{10}} + \frac{54B}{U} \int_0^H \mid g(it) \mid dt + (H + 64B^2) S_1 + 16B^2 Exp\left(-\frac{U}{8B}\right) S_2. \tag{4.1}$$

As in the proof of the first main theorem we can assume $\int_0^H \mid F(it) \mid dt \leq H$ and so $\int_0^H \mid g(it) \mid dt \leq 2H$. We have $\mid g(s) \mid \leq K + 1 = K_2$. Now

$$S_2 \leq H^{\frac{r\epsilon}{4}+2A+3} C^{2A+4},$$

and $U\ log\left(\frac{\lambda_N}{\lambda_M}\right) = U\ log\ \lambda_2 \geq (2C)^{-1} U$,

$$S_1 \leq 2^r S_2 (U\ log\ \lambda_2)^{-r} \leq 2^r\ S_2((2C)^{-1}U)^{-r}.$$

This shows that

$$(r+5)U + \mid J_2 \mid$$

$$\leq (r+5)U + \tfrac{2B^2}{U^{10}} + \tfrac{54B}{U} 2H + \tfrac{(H+64\ B^2)}{(2C^{-1}U)^r} 2^r C^{2A+4} H^{\frac{r\epsilon}{4}+2A+3}$$

$$+16B^2\ Exp\left(-\tfrac{U}{8B}\right) H^{\frac{r\epsilon}{4}+2A+3} C^{2A+4}$$

$$\leq 100rB\ loglog\ K_2 + rH^{\frac{7}{8}} \left\{ \tfrac{r+5}{r} + \tfrac{2B^2}{rH^{\frac{69}{8}}} + \tfrac{108B}{H^{\frac{1}{4}}} \right.$$

$$+(H + 64B^2) 2^r C^{2A+4} H^{\frac{r\epsilon}{4}+2A+3+\frac{r}{16}-(r+1)\frac{7}{8}}$$

$$+16B^2C^{2A+4}(8B)^r r! H^{\frac{rc}{4}+2A+3-\frac{7r}{8}}$$

$$\leq 10rH^{\frac{7}{8}} + 100rB \text{ loglog } K_2,$$

when H satisfies the inequality of the theorem.

§ 5. **THIRD MAIN THEOREM.** *Let $\{a_n\}$ and $\{\lambda_n\}$ be as in the introduction and $|a_n| \leq (nH)^A$ where $A \geq 1$ is an integer constant. Then $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ is analytic in $\sigma \geq A+2$. Suppose $F(s)$ is analytically continuable in $\sigma \geq 0$. Assume that (for some $K \geq 30$) there exist T_1 and T_2 with $0 \leq T_1 \leq H^{\frac{7}{8}}, H - H^{\frac{7}{8}} \leq T_2 \leq H$ such that $|F(\sigma+iT_1)| + |F(\sigma+iT_2)| \leq K$ uniformly in $0 \leq \sigma \leq A+2$. Let*

$$H \geq (4C)^{9000A^2} + 520000\, A^2 \text{ loglog } K_3.$$

Then

$$\int_0^H |F(it)|^2\, dt \geq \sum_{n \leq \alpha H} (H - (3C)^{1000A} H^{\frac{7}{8}} - 130000\, A^2 \text{ loglog } K_3 - 100C^2 n)\, |a_n|^2,$$

where $\alpha = (200C^2)^{-1} 2^{-8A-20}$ and

$$K_3 = \left(\sum_{n \leq H} |a_n| \lambda_n^B \right) K + \left(\sum_{n \leq H} |a_n| \lambda_n^B \right)^2.$$

To prove this theorem we need the following two lemmas.

LEMMA 5.1. *In the interval $[\alpha H, (1600C^2)^{-1}H]$ there exists an X such that*

$$\sum_{X \leq n \leq X + H^{\frac{1}{4}}} |a_n|^2 \leq H^{-\frac{1}{4}} \sum_{n \leq X} |a_n|^2,$$

provided $H \geq 2^{1000A^2} C^{50A}$.

