*Hardy-Ramanujan Journal*
Vol.14 (1991) 1-20

# PROOF OF SOME CONJECTURES ON THE MEAN-VALUE OF TITCHMARSH SERIES-II

BY

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§ 1. **INTRODUCTION.** One of the crowning achievements in the Theory of TITCHMARSH SERIES (introduced by the second of us $[R]_1$) is the following theorem discovered by R. Balasubramanian and K. Ramachandra [BR].

**THEOREM 1.** *Let* $\{a_n\}(n = 1,2,3,\cdots)$ *be a sequence of complex numbers and* $\{\lambda_n\}(n = 1,2,3,\cdots)$ *a sequence of real numbers with* $a_1 = \lambda_1 = 1$ *and* $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C(n = 1,2,3,\cdots)$ *where* $C$ *is a positive constant. Let* $H \geq 10$ *be a real parameter and* $|\, a_n \,| \leq (nH)^A$ *where* $A$ *is a positive integer constant. Suppose that* $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}, (s = \sigma + it)$, *can be continued from* $\sigma \geq A + 2$ *analytically in* $\{\sigma \geq 0, 0 \leq t \leq H\}$. *Assume that (for some* $K \geq 30$*) there exist* $T_1, T_2$ *with* $0 \leq T_1 \leq H^{\frac{1}{8}}, H - H^{\frac{1}{8}} \leq T_2 \leq H$, *such that* $|\, F(\sigma + iT_1) \,| + |\, F(\sigma + iT_2) \,| \leq K$ *uniformly in* $\sigma \geq 0$. *Then for all*

*$H \geq (4C)^{9000A^2} + 520000A^2 loglog\ K$ there holds*

*(a)* $\int_0^H | F(it) |^2 dt$
$\geq \sum_{n \leq \alpha H} (H - (3C)^{1000A} H^{\frac{7}{8}} - 130000A^2 loglog\ K - 100C^2 n) | a_n |^2,$

*where $\alpha = (200C^2)^{-1} 2^{-8A-20}$. Also, there holds,*

$$(b) \qquad \frac{1}{H} \int_0^H | F(it) | dt \geq 1 - \frac{C_1}{H^{\frac{1}{8}}} - \frac{C_2 loglog\ K}{H}$$

*where $C_1$ and $C_2$ are effectively computable positive constants depending only on $A$ and $C$.*

**REMARK 1.** In (a) the case $a_1 = 1$ and $a_n = 0$ for $n > 1$ shows that it is not possible to replace the RHS by $(1 + \varepsilon)$ times its present form. Also the example $F(it) = \zeta(\sigma + it + iT)$ where $\sigma$ is a large negative constant and $H$ is a large constant times $T$, shows that in the most general case we cannot have $\alpha > \frac{1}{2\pi}$ and RHS replaced by $\beta\ H \sum_{n \leq \alpha H} (1 - \frac{n}{\alpha H}) | a_n |^2$ (where $\beta > 0$ is any absolute constant).

**REMARK 2.** (b) is our second main theorem and (a) our third main theorem in [BR]. This theorem covers every possible application to the Riemann zeta-function on its mean square lower bounds and also to $\Omega$ theorems (except the $\Omega$ theorem for $\zeta(\sigma + it)(\frac{1}{2} < \sigma < 1)$ of H.L. Montgomery [M]). However another important result on TITCHMARSH SERIES is $[R]_3$. (This covers some very important applications to $\zeta(1 + it)$ and so on). Having reached our goal thus, we now turn to the question "How much can we relax the conditions on TITCHMARSH SERIES $F(s)$ and still prove worthwhile results?" Of course our results are of interest for their own sake and we do not envisage any fresh applications from the results of the present paper. (We have proved five main theorems on TITCHMARSH SERIES in [BR] by R. Balasubramanian and K. Ramachandra and $[R]_3$ by K. Ramachandra). Our sixth main theorem is

**SIXTH MAIN THEOREM.** *Let $0 \leq \varepsilon < 1, C \geq 1, D \geq 1, E = \frac{1}{1-\varepsilon}, a_1 = \lambda_1 = 1, \frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$ (for $n = 1, 2, 3, \cdots$) and for $n \geq 2$ let $| a_n | \leq$*

