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ON THE FREQUENCY OF TITCHMARSH'S PHENOMENON FOR $\zeta(s)-\Pi$

BY

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§ 1. INTRODUCTION. In a previous paper [R], I proved the following result on $\zeta(1 + it)$.

**THEOREM 1.** Let $0 \leq \theta < 2\pi, z = e^{i\theta}$ and

$$ f(H) = \min_{T \geq 1} \max_{T \leq t \leq T + H} |(\zeta(1 + it))^e| $$

(1)

Then

$$ f(H) \geq e^\gamma \lambda(\theta)(\log \log H - \log \log \log H) + O(1) $$

(2)

where $H \geq \text{Exp}(e^e)$,

$$ \lambda(\theta) = \prod_p \lambda_p(\theta), $$

$$ \lambda_p(\theta) = \left\{ \left(1 - \frac{1}{p}\right) \left(\sqrt{\frac{\sin^2 \theta}{p^2} - \frac{\cos \theta}{p}}\right)^{-\cos \theta} \text{Exp} \left(\sin \theta \sin^{-1} \left(\frac{\sin \theta}{p}\right)\right) \right\}. $$

(3)

In the present paper I prove that

$$ f(H) \leq e^\gamma \lambda(\theta)(\log \log H + \log \log \log H) + O(1). $$

(4)

This result together with Theorem 1 gives the following Theorem.

**THEOREM 2.** We have

$$ |f(H)e^{-\gamma(\lambda(\theta))^{-1}} - \log \log H| \leq \log \log \log H + O(1), $$

(5)

where $H \geq \text{Exp}(e^e)$.

**REMARK.** It is interesting to prove (or disprove!) $f(H)e^{-\gamma(\lambda(\theta))^{-1}} = $
loglog $H + O(1)$.

§ 2. PROOF OF (4). We begin by

**Lemma 1.** Let $T = \text{Exp}((\log H)^2)$ where $H$ exceeds an absolute constant. Then there exists a sub-interval $I$ of $[T, 2T]$ of length $H + 2(\log H)^{10}$, such that the rectangle $(\sigma \geq \frac{3}{4}, t \in I)$ does not contain any zero of $\zeta(s)$ and moreover

$$\max | \log \zeta(\sigma + it) | = O((\log H)^{\frac{1}{4}}(\log \log H)^{-\frac{3}{4}}) \quad (6)$$

the maximum being taken over the rectangle referred to.

**Proof.** Follows from [BR] and the result (due to A.E. Ingham [I], see also [T] page 236 and p. 293-295 [AI]) that the number of zeros of $\zeta(s)$ in $(\sigma \geq \frac{3}{4}, T \leq t \leq 2T)$ is $O(T^{\frac{1}{4}})$.

**Lemma 2.** Let $J$ be the interval obtained by removing from $I$ intervals of length $(\log H)^{10}$ from both ends. Then for $t \in J$, we have,

$$\log \zeta(1 + it) = \sum_{m \geq 1, p} (\sum_{p} p^m s)^{-1} \text{Exp} \left( -\frac{p^m}{X} \right) + O((\log \log H)^{-1}) \quad (7)$$

where $X = \log H \log \log H$ and $s = 1 + it$.

**Proof.** The lemma follows from the fact that the double sum on the right is

$$\frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \log \zeta(s + w) X^w \Gamma(w) dw \quad (8)$$

where $w = u + iv$ is a complex variable. Here we break off the portion $|v| \geq (\log H)^9$ with an error $O((\log \log H)^{-1})$ and move the line of integration to $u = -\frac{1}{4}$. Using Lemma 1 it is easily seen that the horizontal portions and the main integral contribute together $O((\log \log H)^{-1})$.

**Lemma 3.** Denote the double sum in (7) by $S$. Then

$$S = \log \prod_{p \leq X} (1 - p^{-s})^{-1} + O((\log \log H)^{-1}). \quad (9)$$
PROOF. We use the fact that \( \text{Exp}(-p^m X^{-1}) = 1 + O(p^m X^{-1}) \) if \( p^m \leq X \) and \( = O(Xp^{-m}) \) if \( p^m \geq X \). Using this it is easy to see that

\[
S = \sum_{p^m \leq X} \sum_{X^{-1}} (mp^m)^{-1} + O \left( \sum_{p^m \leq X} \sum_{X^{-1}} X (mp^m)^{-1} \right) + O \left( \sum_{p^m \geq X} X (mp^m)^{-1} \right)
\]

Denoting the last double sum by \( S_0 \), we have,

\[
S_0 - \sum_{p \leq X} \log (1 - p^{-s})^{-1} = O \left( \sum_{p \geq X, m \geq 2} (mp^m)^{-1} \right) = O((\log \log H)^{-1}).
\]

**Lemma 4.** We have, for \( t \in J \),

\[
\log \zeta(1 + it) = \sum_{p \leq X} \log (1 - p^{-s})^{-1} + O((\log \log H)^{-1}),
\]

where \( s = 1 + it \).