PROOF. Assume that such an X does not exist. Then for all X in $[\alpha H, (1600C^2)^{-1}H]$,

$$\sum_{X \leq n \leq X + H^{\frac{1}{4}}} |a_n|^2 > H^{-\frac{1}{4}} \sum_{n \leq X} |a_n|^2. \tag{5.1}$$

Let $L = \alpha H, I_j = [2^{j-1}L, 2^j L]$ for $j = 1, 2, \cdots, 8A+17$. Also let $I_0 = [1, L]$. Put $S_j = \sum_{n \in I_j} | a_n |^2$ $(j = 0, 1, 2, \cdots, 8A+17)$. For $j \geq 1$ divide the interval I_j into maximum number of disjoint sub-intervals each of length $H^{\frac{1}{4}}$ (discarding the bit at one end). Since the lemma is assumed to be false the sum over each sub-interval is $\geq H^{-\frac{1}{4}} S_{j-1}$. The number of sub-intervals is $\geq [2^{j-1}LH^{-\frac{1}{4}}] - 1 \geq 2^{j-2}LH^{-\frac{1}{4}}$ (provided $2^{j-1}LH^{-\frac{1}{4}} - 2 \geq 2^{j-2}LH^{-\frac{1}{4}}$, i.e. $2^{j-2}LH^{-\frac{1}{4}} \geq 2$ i.e. $\alpha H^{\frac{3}{4}} \geq 4$ i.e. $H \geq (4\alpha^{-1})^{\frac{4}{3}}$). It follows that $S_j \geq 2^{j-2}LH^{-\frac{1}{2}} S_{j-1}$. By induction $S_j \geq (\frac{1}{2}LH^{-\frac{1}{2}})^j S_0$. Since $S_0 \geq 1$ we have in particular

$$S_{8A+17} \geq \left(\frac{1}{2}\alpha H^{\frac{1}{2}}\right)^{8A+17} \geq \left(\frac{1}{2}\alpha\right)^{8A+17} H^{4A+\frac{1}{2}\cdot 17}.$$

On the other hand

$$S_{8A+17} = \sum_{\alpha_1 H \leq n \leq \alpha_2 H} | a_n |^2 \leq \sum_{n \leq \alpha_2 H} (nH)^{2A},$$

where $\alpha_1 = 16^{-1}(200C^2)^{-1}$ and $\alpha_2 = 8^{-1}(200C^2)^{-1}$. Thus $S_{8A+17} \leq H^{4A+1}$. Combining the upper and lower bounds we are led to

$$H^{\frac{1}{2}\cdot 15} \leq (2\alpha^{-1})^{8A+17} \tag{5.2}$$

provided $H \geq (4\alpha^{-1})^{\frac{4}{3}}$ (the latter condition is satisfied by the inequality for H prescribed by the Lemma). But (5.2) contradicts the inequality prescribed for H by the lemma. This contradiction proves the Lemma.

From now on we assume that X is as given by Lemma 5.1.

LEMMA 5.2. *Let* $\overline{A}(s) = \sum_{n \leq X} a_n \lambda_n^{-s}$, $E(s) = \sum_{X \leq n \leq X + H^{\frac{1}{4}}} a_n \lambda_n^{-s}$ *and* $B(s) = F(s) - \overline{A}(s) - E(s)$. *Clearly in* $\sigma \geq A+2$ *we have* $B(s) = \sum_{n \geq X + H^{\frac{1}{4}}} a_n \lambda_n^{-s}$. *Let* $H \geq 2^{1000A^2} C^{50A}$, $U = H^{\frac{7}{8}} + 100\ B\ \log\log\ K_3$, $K_3 \geq 30$ *and* $H \geq (2r+5)U$. *Then we have the following five inequalities.*

(a) $\int_0^H | \overline{A}(it) |^2\, dt \leq 100 C^2 H \sum_{n \leq X} | a_n |^2$,

