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PROOF OF SOME CONJECTURES ON THE MEAN-VALUE OF TITCHMARSH SERIES WITH APPLICATIONS TO TITCHMARSH'S PHENOMENON

BY

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§ 1. **INTRODUCTION.** This is a continuation of $[R]_1$ but no previous knowledge of this paper is necessary. In fact we improve these results a good deal. However for some applications of the main results of the present paper, a knowledge of the results of [BRS] is assumed. The main results of the present paper are the following two theorems on what I call weak Titchmarsh series. We begin with a definition.

WEAK TITCHMARSH SERIES. *Let* $0 \leq \varepsilon < 1, D \geq 1, C \geq 1$ *and* $H \geq 10$. *Put* $R = H^{\varepsilon}$. *Let* $a_1 = \lambda_1 = 1$ *and* $\{\lambda_n\}$ $(n = 1,2,3,\cdots)$ *be any sequence of real numbers with* $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$ $(n = 1,2,3,\cdots)$ *and* $\{a_n\}$ $(n = 1,2,3,\cdots)$ *any sequence of complex numbers satisfying*

$$\sum_{\lambda_n \leq X} | a_n | \leq D(\log X)^R$$

for all $X \geq 3C$. *Then for complex* $s = \sigma + it (\sigma > 0)$ *we define the analytic function* $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ *as a weak Titchmarsh series associated with the parameters occuring in the definition.*

THEOREM 1 (FOURTH MAIN THEOREM). *For a weak Titchmarsh series* $F(s)$ *with* $H \geq 36C^2H^{\varepsilon}$, *we have*

$$\liminf_{\sigma \to +0} \int_0^H | F(\sigma + it) | \, dt \geq H - 36C^2H^{\varepsilon} - 12\, CD.$$

THEOREM 2 (FIFTH MAIN THEOREM). *For a weak Titchmarsh series $F(s)$ with $\log H \geq 4320\, C^2(1-\varepsilon)^{-5}$, we have,*

$$\liminf_{\sigma \to +0} \int_0^H |F(\sigma+it)|^2\, dt \geq \sum_{n \leq M} \left(H - \frac{H}{\log H} - 100C^2 n\right) |a_n|^2 - 2D^2$$

where $M = (36C^2)^{-1} H^{1-\varepsilon}(\log H)^{-4}$.

REMARKS. Theorems 1 and 2 have been refered to as the fourth and the fifth main theorems in $[\mathrm{R}]_2$. Also we remark that it is not difficult to improve the conditions in the theorems slightly.

§ 2. **PROOF OF THEOREM 1.** We can argue with $\sigma > 0$ and then pass to the limit as $\sigma \to +0$. But formally the notation is simplified if we treat as though $F(s)$ is convergent absolutely if $\sigma = 0$ and there is no loss of generality. Let r be a positive integer and $0 < U \leq r^{-1}H$. Then since $|F(s)| \geq 1 + Re(F(s))$, we have (with $\lambda = u_1 + \cdots + u_r$),

$$\begin{aligned} \int_0^H |F(it)|\, dt &\geq U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_\lambda^{H-rU+\lambda} |F(it)|\, dt \\ &\geq U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_\lambda^{H-rU+\lambda} \{1 + Re(F(it))\} dt \\ &\geq H - rU - 2^{r+1} U^{-r} J \end{aligned}$$

where $J = \sum_{n=2}^{\infty} |a_n| (\log \lambda_n)^{-r-1}$. Now $J = S_0 + \sum_{j=1}^{\infty} S_j$ where $S_0 = \sum_{\lambda_n \leq 3C} |a_n| (\log \lambda_n)^{-r-1}$ and $S_j = \sum_{3^j C \leq \lambda_n \leq 3^{j+1} C} |a_n| (\log \lambda_n)^{-r-1}$. In S_0 we use $\lambda_n \geq \lambda_2 \geq 1 + C^{-1}$ and so $(\log \lambda_n)^{-r-1} \leq (2C)^{r+1}$ and we obtain $S_0 \leq D(2C)^{r+1}(3C)^R$. Also, we have,

$$\begin{aligned} S_j &\leq D(\log(3^{j+1}C))^R (\log(3^j C))^{-r-1} \\ &\leq D2^R (\log(3^j C))^{R-r-1}, \text{(since } 3^{j+1}C \leq (3^j C)^2\text{)}, \\ &\leq D2^R j^{-2} \text{ by fixing } r = [3R]. \end{aligned}$$

Thus for $r = [3R]$ we have

$$\begin{aligned} J &\leq D(2C)^{r+1}(3C)^R + 2D2^R, \text{(since } \sum_{j=1}^{\infty} j^{-2} < 2\text{)}, \\ &\leq 3D(2C)^{r+1}(3C)^R. \end{aligned}$$

Collecting we have,

$$\begin{aligned} 2^{r+1}U^{-r}J &\le 12CD(3C)^R\left(\tfrac{4C}{U}\right)^r \\ &\le 12CD\left(\tfrac{12C^2}{U}\right)^R \text{ if } U\ge 4C \\ &\le 12CD \text{ by fixing } U=12C^2. \end{aligned}$$

The only condition which we have to satisfy is $rU\le H$ which is secured by $H\ge 36C^2H^{\epsilon}$. This completes the proof of Theorem 1.

