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ON THE FREQUENCY OF TITCHMARSH'S PHENOMENON FOR $\zeta(s)$-IX

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§ 1. **INTRODUCTION.** In a previous paper [R], I proved the following result on $\zeta(1+it)$.

THEOREM 1. *Let* $0 \leq \theta < 2\pi, z = e^{i\theta}$ *and*

$$f(H) = \min_{T\geq 1} \max_{T\leq t\leq T+H} \mid (\zeta(1+it))^z \mid \tag{1}$$

Then

$$f(H) \geq e^{\gamma}\lambda(\theta)(loglog\ H - logloglog\ H) + O(1) \tag{2}$$

where $H \geq Exp(e^e)$,

$$\lambda(\theta) = \prod_p \lambda_p(\theta),$$

$$\lambda_p(\theta) = \left\{\left(1-\frac{1}{p}\right)\left(\sqrt{1-\frac{Sin^2\theta}{p^2}}-\frac{Cos\ \theta}{p}\right)^{-Cos\ \theta} Exp\left(Sin\ \theta\ Sin^{-1}\left(\frac{Sin\ \theta}{p}\right)\right)\right\}. \tag{3}$$

In the present paper I prove that

$$f(H) \leq e^{\gamma}\lambda(\theta)(loglog\ H + logloglog\ H) + O(1). \tag{4}$$

This result together with Theorem 1 gives the following Theorem.

THEOREM 2. *We have*

$$\mid f(H)e^{-\gamma}(\lambda(\theta))^{-1} - loglog\ H \mid \leq logloglog\ H + O(1), \tag{5}$$

where $H \geq Exp(e^e)$.

REMARK. It is interesting to prove (or disprove!) $f(H)e^{-\gamma}(\lambda(\theta))^{-1} =$

loglog $H + O(1)$.

§ 2. **PROOF OF (4).** We begin by

LEMMA 1. *Let* $T = Exp((\log H)^2)$ *where* H *exceeds an absolute constant. Then there exists a sub-interval* I *of* $[T, 2T]$ *of length* $H + 2(\log H)^{10}$, *such that the rectangle* $(\sigma \geq \frac{3}{4}, t \in I)$ *does not contain any zero of* $\zeta(s)$ *and moreover*

$$\max \mid \log \zeta(\sigma + it) \mid = O((\log H)^{\frac{1}{4}}(\log\log H)^{-\frac{3}{4}}) \quad (6)$$

the maximum being taken over the rectangle referred to.

PROOF. Follows from [BR] and the result (due to A.E. Ingham [I], see also [T] page 236 and p. 293-295 [AI]) that the number of zeros of $\zeta(s)$ in $(\sigma \geq \frac{3}{4}, T \leq t \leq 2T)$ is $O(T^{\frac{3}{4}})$.

LEMMA 2. *Let* J *be the interval obtained by removing from* I *intervals of length* $(\log H)^{10}$ *from both ends. Then for* $t \in J$, *we have,*

$$\log \zeta(1 + it) = \sum_{m \geq 1, p} \sum (mp^{ms})^{-1} Exp\left(-\frac{p^m}{X}\right) + O((\log\log H)^{-1}) \quad (7)$$

where $X = \log H \log\log H$ *and* $s = 1 + it$.

PROOF. The lemma follows from the fact that the double sum on the right is

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \log \zeta(s + w) X^w \Gamma(w) dw \quad (8)$$

where $w = u + iv$ is a complex variable. Here we break off the portion $\mid v \mid \geq (\log H)^9$ with an error $O((\log\log H)^{-1})$ and move the line of integration to $u = -\frac{1}{4}$. Using Lemma 1 it is easily seen that the horizontal portions and the main integral contribute together $O((\log\log H)^{-1})$.