$Exp\{(DH^{\varepsilon}-100C-1)\log\lambda_n\},(H\geq 10)$. *Let* $F(s)=\sum_{n=1}^{\infty}a_n\lambda_n^{-s}$ *(convergent absolutely in* $\sigma\geq DH^{\varepsilon}$*) admit an analytic continuation in* $\{\sigma\geq 0, 0\leq t\leq H\}$. *Assume that there exist* $T_1,T_2$ *with* $0\leq T_1\leq\frac{1}{8}H,\frac{7}{8}H\leq T_2\leq H$, *such that* $|F(\sigma+iT_1)|+|F(\sigma+iT_2)|\leq K$ *(where* $K\geq 30$ *holds uniformly in* $\sigma\geq 0$. *Let finally* $H\geq max\{(100D\ \log\log K)^E, 100D(100DE)^{3E}\}$. *Then there holds*

$$200\int_0^H |F(it)|\,dt\geq H.$$

Our next main theorem is as follows

**SEVENTH MAIN THEOREM.** *Let* $0\leq\varepsilon<1, C\geq 1, D\geq 2560C^2, E=\frac{1}{1-\varepsilon}, a_1=\lambda_1=1,\frac{1}{C}\leq\lambda_{n+1}-\lambda_n\leq C$ *(for* $n=1,2,3,\cdots$*) and for* $n\geq 2$ *let* $|a_n|\leq Exp\{(DH^{\varepsilon}-100C-1)\log\lambda_n\}$. *We assume*

$$H\geq max\{(256\ D\ \log\log\ K)^E,(24000C^6DE)^{3E}\}$$

*where* $F(s)=\sum_{n=1}^{\infty}a_n\lambda_n^{-s}$ *has the proporties stated in the sixth main theorem and with* $K$ *defined exactly as in the sixth main theorem. Then there holds*

$$\frac{10^8}{H}\int_0^H |F(it)|^2\,dt\geq\sum_{n\leq M_1}|a_n|^2$$

*where* $M_1=[(8000C^6D)^{-1}H^{1-\varepsilon}]$.

§ 2. **PROOF OF THE SIXTH MAIN THEOREM.** We begin with some lemmas. We put $B=DH^{\varepsilon}$.

**LEMMA 2.1.** *For* $\sigma\geq B$, *we have* $|F(s)-1|\leq\frac{1}{100}$.

**PROOF.** We observe that $\lambda_n\geq 1+\frac{n-1}{C}$ and hence, for $\sigma\geq B$, $|a_n\lambda_n^{-s}|\leq\left(1+\frac{n-1}{C}\right)^{-100C-1}$ and so

$$\begin{aligned}|F(s)-1| &\leq \sum_{n=1}^{\infty}\left\{C^{100C+1}(n+C)^{-100C-1}\right\}\\ &\leq C^{100C+1}\textstyle\int_0^{\infty}(u+C)^{-100C-1}du=\frac{1}{100}.\end{aligned}$$

**LEMMA 2.2.** *Let $z = x + iy$ be a complex variable with $|x| \leq \frac{1}{4}$. Then for all $y$ we have, $|Exp((Sin\ z)^2)| \leq e^{\frac{1}{2}} \leq 2$. Moreover if $|y| \geq 2$, we have,*

$$|Exp((Sin\ z)^2)| \leq e^{\frac{1}{2}}(ExpExp\ |y|)^{-1} \leq 2(ExpExp\ |y|)^{-1}.$$

**PROOF.** See Lemma 2.2.1 of the previous paper I of this series by us [BR].

**LEMMA 2.3.** *Let $B_0 > 0, k$ and $\sigma$ real with $0 < |\sigma| \leq B_0$. Then, we have,*

$$\int_{-\infty}^{\infty} \left| Exp\left(Sin^2\left(\frac{ik-\sigma-iu_1}{4B_0}\right)\right) \frac{du_1}{ik-\sigma-iu_1} \right| \leq 12 + 4\ log\ \frac{B_0}{|\sigma|}.$$

**PROOF.** See Lemma 2.2.2 of the paper referred to in the proof of Lemma 2.2.

**LEMMA 2.4.** *Put $s_0 = B + it_0$ where $B = DH^{\epsilon}$. Then subject to $\frac{1}{4}H \leq t_0 \leq \frac{3}{4}H$, we have,*

$$F(s_0) = \frac{1}{2\pi i}\int F(w)X^{w-s_0}Exp\left(Sin^2\left(\frac{w-s_0}{4B}\right)\right)\frac{dw}{w-s_0}, \tag{2.1}$$

*the contour being the (anti-clockwise) boundary of the rectangle bounded by the lines $Re\ w = 0, Re\ w = 2B, Im\ w = T_1, Im\ w = T_2$.*

**PROOF.** Follows by Cauchy's theorem.

**LEMMA 2.5.** *Let $I_1, I_2$ be the integrals over the horizontal boundaries in (2.1) and $J_1$ that over the left vertical boundary and $J_2$ that over the right vertical boundary. Then*

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} (|I_1| + |I_2|)dt_0$$

$$\leq 2\cdot\frac{H}{2}\cdot\left\{\frac{1}{2\pi}K(X^B + X^{-B})\frac{8}{H}\cdot 2\cdot\left(ExpExp\ \frac{H}{32B}\right)^{-1}\cdot 2B\right\}, \tag{2.2}$$

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} |J_1|\ dt_0$$

$$\leq \frac{1}{2\pi}\left\{\int_0^H |F(iv)| \int_{(t_0)} X^{-B} \left| Exp\left(Sin^2\left(\frac{iv-B-it_0}{4B}\right)\right)\right| \left|\frac{dt_0 dv}{iv-B-it_0}\right|\right\}$$

$$\leq \frac{1}{2\pi}\left(\int_0^H |F(iv)|\ dv\right)X^{-B}\cdot 12, \tag{2.3}$$

*and*

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} \mid J_2 \mid dt_0 \leq \frac{1}{2\pi}\left(H \cdot \frac{101}{100}\right) X^B \cdot 12 \leq 2HX^B, \tag{2.4}$$

*provided* $H \geq 64B$ *i.e.* $H \geq 64DH^{\varepsilon}$ *holds for the validity of (2.2). We have employed* $(t_0)$ *to mean integration over* $-\infty < t_0 < \infty$.