§ 3. **PROOF OF THEOREM 2.** We write $\lambda = u_1+\cdots+u_r$, where $0\le u_i\le U$ and $0<U\le r^{-1}H$. We put $M_1=[M], A(s)=\sum_{m\le M_1} a_m\lambda_m^{-s}$ and $B(s)=\sum_{n\ge M_1+1} a_n\lambda_n^{-s}$ so that $F(s)=A(s)+B(s)$. For the moment we suppose M to be a free parameter with the restriction $3\le M\le H$. We use

$$|F(it)|^2 \ge |A(it)|^2 + 2\,Re(A(it)\overline{B(it)}).$$

Now by a well-known theorem of H.L. Montgomery and R.C. Vaughan we have

$$\int_{\lambda}^{H-rU+\lambda} |A(it)|^2\,dt \ge \sum_{n\le M}(H-rU-100C^2n)\,|a_n|^2$$

Next the absolute value of

$$2U^{-r}\int_0^U du_r\cdots\int_0^U du_1\int_{\lambda}^{H-rU+\lambda}(A(it)\overline{B(it)})dt \tag{3.1}$$

does not exceed

$$2^{r+2}U^{-r}\sum_{m\le M_1, n\ge M_1+1} |a_m\overline{a}_n|\left(log\frac{\lambda_n}{\lambda_m}\right)^{-r-1}$$

$$\le 2^{r+2}U^{-r}\left(\sum_{m\le M_1}|a_m|\right)\left(\sum_{n\ge M_1+1}|a_n|\left(log\frac{\lambda_n}{\lambda_{M_1}}\right)^{r+1}\right)$$

Here the m-sum is $\le D(log\ \lambda_{M_1})^R\le D(log(3MC))^R$, since $\lambda_{M_1}\le M_1C\le MC$. It is enough to choose $M\ge 1$ for the bound for the m-sum. The n-sum can be broken up into $\lambda_n\le 3\lambda_{M_1}$ and $3^j\lambda_{M_1}<\lambda_n\le 3^{j+1}\lambda_{M_1}$ ($j=$

$1,2,3,\cdots$). Let us denote these sums by S_0 and S_j. Now since $\left(log\frac{\lambda_n}{\lambda_{M_1}}\right) \geq \left(log\frac{\lambda_{M_1+1}}{\lambda_{M_1}}\right) \geq log\left(1+\frac{1}{C\lambda_{M_1}}\right) \geq (2C\lambda_{M_1})^{-1} \geq (2C^2M)^{-1}$, we obtain

$$S_0 \leq D\left(log\left(3\lambda_{M_1}\right)\right)^R (2C^2M)^{r+1} \leq D(log(3MC))^R(2C^2M)^{r+1}.$$

Also

$$\begin{aligned} S_j &\leq D\left(log\left(3^{j+1}\lambda_{M_1}\right)\right)^R (j\log\ 3)^{-r-1} \\ &\leq D(j\ log\ 3 + log(3MC))^R j^{-r-1} \\ &\leq 2^R D(j\ log 3)^R (log(3MC))^R j^{-r-1} \\ &\leq 4^R D(log(3MC))^R j^{-2}, \text{ if } r \geq R+1, \end{aligned}$$

and so (since $\sum_{j=1}^{\infty} j^{-2} < 2$),

$$\left(\sum_m \cdots\right)\left(\sum_n \cdots\right) \leq D^2(log(3MC))^R(log(3MC))^R Y$$

(where $Y = (2C^2M)^{r+1} + 2(4^R)$)

$$\leq D^2(log(3MC))^{2R}((2C^2M)^{r+1} + 2(4^R)).$$

Hence the absolute value of the expression (3.1) does not exceed

$$D^2(log(3MC))^{2R}\left((8C^2M)\left(\frac{4C^2M}{U}\right)^r + 2\left(\frac{4^R}{U^r}\right)\right) \tag{3.2}$$

$$\leq D^2\left\{8C^2M\left(\frac{4C^2M(log(3MC))^2}{U}\right)^{R+log(8C^2M)} + 2\left(\frac{4}{U}\right)^R\right\}$$

if $U \geq 4C^2M$ and $r \geq R + log(8C^2M)$. We put $U = 12C^2M(log(3MC))^2$ and obtain for (3.2) the bound $D^2\{1+1\} \leq 2D^2$. The conditions to be satisfied are $M \geq 1$ and