LEMMA 3. *Denote the double sum in (7) by* S. *Then*

$$S = \log \prod_{p \leq X} (1 - p^{-s})^{-1} + O((\log\log H)^{-1}). \quad (9)$$

PROOF. We use the fact that $Exp(-p^m X^{-1}) = 1 + O(p^m X^{-1})$ if $p^m \leq X$ and $= O(Xp^{-m})$ if $p^m \geq X$. Using this it is easy to see that

$$\begin{aligned} S &= \sum\sum_{p^m \leq X} (mp^{ms})^{-1} + O\left(\sum\sum_{p^m \leq X} X^{-1}\right) + O\left(\sum\sum_{p^m \geq X} X(mp^{2m})^{-1}\right) \\ &= \sum\sum_{p^m \leq X} (mp^{ms})^{-1} + O((loglog\ H)^{-1}). \end{aligned}$$

Denoting the last double sum by S_0, we have,

$$S_0 - \sum_{p \leq X} log(1-p^{-s})^{-1} = O\left(\sum\sum_{p^m \geq X, m \geq 2} (mp^m)^{-1}\right) = O((loglog\ H)^{-1}).$$

LEMMA 4. *We have, for $t \in J$,*

$$log\ \zeta(1+it) = \sum_{p \leq X} log(1-p^{-s})^{-1} + O((loglog\ H)^{-1}), \tag{10}$$

where $s = 1 + it$.

PROOF. Follows from Lemmas 1,2 and 3.

LEMMA 5. *Let $0 \leq r < 1, 0 \leq \phi < 2\pi$. Then, we have,*

$$log \mid (1-re^{i\phi})^{-z} \mid \leq -Cos\,\theta\ log\left(\sqrt{1-r^2\ Sin^2\theta} - r\ Cos\ \theta\right) + Sin\,\theta\ Sin^{-1}(r\ Sin\,\theta). \tag{11}$$

REMARK. Put

$$\lambda_p(\theta) = (1-p^{-1})\left(\sqrt{1-p^{-2}Sin^2\theta} - p^{-1}Cos\ \theta\right)^{-Cos\ \theta} Exp\left(Sin\ \theta\ Sin^{-1}\left(\frac{Sin\ \theta}{p}\right)\right). \tag{12}$$

In the lemma replace $re^{i\phi}$ by p^{-s}. Lemmas 4 and 5 complete the proof of (4) and hence that of Theorem 2 since $\sum_{p \geq X} log\ \lambda_p(\theta) = O(X^{-1})$ and $\prod_{p \leq X} (1-p^{-1})^{-1} = e^{\gamma}\ log\ X + O(1)$. (See [P] page 81).

PROOF OF LEMMA 5. Denote the LHS of (11) by $g(\phi)$. Then

$$\begin{aligned} g(\phi) &= \sum_{n=1}^{\infty} n^{-1} r^n Cos(n\phi + \theta) \\ g'(\phi) &= -\sum_{n=1}^{\infty} r^n\ Sin(n\phi + \theta) \\ &= Im\left\{\frac{-re^{i(\phi+\theta)}(1-re^{-i\phi})}{(1-re^{i\phi})(1-re^{-i\phi})}\right\}. \end{aligned}$$

Hence $g'(\phi) = 0$ if $Sin(\phi + \theta) = r\ Sin\ \theta$, i.e. if

$$\phi = -\theta + Sin^{-1}(r\ Sin\ \theta). \tag{13}$$

At this point $g(\phi)$ attains the maximum as we shall show in the end. Now

$$g(\phi) = Re\left\{-e^{i\theta}\left(log\sqrt{1 - 2r\ Cos\ \phi + r^2} - i\ Sin^{-1}\ \frac{r\ Sin\ \phi}{\sqrt{1 - 2r\ Cos\ \phi + r^2}}\right)\right\}$$

$$= -Cos\ \theta\ log\sqrt{1 - 2r\ Cos\ \phi + r^2} - Sin\ \theta Sin^{-1}\left(\frac{r\ Sin\ \phi}{\sqrt{1 - 2r\ Cos\ \phi + r^2}}\right) \tag{14}$$