**PROOF.** Follows by Lemmas 2.2 and 2.3.

**LEMMA 2.6.** *Let* $X$ *be chosen by* $X^B = X^{DH^{\varepsilon}} = \frac{1}{5}$ *and let* $H \geq (64D)^E$. *Then, we have,*

$$\begin{aligned} \tfrac{H}{2}\left(\tfrac{99}{100}\right) \quad \leq \quad & \textstyle\int_{\frac{H}{4}}^{\frac{3H}{4}} \mid F(s_0) \mid dt_0 \leq \frac{2H}{5} + 10 \int_0^H \mid F(iv) \mid dv \\ & +30DH^{\varepsilon}K\left(ExpExp\, \tfrac{H^{1-\varepsilon}}{32D}\right)^{-1}. \end{aligned} \tag{2.5}$$

**PROOF.** Follows from Lemma 2.5 on observing that

$$2\left(\frac{1}{2}\right)\left(\frac{1}{2\pi}\right)\left(\frac{26}{5}\right)(8)(2)(2) \leq 30.$$

We can now complete the proof of the sixth main theorem with the help of (2.5). Let $H \geq (64D)^E$ so that $H^{1-\varepsilon}(32D)^{-1} \geq 2$ and so $ExpExp\frac{H^{1-\varepsilon}}{32D} \geq \left(ExpExp\, \frac{H^{1-\varepsilon}}{64D}\right)^2$. Let $ExpExp\frac{H^{1-\varepsilon}}{64D} \geq max(K, 30DH)$
i.e. $H \geq (64D\ loglog\ K)^E$ and since $Exp\, \frac{H^{1-\varepsilon}}{64D} \geq \frac{1}{n!}\left(\frac{H^{1-\varepsilon}}{64D}\right)^n$ for all integers $n \geq 1$,

$$\left(\frac{H^{1-\varepsilon}}{64D}\right)^{2E} \cdot \frac{1}{(3E)^{3E}} \geq 30\ DH$$

would suffice to secure what we want. This requires

$$H \geq 30D\left(\frac{3(64D)^{\frac{2}{3}}}{1-\varepsilon}\right)^{3E},$$

which is secured by

$$H \geq 30D(48DE)^{3E}.$$

Hence under the condition $H \geq max((64D\ loglog\ K)^E, 30D(48\ DE)^{3E})$ we have

$$\begin{aligned} \textstyle\int_0^H \mid F(iv) \mid dv \quad & \geq \quad \tfrac{H}{10}\left(\tfrac{99}{200} - \tfrac{2}{5}\right) - \tfrac{1}{10} \\ & \geq \quad \tfrac{H}{112} \text{ if } H \geq 20000. \end{aligned}$$

The last condition is clearly satisfied by the condition $H \geq 300(48\ DE)^{3E}$ imposed already. So we get better constants in lower bounds for $H$. In the theorem we have rounded off these constants.

The sixth main theorem is completely proved.

## § 3. PROOF OF THE SEVENTH MAIN THEOREM.

The proof of this theorem is more involved. It consists of four steps. Step III deals with a convexity question and we prove a convexity theorem of independent interest. In all there are ten lemmas and the tenth is essentially the theorem with sharper constants than those in the theorem. In the theorem we have rounded off the constants.

**STEP I.** We put $B = DH^{\varepsilon}, s_0 = \frac{4}{\log H} + it_0$, where $\frac{1}{4}H \leq t_0 \leq \frac{3}{4}H$. Since $F(s)$ converges absolutely in $\sigma \geq B$, we have, with $Y = H^{\log\log H}$,

$$\frac{1}{2\pi i}\int_{Re\ w=2B} F(w)Y^{w-s_0}Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right)\frac{dw}{w-s_0}$$

$$= \sum_{n=1}^{\infty}\frac{a_n}{\lambda_n^{s_0}}\Delta\left(\frac{Y}{\lambda_n}, B, s_0\right), \tag{3.1}$$

where

$$\Delta\left(\frac{Y}{\lambda_n}, B, s_0\right) = \frac{1}{2\pi i}\int_{Re\ w=2B}\left(\frac{Y}{\lambda_n}\right)^{w-s_0} Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right)\frac{dw}{w-s_0}.$$