(b) $\int_{2U+rU}^{H-(r+3)U} |\overline{A}(it)|^2\, dt \geq \sum_{n\leq X} (H-(2r+5)U-100C^2n) \mid a_n \mid^2,$

(c) $\int_0^H \mid E(it)\mid^2 dt \leq 100C^2H^{\frac{3}{4}} \sum_{n\leq X} \mid a_n\mid^2,$

(d) $\int_0^H \mid B(it)\mid^2 dt \leq 1000C^2H \sum_{n\leq X} \mid a_n\mid^2,$

and finally

(e) $\int_0^H \mid A(-it)B(it)\mid dt \leq 400C^2H \sum_{n\leq X} \mid a_n\mid^2,$

where (d) *and* (e) *are true provided*

$$\int_0^H \mid F(it)\mid^2 dt \leq H \sum_{n\leq X} \mid a_n\mid^2 .$$

PROOF. The inequalities (a) and (b) follow from the Montgomery-Vaughan theorem. From the same theorem

$$\begin{aligned}\int_0^H \mid E(it)\mid^2 dt &\leq \sum_{X\leq n\leq X+H^{\frac{1}{4}}} (H+100C^2n)\mid a_n\mid^2\\ &\leq 100C^2H \sum_{X\leq n\leq X+H^{\frac{1}{4}}} \mid a_n\mid^2\end{aligned}$$

and hence (c) follows from Lemma 5.1. Since

$$\mid B(it)\mid^2 \leq 9(\mid F(it)\mid^2 + \mid \overline{A}(it)\mid^2 + \mid E(it)\mid^2)$$

the inequality (d) follows from (a) and (c). Lastly (e) follows from (a) and (d). Thus the lemma is completely proved.

We are now in a position to prove the theorem. We write (with $\lambda = u_1 + u_2 + \cdots + u_r$ as usual)

$$\int_0^H \mid F(it)\mid^2 dt \geq U^{-r}\int_0^U du_r \cdots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} \mid F(it)\mid^2 dt$$

(where $(r+5)U \leq H$ and $0 \leq u_i \leq U$. In fact we assume $(2r+5)U \leq H$).

Now

$$| F(it) |^2 \geq | \overline{A}(it) |^2 + 2Re(A(-it)B(it)) + 2Re(A(-it)E(it)) + 2\ Re(\overline{B}(-it)E(it)),$$

where $\overline{B}(s)$ is the analytic continuation of $\sum_{n \geq X+H^{\frac{1}{4}}} a_n \lambda_n^{-s}$. Accordingly

$$\int_0^H | F(it) |^2\, dt \geq J_1 + J_2 + J_3 + J_4 \tag{5.3}$$

where

$$J_1 = \int_0^H | \overline{A}(it) |^2\, dt,\ J_2 = 2\ Re \int_0^H (A(-it)B(it))dt,$$

$$J_3 = 2\ Re \int_0^H (A(-it)E(it))dt \text{ and } J_4 = 2\ Re \int_0^H (\overline{B}(-it)E(it))dt.$$

By Lemma 5.2(b), we have,

$$J_1 \geq \sum_{n \leq X} (H - (2r+5)U - 100C^2 n) \mid a_n \mid^2 .$$

Also by Lemma 5.2 ((a) and (c)), we have,

$$| J_3 | \leq 2 \int_0^H | A(-it)E(it) |\, dt \leq 200 C^2 H^{\frac{7}{8}} \sum_{n \leq X} | a_n |^2 .$$

Similarly by Lemma 5.2 ((c) and (d)),

$$| J_4 | \leq 800 C^2 H^{\frac{7}{8}} \sum_{n \leq X} | a_n |^2 .$$

For J_2 we use the main lemma. We choose $U = H^{\frac{7}{8}} + 100\ B\ loglog K_3$. We have $g(s) = A(-s)B(s)$. We have

$$| g(s) | \leq \left(\sum_{n \leq H} | a_n | \lambda_n^B \right) K + \left(\sum_{n \leq H} | a_n | \lambda_n^B \right)^2 = K_3.$$