From (13) we have

$$\begin{aligned} Sin\ \phi &= r\ Sin\ \theta\ Cos\ \theta - \sqrt{1 - r^2\ Sin^2\theta}\ Sin\ \theta \\ &= -Sin\ \theta\left(\sqrt{1 - r^2Sin^2\theta} - r\ Cos\ \theta\right), \\ Cos\ \phi &= \sqrt{1 - r^2Sin^2\theta}\ Cos\ \theta + r\ Sin^2\theta, \\ 1 - 2r\ Cos\ \phi + r^2 &= 1 - 2r\ Cos\ \theta\sqrt{1 - r^2Sin^2\theta} - 2r^2Sin^2\theta + r^2 \\ &= \left(\sqrt{1 - r^2Sin^2\theta} - r\ Cos\ \theta\right)^2, \end{aligned}$$

since $-r^2Sin^2\theta + r^2Cos^2\theta = -2r^2Sin^2\theta + r^2$. Hence

$$g(\phi) \leq h(\theta) \tag{15}$$

where $h(\theta)$ is the RHS of (11), provided $g(\phi)$ attains its maximum for the value ϕ given by (13). We now show that

(a) If $Cos\ \theta \geq 0$ then $g(\pi) < h(\theta)$

and

(b) If $Cos\ \theta < 0$ then $g(0) < h(\theta)$.

Note that $Sin\ \theta\ Sin^{-1}(r\ Sin\ \theta) \geq 0$. Hence it suffices to prove (in case (a))

$$\begin{aligned} g(\pi) &= Re\ log\left\{(1 - re^{i\phi})^{-z}\right\}_{\phi=\pi} \\ &= -Cos\ \theta\ log(1 + r) < -Cos\ \theta\ log(\sqrt{1 - r^2Sin^2\theta} - r\ Cos\ \theta) \end{aligned}$$

i.e. $log(1+r) > log(\sqrt{1-r^2Sin^2\theta} - r\ Cos\ \theta)$

i.e. $(1+r+r\ Cos\ \theta)^2 > 1-r^2Sin^2\theta$

i.e. $(1+r)^2 + 2r(1+r)Cos\ \theta > 1-r^2$

i.e. $1+r+2r\ Cos\ \theta > 1-r$ (true since $Cos\ \theta \geq 0$)

In case (b) it suffices to prove

$$\begin{aligned} g(0) &= Re\ log\{(1-re^{i\phi})^{-z}\}_{\phi=0} \\ &= -Cos\ \theta\ log(1-r) < -Cos\ \theta\ log(\sqrt{1-r^2Sin^2\theta} - r\ Cos\ \theta) \end{aligned}$$

i.e. $log(1-r) < log(\sqrt{1-r^2Sin^2\theta} - r\ Cos\ \theta)$

i.e. $1-r < \sqrt{1-r^2Sin^2\theta} - r\ Cos\ \theta$

i.e. $(1-r+r\ cos\ \theta)^2 < 1-r^2Sin^2\theta$

i.e. $(1-r)^2 + 2r(1-r)Cos\ \theta < 1-r^2$

i.e. $1-r+2r\ Cos\ \theta < 1+r$ (which is true).

Thus Lemma 5 is completely proved and hence (4) and Theorem 2 are completely proved.

REMARK 1. In the notation of [BR] our method gives $\gg TK^{-E}$ disjoint sub-intervals I (of length K) of $[T, 2T]$ for which $\max_{t\in I} \mid (\zeta(1+it))^z \mid$ lies between $e^\gamma\lambda(\theta)(loglog\ K - logloglog\ K) + O(1)$ and $e^\gamma\lambda(\theta)(loglog\ K + logloglog\ K) + O(1)$.

REMARK 2. By our method we can show that if $\zeta(s) \neq 0$ in the open half plane $\sigma > \frac{1}{2}$ then for $t \geq 100$ we have

$$\mid (\zeta(1+it))^z \mid \leq 2e^\gamma\lambda(\theta)loglog\ t + O(1).$$

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