For brevity we sometimes write $\Delta$ for $\Delta\left(\frac{Y}{\lambda_n}, B, s_0\right)$. Initially we set $H \geq 100, D \geq 2560C^2$ for some reasons to follow. We begin with

**LEMMA 3.1.** *We have,*

(a) $|\Delta| \leq \frac{1}{2\pi}\left(\frac{Y}{\lambda_n}\right)^{2B}\cdot 12,$

(b) $|\Delta - 1| \leq \frac{1}{2\pi}\left(\frac{\lambda_n}{Y}\right)^{2B}\cdot 12.$

**REMARK.** We use (a) for $\lambda_n \geq Y$ and (b) for $\lambda_n \leq Y$, although both are valid whether $\lambda_n \geq Y$ or not.

**PROOF.** For (a) we move the line of integration to $Re\ w = 2B + \frac{4}{\log H}$. For (b) we move the line of integration to $Re\ w = -2B + \frac{4}{\log H}$. In both the cases we use Lemma 2.3 with $B_0 = 2B$. The only condition that we need is $2B > 1 + \frac{4}{\log H}$ which is clearly satisfied.

**LEMMA 3.2.** *We have, if* $Y^2 \geq 10C$,

$$\left| \sum_{\lambda_n \geq Y^2} \frac{a_n}{\lambda_n^{s_0}} \Delta \right| \leq \frac{1}{100} C^{100C} \left( \frac{Y^2}{C} - 2C \right)^{-100C}.$$

**PROOF.** By (a) of Lemma 3.1 we have, since $\sqrt{\lambda_n} \geq Y$ and so $| \Delta | \leq 2\lambda_n^{-B}$,

$$\begin{aligned} \text{LHS} &\leq 2 \sum_{\lambda_n \geq Y^2} \frac{|a_n|}{\lambda_n^B} \leq 2 \sum_{\lambda_n \geq Y^2} \lambda_n^{-100C-1} \\ &\leq \sum_{nC \geq Y^2} \left(1 + \frac{n-1}{C}\right)^{-100C-1} \leq C^{100C+1} \sum_{n \geq \frac{Y^2}{C}} (C+n-1)^{-100C-1} \\ &\leq C^{100C+1}(100C)^{-1} \left( \left[\frac{Y^2}{C}\right] - 2 + C \right)^{-100C}, \text{ since } Y^2 \geq 10C, \\ &\leq C^{100C} \left( \frac{Y^2}{C} - 2C \right)^{-100C}. \end{aligned}$$

This proves the lemma completely.

In the LHS of (3.1) we would like to cut off the portions $Im\ w \leq T_1, Im\ w \geq T_2$ and move the line of integration in the rest to $Re\ w = 0$. The horizontal bits contribute two terms the sum of whose absolute values is

$$\leq 2 \cdot \frac{1}{2\pi} K \cdot Y^{2B} \cdot 2B \cdot 2 \left( ExpExp\ \frac{H}{64B} \right)^{-1} \cdot \frac{8}{H}, \text{ if } \quad H \geq (128D)^E.$$

Also the infinite vertical bits do not together exceed in absolute value

$$2 \cdot \frac{1}{2\pi} \cdot \frac{101}{100} Y^{2B} \left( ExpExp\ \frac{H}{128B} \right)^{-1} \times$$

$\int_{Im(w-s_0) \geq \frac{H}{8}} \left| Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right) \right|^{\frac{1}{2}} \left| \frac{dw}{w-s_0} \right|$. Since for $H \geq (128\ D)^E$,

$$Exp \left| Sin^2\left(\frac{w-s_0}{8B}\right) \right| = \left| Exp\left(\tfrac{1}{2}\ Sin^2\left(\frac{w-s_0}{8B}\right)\right) \right|^2$$

$\leq \left| Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right) \right|^{\frac{1}{2}} \left| Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right) \right|^{\frac{1}{2}}$ and $\left| Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right) \right|^{\frac{1}{2}}$

$\leq \left( ExpExp\ \frac{H}{64B} \right)^{-\frac{1}{2}} \leq \left( ExpExp\ \frac{H}{128H} \right)^{-1}$. Also putting $Im\ w = v$,

$$\left| Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right) \right|^{\frac{1}{2}} < \left( ExpExp \frac{|v-t_0|}{8B} \right)^{-\frac{1}{2}}$$

$\leq \left(Exp \mid \frac{v-t_0}{16B} \mid\right)^{-1}$ and $\int_0^\infty Exp\left(-\frac{v}{16B}\right)dv = 16B$.