**Proof.** Follows from Lemmas 1, 2 and 3.

**Lemma 5.** Let \( 0 \leq r < 1, 0 \leq \phi < 2\pi \). Then, we have,

\[
\log |1 - re^{i\phi}|^{-2} \leq -\cos \theta \log \left( \sqrt{1 - r^2 \sin^2 \theta - r \cos \theta} \right) + \sin \theta \sin^{-1}(r \sin \theta).
\]

**Remark.** Put

\[
\lambda_p(\theta) = (1-p^{-1}) \left( \sqrt{1 - p^{-2} \sin^2 \theta - p^{-1} \cos \theta} \right)^{-\cos \theta} \text{Exp} \left( \sin \theta \sin^{-1} \left( \frac{\sin \theta}{p} \right) \right).
\]

In the lemma replace \( re^{i\phi} \) by \( p^{-s} \). Lemmas 4 and 5 complete the proof of (4) and hence that of Theorem 2 since \( \sum_{p \geq X} \log \lambda_p(\theta) = O(X^{-1}) \) and

\[
\prod_{p \leq X} (1 - p^{-1})^{-1} = e^\gamma \log X + O(1). \quad \text{(See [P] page 81)}.
\]

**Proof of Lemma 5.** Denote the LHS of (11) by \( g(\phi) \). Then

\[
g(\phi) = \sum_{n=1}^{\infty} n^{-1} r^n \cos(n\phi + \theta)
\]

\[
g'(\phi) = -\sum_{n=1}^{\infty} r^n \sin(n\phi + \theta)
\]

\[
= \text{Im} \left\{ \frac{-r e^{i(\phi + \theta)} (1 - re^{-i\phi})}{(1 - re^{i\phi}) (1 - re^{-i\phi})} \right\}.
\]
Hence $g'(\phi) = 0$ if $\sin(\phi + \theta) = r \sin \theta$, i.e. if

$$\phi = -\theta + \sin^{-1}(r \sin \theta). \quad (13)$$

At this point $g(\phi)$ attains the maximum as we shall show in the end. Now

$$g(\phi) = \Re \left\{ -e^{i\phi} \left( \log \sqrt{1 - 2r \cos \phi + r^2} - i \sin^{-1} \frac{r \sin \phi}{\sqrt{1 - 2r \cos \phi + r^2}} \right) \right\}$$

$$= -\cos \theta \log \sqrt{1 - 2r \cos \phi + r^2} - \sin \theta \sin^{-1} \left( \frac{r \sin \phi}{\sqrt{1 - 2r \cos \phi + r^2}} \right) \quad (14)$$

From (13) we have

$$\sin \phi = r \sin \theta \cos \theta - \sqrt{1 - r^2 \sin^2 \theta} \sin \theta$$

$$= -\sin \theta \left( \sqrt{1 - r^2 \sin^2 \theta} - r \cos \theta \right),$$

$$\cos \phi = \sqrt{1 - r^2 \sin^2 \theta} \cos \theta + r \sin^2 \theta,$$

$$1 - 2r \cos \phi + r^2 = 1 - 2r \cos \theta \sqrt{1 - r^2 \sin^2 \theta} - 2r^2 \sin^2 \theta + r^2$$

$$= \left( \sqrt{1 - r^2 \sin^2 \theta} - r \cos \theta \right)^2,$$

since $-r^2 \sin^2 \theta + r^2 \cos^2 \theta = -2r^2 \sin^2 \theta + r^2$. Hence

$$g(\phi) \leq h(\theta) \quad (15)$$

where $h(\theta)$ is the RHS of (11), provided $g(\phi)$ attains its maximum for the value $\phi$ given by (13). We now show that

(a) If $\cos \theta \geq 0$ then $g(\pi) < h(\theta)$

and

(b) If $\cos \theta < 0$ then $g(0) < h(\theta)$.

Note that $\sin \theta \sin^{-1}(r \sin \theta) \geq 0$. Hence it suffices to prove (in case (a))

$$g(\pi) = \Re \log \left\{ (1 - re^{i\phi})^{-2} \right\}_{\phi=\pi}$$

$$= -\cos \theta \log(1 + r) < -\cos \theta \log(\sqrt{1 - r^2 \sin^2 \theta} - r \cos \theta)$$
i.e. \( \log(1 + r) > \log(\sqrt{1 - r^2 \sin^2 \theta} - r \cos \theta) \)

i.e. \((1 + r + r \cos \theta)^2 > 1 - r^2 \sin^2 \theta\)

i.e. \((1 + r)^2 + 2r(1 + r) \cos \theta > 1 - r^2\)

i.e. \(1 + r + 2r \cos \theta > 1 - r\) (true since \(\cos \theta \geq 0\))

In case (b) it suffices to prove

\[
g(0) = \Re \log \{(1 - re^{i\phi})^{-2}\}_{\phi=0}
\]

\[
= -\cos \theta \log(1 - r) < -\cos \theta \log(\sqrt{1 - r^2 \sin^2 \theta} - r \cos \theta)
\]

i.e. \(\log(1 - r) < \log(\sqrt{1 - r^2 \sin^2 \theta} - r \cos \theta)\)

i.e. \(1 - r < \sqrt{1 - r^2 \sin^2 \theta} - r \cos \theta\)

i.e. \((1 - r + r \cos \theta)^2 < 1 - r^2 \sin^2 \theta\)

i.e. \((1 - r)^2 + 2r(1 - r) \cos \theta < 1 - r^2\)

i.e. \(1 - r + 2r \cos \theta < 1 + r\) (which is true).

Thus Lemma 5 is completely proved and hence (4) and Theorem 2 are completely proved.

**Remark 1.** In the notation of [BR] our method gives \(\gg TK^{-E}\) disjoint sub-intervals \(I\) (of length \(K\)) of \([T, 2T]\) for which \(\max_{i \in I} |(\zeta(1 + it))^{-2}|\) lies between \(e^\gamma \lambda(\theta)(\log \log K - \log \log \log K) + O(1)\) and \(e^\gamma \lambda(\theta)(\log \log K + \log \log \log K) + O(1)\).

**Remark 2.** By our method we can show that if \(\zeta(s) \neq 0\) in the open half plane \(\sigma > \frac{1}{2}\) then for \(t \geq 100\) we have

\[
| (\zeta(1 + it))^{-2} | \leq 2e^\gamma \lambda(\theta) \log \log t + O(1).
\]
REFERENCES


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