By Lemma 5.2 ((e)) we have

$$\int_0^H | g(it) |\, dt \leq 400 C^2 H \sum_{n \leq X} | a_n |^2$$

Again

$$\begin{aligned} S_2 &\leq \sum_{m\leq X, n\geq X+H^{\frac{1}{4}}} |a_m||a_n| \left(\tfrac{\lambda_m}{\lambda_n}\right)^{A+2} \\ &\leq \sum_{m\leq X, n\geq X+H^{\frac{1}{4}}} (mH)^A(nH)^A(C^2mn^{-1})^{A+2} \\ &\leq C^{2A+4}H^{4A+3}. \end{aligned}$$

Put $x = \frac{\lambda_N}{\lambda_M} - 1$ where $N = \left[X + H^{\frac{1}{4}}\right], M = [X]$. Then $0 < x < \frac{2C(N-M)}{C^{-1}M} < \frac{3C^2H^{\frac{1}{4}}}{\alpha H} < \frac{1}{2}$ under the conditions on H imposed in the theorem. Hence

$$U\, log\left(\frac{\lambda_N}{\lambda_M}\right) \geq \frac{U}{2}\left(\frac{\lambda_N - \lambda_M}{\lambda_M}\right) \geq \frac{U}{2}\left(\frac{N-M-3}{C^2M}\right) \geq \frac{1}{2}H^{\frac{7}{8}}\left(\frac{H^{\frac{1}{4}}-3}{C^2H}\right) \geq \frac{H^{\frac{1}{8}}}{3C^2},$$

(under the conditions on H imposed in the theorem). Thus

$$S_1 \leq 2^r S_2 H^{-\frac{r}{8}}(3C^2)^r.$$

We choose $r = 100A + 100$ and check that $U \geq 2^{70}(16B)^2$, and that $H \geq (2r+5)U$. Thus by applying the main Lemma we obtain

$$\left|\frac{1}{2}J_2\right| \leq \left\{\frac{2B^2}{U^{10}} + \frac{54B}{U}(400C^2H) + \frac{(H+64B^2)2^rC^{2A+4}H^{4A+3}}{((3C^2)^{-1}H^{\frac{1}{8}})^r} + 16B^2Exp\left(-\frac{U}{8B}\right)C^{2A+4}H^{4A+3}\right\}\sum_{n\leq X}|a_n|^2.$$

Hence

$$\int_0^H |F(it)|^2 \geq \sum_{n\leq \alpha H}(H - D - 100C^2n)|a_n|^2,$$

where

$$\begin{aligned} D &= (2r+5)U + 1000C^2H^{\frac{7}{8}} + \tfrac{4B^2}{U^{10}} + \tfrac{43200C^2BH}{U} \\ &+ (H+64B^2)2^{r+1}C^{2A+4}(3C^2)^rH^{4A+3-\frac{r}{8}} \\ &+ 32B^2\ Exp\left(-\tfrac{U}{8B}\right)C^{2A+4}H^{4A+3} \\ &< 130000A^2 loglog\ K_3 + 405AH^{\frac{7}{8}} + 1000C^2H^{\frac{7}{8}} + 36A^2H^{\frac{7}{8}} \end{aligned}$$

$$+43200C^2(3A)H^{\frac{7}{8}}$$

$$+600A^2H(2^{100A+101})C^{2A+4}3^{100A+100}C^{200A+200}H^{4A+3-12A-12}$$

$$+300A^2C^{2A+4}(720)(56)(24A)^8H^{\frac{7}{8}}$$

$$\leq 130000A^2 loglog\ K_3 + H^{\frac{7}{8}}\{405A + 1000C^2 + 36A^2 + 129600AC^2 + 600A^2C^{406A}3^{401A} + 3^{58}A^{10}C^{6A}\}$$

$$\leq 130000A^2 loglog\ K_3 + (3C)^{1000A}.$$

This proves the theorem completely.

The next two theorems due to K. Ramachandra belong to a different class in the sense that restrictions of bounds like those involving K do not appear. His paper follows ours.

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