Thus the contribution from the infinite vertical bits do not exceed $\left(\text{since in } Im(w-s_0) \geq \frac{H}{8}, \mid w - s_0 \mid \geq \frac{H}{8}\right)$
$2 \cdot \frac{1}{2\pi} \cdot \frac{101}{100} Y^{2B} \left(ExpExp\, \frac{Y}{128B}\right)^{-1} 16B \cdot \frac{8}{H}$ (by Lemma 2.1). Thus we have

**LEMMA 3.3.** *We have, with some $\theta$'s not necessarily the same ones, with* $\mid \theta \mid \leq 1$

$$F(s_0) = \sum_{\lambda_n \leq Y^2} \frac{a_n}{\lambda_n^{s_0}} \Delta + \frac{\theta}{100} C^{100C} \left(\frac{Y^2}{C} - 2C\right)^{-100C}$$

$$+\theta \cdot 2 \cdot \frac{1}{2\pi} \cdot K \cdot Y^{2B} 4B \cdot 2 \cdot \left(ExpExp\, \frac{Y}{128B}\right)^{-1} \frac{8}{H}$$

$$+\frac{\theta}{2\pi} \int_{\substack{T_1 \leq Im\, w \leq T_2 \\ Re\, w = 0}} \mid F(w) Y^{w-s_0} \mid Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right) \mid\mid \frac{dw}{w-s_0} \mid$$

*provided* $H \geq (256D)^E$.

**PROOF.** Follows since $K \geq 30$ from the arguments preceeding the lemma.

**LEMMA 3.4.** *We have,*

$$\mid \sum_{\lambda_n \leq Y^2} \frac{a_n}{\lambda_n^{s_0}} \Delta \mid^2 \leq 16 \left\{ \mid F(s_0) \mid^2 + \frac{1}{10000} C^{200C} \left(\frac{Y^2}{C} - 2C\right)^{-200C} \right.$$

$$+ \left(\frac{128}{2\pi}\right)^2 K^2 Y^{4B} B^2 \left(ExpExp\, \frac{H}{128B}\right)^{-2} \frac{1}{H^2}$$

$$\left. + \frac{Y^{-\frac{8}{\log H}}}{(2\pi)^2} \int_{\substack{T_1 \leq Im\, w \leq T_2 \\ Re\, w = 0}} \mid F(w) \mid^2 \mid Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right) \mid\mid \frac{dw}{w-s_0} \mid J \right\}$$

*where*

$$J = \int_{\substack{-\infty \leq Im\, w \leq \infty \\ Re\, w = 0}} \mid Exp\left(Sin^2\left(\frac{w-s_0}{8B}\right)\right) \mid\mid \frac{dw}{w-s_0} \mid .$$

*We have used $H \geq (256D)^E$ and $Y^2 \geq 10C$. Also since $\frac{4}{\log H} \leq 2B$ we have $J \leq 12 + 4\log(2B \log H)$ by Lemma 2.3, with $B_0 = 2B$. In the last but one integral note that $\mid w - s_0 \mid \geq \frac{4}{\log H}$.*

**PROOF.** Follows from Lemma 3.3.

**STEP II.** In this step we obtain a lower bound for

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} \mid \varphi(s_0) \mid^2 dt_0 \text{ where } \varphi(s_0) = \sum_{\lambda_n \le Y^2} \frac{a_n}{\lambda_n^{s_0}} \Delta$$

and $Y = H^{loglog\, H}$. Put $\sigma_0 = \frac{4}{log\, H}$ and $M_1 = \left[\frac{H^{1-\varepsilon}}{8000C^6 D}\right]$ and assume that $H^{1-\varepsilon} \ge 24000C^6 D$. Also we put

$$A(s_0) = \sum_{m \le M_1} \left(\frac{a_m}{\lambda_m^{s_0}} \Delta\right), B(s_0) = \sum_{Y^2 \ge \lambda_n \ge M_1+1} \left(\frac{a_n}{\lambda_n^{s_0}} \Delta\right)$$

and assume $H \ge Exp(e^e)$ so that $loglog\, H \ge e$. To start with observe that $\mid \Delta \mid \le 3$ and that

$$\mid \varphi(s_0) \mid^2 \ge \mid A(s_0) \mid^2 + 2Re(A(s_0)\overline{B(s_0)}).$$

Hence by putting $\lambda = u_1 + u_2 + \cdots + u_r$ where $0 \le u_j \le U, j = 1, 2, 3, \cdots, r$ and $2rU \le \frac{1}{2}H$ we have

$$\begin{aligned} \int_{\frac{H}{4}}^{\frac{3H}{4}} \mid \varphi(s_0) \mid^2 dt_0 &\ge U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{\frac{H}{4}+\lambda}^{\frac{3H}{4}-U_r+\lambda} \mid \varphi(s_0) \mid^2 dt_0 \\ &\ge I_1 + Re(2I_2), \end{aligned}$$

where $I_1 = \int_{\frac{H}{4}+rU}^{\frac{3}{4}H-rU} \mid A(s_0) \mid^2 dt_0$ and

$$I_2 = U^{-r} \int_0^U du_r \cdots \int_0^U du_1 \int_{\frac{H}{4}+\lambda}^{\frac{3}{4}H-rU+\lambda} A(s_0)\overline{B(s_0)} dt_0.$$