$$12C^2M(log(3MC))^2(R + log(8C^2M) + 1) \leq H.$$

In fact we can satisfy $Ur \leq \frac{H}{log\ H}$ by requiring

$$12C^2M(log(3MC))^2(R + log(8C^2M) + 1) \leq \frac{H}{log\ H}.$$

This is satisfied if

$$36C^2M(log(8C^2M))^3R \leq H(log\ H)^{-1}$$

Let $8C^2M \leq H$. Then $36C^2MR \leq H(log\ H)^{-4}$ gives what we want. We choose $M = (36C^2)^{-1}H^{1-\varepsilon}(log\ H)^{-4}$. Clearly this satisfies $8C^2M \leq H$. In order to satisfy $M \geq 1$ we have to secure that

$$(36C^2)^{-1}\frac{((1-\varepsilon)(log\ H))^5}{120}(log\ H)^{-4} \geq 1$$

i.e. $log\ H \geq 4320C^2(1-\varepsilon)^{-5}$.

This completes the proof of Theorem 2.

§ 4. **APPLICATIONS OF THEOREMS 1 AND 2.** An immediate application of Theorem 1 is

THEOREM 3. *Let* $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$ *(where* $(0 < a \leq 1)$*) be the Hurwitz zeta-function in* $\sigma > 1$ *and consider its analytic continuation in* $\sigma \geq 1$. *Then*

$$\min_{T\geq 1}\int_T^{T+H} |\ \zeta(1+it,a)\ |\ dt \geq \frac{1}{a}H + o(H).$$

Let $(\zeta(s))^u = \sum_{n=1}^{\infty} d_u(n)n^{-s}$ where u is any complex constant. Consider the analytic continuation of $(\zeta(s))^u$ in $\sigma \geq 1, t \geq 1$. An immediate corollary to Theorem 2 is

THEOREM 4. *We have,*

$$\min_{T\geq 1}\left(\frac{1}{H}\int_T^{T+H} |\ \zeta(1+it)^u\ |^2\ dt\right) \geq \sum_{n=1}^{\infty}\frac{|\ d_u(n)\ |^2}{n^2} + o(1),$$

and in particular for $u = 1$, *we have,*

$$\min_{T\geq 1}\left(\frac{1}{H}\int_T^{T+H} |\ \zeta(1+it)\ |^2\ dt\right) \geq \frac{\pi^2}{6} + o(1).$$

It is possible to prove by using Theorem 2 a very nice Ω theorem for $|\ (\zeta(1+it))^z\ |$, where $z = e^{i\theta}(0 \leq \theta < 2\pi, \theta \text{ fixed})$ as $t \to \infty$. It is

THEOREM 5. *We have,*

$$\min_{T\geq 1}\max_{T\leq t\leq T+H}|((\zeta(1+it))^z|\geq e^{\gamma}\lambda(\theta)(loglog\ H - logloglog\ H)+O(1), \quad (4.1)$$

where

$$\lambda(\theta)=\prod_p\left\{\left(1-\frac{1}{p}\right)\left(\frac{\sqrt{p^2-Sin^2\theta}+Cos\theta}{p-p^{-1}}\right)^{Cos\ \theta} Exp\left(Sin\ \theta\ Sin^{-1}\left(\frac{Sin\ \theta}{p}\right)\right)\right\}$$

PROOF. By Theorem 2 we have with $\varepsilon=\frac{1}{3}, u=kz$,

$$\frac{1}{H}\int_T^{T+H}|((\zeta(1+it))^u|^2\,dt\geq\frac{1}{2}\sum_{n\leq H^{\frac{1}{4}}}\frac{|d_u(n)|^2}{n^2} \quad (4.2)$$

uniformly in $T\geq 1$, and k any positive integer satisfying $1\leq k\leq log\ H$, provided H exceeds an absolute constant. Denote by S the RHS in (4.2). Then $S^{\frac{1}{2k}}$ has been studied in [BRS] as a function of H as k runs over $1\leq k\leq log\ H$. It has been proved (by considering the maximum term of the sum in S) that

$$\max_{1\leq k\leq log\ H}\left(S^{\frac{1}{2k}}\right)\geq e^{\gamma}\lambda(\theta)(loglog\ H - logloglog\ H)+O(1).$$

This completes the proof of Theorem 5.

These ideas are quite general (applicable to zeta and L-functions of algebraic number fields). For example for ordinary L-series $L(s)=L(s,\chi)$ where χ is a non-principal character *mod* q, we can prove

THEOREM 6. *We have,*

$$\min_{T}\max_{T\leq t\leq T+H}|(L(1+it))^z|\geq e^{\gamma}\lambda(\theta)\frac{\varphi(q)}{q}\{(loglog\ H - logloglog\ H)+O(1)\}$$

uniformly in $q\geq 3$.

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P.S. The theorems of § 4 can be stated for σ satisfying $|\sigma - 1| \leq \psi(t)$ for suitable $\psi(t) \to 0$ as $t \to \infty$.

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