By a well-known theorem of H.L. Montgomery and R.C. Vaughan we have

$$I_1 \ge \sum_{m \le M_1} (\frac{H}{2} - 2rU - 100C^2 m) \mid a_m \mid^2 \lambda_m^{2\sigma_0} \mid \Delta \mid^2.$$

We now assume that $\frac{H}{4} \ge 2rU + 100C^2 M_1$, so that we can replace the quantity in the common bracket here by the lower bound $\frac{1}{4}H$. Note that for $n \le M_1$

$$\mid \Delta \mid \ge 1 - 6\left(\frac{\lambda_n}{Y}\right)^{2B} \ge 1 - 6\left(\frac{CM_1}{H^\varepsilon}\right)^{DH^\varepsilon} \ge 1 - 6\left(\frac{1}{H}\right)^{DH^\varepsilon}$$

(if $H \geq 12$) $\geq \frac{1}{2}$ and that $\lambda_n^{-2\sigma} > (M_1 C)^{-\frac{8}{\log H}} > e^{-8}$. Thus with $M_1 = [H^{1-\varepsilon}(8000C^6 D)^{-1}]$ and $H \geq 12C$, we have,

$$I_1 \geq \frac{H}{16} + \frac{H}{16} e^{-8} \sum_{2 \leq n \leq M_1} |\, a_n \,|^2 .$$

Now we turn to $I_2$. We have

$$|\, I_2 \,| \leq U^{-r} \sum_{\substack{m \leq M_1, n \geq M_1+1 \\ \lambda_n \leq Y^2}} \left\{ |\, a_m \,||\, a_n \,| \left( \log \frac{\lambda_n}{\lambda_m} \right)^{-r-1} \Delta_m \Delta_n 2^{r+1} \right\}$$

(where we have written $\Delta_m$ and $\Delta_n$ with an obvious meaning namely the $\Delta$'s associated with $\lambda_m$ and $\lambda_n$)

$$\leq U^{-r} \left( \sum_{m \leq M_1} |\, \Delta_m a_m \,| \right) \left( \sum_{\substack{n \geq M_1+1 \\ \lambda_n \leq Y^2}} |\, \Delta_n a_n \,| \left( \log \frac{\lambda_n}{\lambda_{M_1}} \right)^{-r-1} \right) 2^{r+1}.$$

Here the $m$-sum is

$$\leq 3\ Exp(DH^{\varepsilon}) \sum_{n=1}^{\infty} \lambda_n^{-100C-1} \leq \frac{303}{100} Exp(DH^{\varepsilon})$$

by Lemma 2.1. The $n$-sum is

$$\leq 3 \sum_{\lambda_{M_1} < \lambda_n \leq Y^2} |\, a_n \,| \left( \log \frac{\lambda_n}{\lambda_{M_1}} \right)^{-r-1}$$

$$\leq 3 \sum_{\lambda_{M_1} < \lambda_n \leq 2\lambda_{M_1}} |\, a_n \,| \left( \log \frac{\lambda_{M_1+1}}{\lambda_{M_1}} \right)^{-r-1}$$

$$+3\ Exp(DH^{\varepsilon}) \sum_{\lambda_n > 2\lambda_{M_1}} \lambda_n^{-100C-1} (\log 2)^{-r-1}$$

$$\leq 3\ Exp(DH^{\varepsilon}) \sum_{\lambda_{M_1} < \lambda_n \leq 2\lambda_{M_1}} \lambda_n^{-100C-1} (M_1 C^2)^{r+1}$$

$$+3\ Exp(DH^{\varepsilon}) 2^{r+1} \sum_{\lambda_n > 2\lambda_{M_1}} \lambda_n^{-100C-1} \quad (\text{since } \log \frac{\lambda_{M_1+1}}{\lambda_{M_1}} \geq (C^2 M_1)^{-1}),$$

$$\leq \tfrac{3}{100}(2C)^{2r+2}Exp(DH^{\epsilon})\left(\tfrac{H^{(1-\epsilon)}}{8000C^4D}\right)^{(r+1)}$$

(if $M_1 \geq 2$, i.e. if $H \geq (24000C^6D)^E$)

Here we have used Lemma 2.1.

Thus

$$\begin{aligned} |\, 2I_2 \,| &\leq \tfrac{303}{100}\cdot 2\cdot Exp(2DH^{\epsilon})\cdot \tfrac{3}{100}(8C^2)^{r+1}U^{-r}\left(\tfrac{H^{1-\epsilon}}{8000C^4D}\right)^{r+1} \\ &\leq Exp(2DH^{\epsilon})\left(\tfrac{H}{8000C^2D}\right)\left(\tfrac{H^{1-\epsilon}}{1000UC^2D}\right)^{r}. \end{aligned}$$

We have to satisfy $\frac{H}{4} \geq 2rU + 100C^2M_1$ and by the definition of $M_1$ (viz. $M_1 = [H^{1-\epsilon}(8000C^6D)^{-1}]$) this is satisfied if $U = \frac{H^{1-\epsilon}}{300C^2D}, r = [4DH^{\epsilon}]$ and $\frac{H}{4} \geq \frac{H}{75C^2} + \frac{H}{100}$ which is clearly satisfied. Thus

$$\begin{aligned} |\, 2I_2 \,| &< \tfrac{H}{4000C^2D}Exp(2DH^{\epsilon} - 3DH^{\epsilon}) \\ &< \tfrac{H}{32}. \end{aligned}$$

Of course $M_1 \geq 2$ requires $H \geq (24000C^6D)^E$. Collecting we have the following result.

*Let* $\varphi(s_0) = \sum_{\lambda_n \leq Y^2} (a_n\lambda_n^{-s_0}\Delta)$ *where* $Y = H^{loglog\, H}$ *and* $\Delta$ *is as explained in Step I. Then*

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} |\, \varphi(s_0)\,|^2\, dt_0 \geq \frac{H}{32} + \frac{H}{16}e^{-8}\sum_{2\leq n\leq M_1} |\, a_n\,|^2$$

*where* $M_1 = [H^{1-\epsilon}(8000C^6D)^{-1}]$ *provided* $H \geq (24000C^6D)^E$ *and* $D \geq 2560C^2$.

**STEP III.** (CONVEXITY). We begin by stating a convexity theorem of R.M. Gabriel [G]. *Let $z = x + iy$ be a complex variable. Let $D_0$ be a closed rectangle with sides parallel to the axes and let $L$ be the closed line segment parallel to the y-axis which divides $D_0$ into 2 equal parts. Let $D_1$ and $D_2$ be the two congruent rectangles into which $D_0$ is divided by $L$. Let $K_1$ and $K_2$ be the boundaries of $D_1$ and $D_2$ (with the line $L$ excluded). Let $f(z)$ be* *analytic in the interior of* $D_0$ *and continuous on the boundary of* $D_0$. *Then, we have,*

$$\int_L |\, f(z) \,|^q |\, dz \,| \leq \left(\int_{K_1} |\, f(z) \,|^q |\, dz \,|\right)^{\frac{1}{2}} \left(\int_{K_2} |\, f(z) \,|^q |\, dz \,|\right)^{\frac{1}{2}},$$

*where* $q > 0$ *is any real number.*

(See Theorem 2 in the appendix to $[\mathrm{R}]_2$ for a proof). We now slightly extend this as follows.Consider the rectangle $0 \leq x \leq (2^n + 1)a$ (where $n$ is a non-negative integer and $a$ is a positive number), and $0 \leq y \leq R$. Let $I_x$ denote the integral $\int_0^R |\, f(z) \,|^q \, dy$ where as before $z = x + iy$. Let $Q_\alpha$ denote the maximum of $|\, f(z) \,|^q$ on $\{0 \leq x \leq \alpha, y = 0, R\}$. Then we have as a first application of the theorem of Gabriel.

$$I_a \leq (I_0 + 4aQ_{2a})^{\frac{1}{2}} (I_{2a} + 4aQ_{2a})^{\frac{1}{2}}.$$

We prove by induction that if $b_m = 2^m + 1$, then

$$I_a \leq$$

$$\left(I_0 + 2^{2(m+1)} aQ_{ab_m}\right)^{\frac{1}{2}} \left(I_a + 2^{2(m+1)} aQ_{ab_m}\right)^{\frac{1}{2} - \frac{1}{2^{m+1}}} \left(I_{ab_m} + 2^{2(m+1)} aQ_{ab_m}\right)^{\frac{1}{2^{m+1}}}.$$

We have as a first application of Gabriel's theorem this result with $m = 0$. Assuming this to be true for $m$ we prove it with $m$ replaced by $m + 1$. We apply Gabriel's Theorem to give the bound for $I_{ab_m}$ in terms of $I_a$ and $I_{ab_{m+1}}$. We have

$$I_{ab_m} \leq (I_a + 2b_{m+1} aQ_{ab_{m+1}})^{\frac{1}{2}} (I_{ab_{m+1}} + 2ab_{m+1} Q_{ab_{m+1}})^{\frac{1}{2}}$$

since as we can easily check $b_{m+1} = b_m + b_m - 1$. We add $2^{2(m+1)}$ $a$ $Q_{ab_m}$ to both sides and use that for $A > 0, B > 0, Q > 0$ we have

$$\sqrt{AB} + Q \leq \sqrt{(A + Q)(B + Q)}$$

which on squaring both sides reduces to a consequence of $(\sqrt{A} - \sqrt{B})^2 \geq 0$. Thus

$$I_{ab_m} + 2^{2(m+1)} a Q_{ab_m} \le$$
$$\left(I_a + a\left(2b_{m+1} + 2^{2(m+1)}\right) Q_{ab_{m+1}}\right)^{\frac{1}{2}} \left(I_{ab_{m+1}} + a\left(2b_{m+1} + 2^{2(m+1)}\right) Q_{ab_{m+1}}\right)^{\frac{1}{2}}$$

Now $2b_{m+1} + 2^{2(m+1)} \le 2^{2(m+2)}$ i.e. $2(2^{m+1}+1) \le 3 \cdot 2^{2(m+1)}$ which is true. Since $\frac{1}{2} - \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} = \frac{1}{2} - \frac{1}{2^{m+2}}$ the induction is complete and the required result is proved. We state it as a

**CONVEXITY THEOREM.** *For $m = 0, 1, 2, \cdots, n$ we have*

$$I_a \le \left(I_0 + 2^{2(m+1)} a Q_{ab_m}\right)^{\frac{1}{2}} \left(I_a + 2^{2(m+1)} a Q_{ab_m}\right)^{\frac{1}{2} - \frac{1}{2^{m+1}}} \times \left(I_{ab_m} + 2^{2(m+1)} a Q_{ab_m}\right)^{\frac{1}{2^{m+1}}}.$$

**STEP IV.** (FINAL DEDUCTION).

We now go back to Lemma 3.4. We have $Y \ge H^2$, and so if $H \ge 10C^2$,

$$\frac{C^{200C}}{10000}\left(\frac{H^2}{C} - 2C\right)^{-200C} \le \frac{C^{200C}}{10000}(8HC)^{-200C} \le \frac{(8)^{-200C}}{10000}\frac{1}{H^2} \le \frac{8^{-200}}{10000}\frac{1}{H^2},$$

since $\frac{H^2}{C} - 2C \ge H\left(\frac{H}{C} - 2C\right) \ge 8HC$. Let

$$\left(\frac{128}{2\pi}\right)^2 Y^{4B} B^2 \left(ExpExp\frac{H}{128B}\right)^{-1} \le \frac{1}{16}$$

and

$$K^2\left(ExpExp\frac{H}{128B}\right)^{-1} \le 1.$$

The second is satisfied if (note $B = DH^{\epsilon}$)

$$H \ge (256D \ loglog \ K)^E.$$

The first is satisfied if

$$ExpExp\left(\frac{H^{1-\varepsilon}}{256D}\right) \ge 88BY^{2B} = 88DH^{\epsilon}H^{2DH^{\epsilon} loglog \ H}.$$

This is satisfied (since $88DH^{\varepsilon}H^{2DH^{\varepsilon} loglog \ H} \le e^{H^2+H^3} \le e^{2H^3}$) if

$$ExpExp\left(\frac{H^{1-\varepsilon}}{256D}\right) \ge Exp(H^4)$$

i.e. if $Exp\ \frac{H^{1-\epsilon}}{256D} \geq H^4$. This is satisfied if $Exp\ \frac{H^{1-\epsilon}}{1024D} \geq H$ which is implied by $\left(\frac{H^{1-\epsilon}}{1024D}\right)^{[3E]} \frac{1}{(3E)^{3E}} \geq H$ which is implied by $H \geq (3072DE)^{3E}$. Thus by Lemma 4 and Lemma 2.3 we obtain

$$\frac{H}{32} + \frac{H}{16}e^{-8}\left(\sum_{2\leq n\leq M_1} | a_n |^2\right) \leq 16\int_{\frac{H}{4}}^{\frac{3H}{4}} | F(s_0 |^2\, dt_0 +$$

$$+\frac{2}{H} + \frac{16}{(2\pi)^2}(log\ H)^{-8}(12+16\ log\ H)^2 \int_{\substack{T_1\leq Im\ w\leq T_2 \\ Re\ w=0}} | F(w) |^2 | dw | .$$

Now

$$\frac{16(log\ H)^{-8}}{(2\pi)^2}(12+16\ log\ H)^2 \leq \frac{16}{36}(log\ H)^{-6}(28)^2 \leq 1$$

if $(log\ H)^3 \geq \frac{(28)(6)}{4}$ i.e. if $H \geq 6$. Also if $\frac{H}{32\times 33} \geq \frac{2}{H}$ i.e. if $H \geq 8\sqrt{33}$ then the result in question becomes

$$\frac{H}{33} + \frac{H}{16}e^{-8}\left(\sum_{2\leq n\leq M_1} | a_n |^2\right) \leq 16\int_{\frac{H}{4}}^{\frac{3H}{4}} | F(s_0) |^2\, dt_0 + \int_{T_1}^{T_2} | F(iv) |^2\, dv.$$

If $\int_{T_1}^{T_2} | F(iv) |^2\, dv \geq \frac{H}{33\times 17} + \frac{H}{16\times 17}e^{-8}\left(\sum_{2\leq n\leq M_1} | a_n |^2\right)$ it follows that $\int_0^H | F(iv) |^2\, dv \geq \frac{e^{-8}}{16\times 17}\left(H \sum_{n\leq M_1} | a_n |^2\right)$. Otherwise it follows that

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} | F(s_0) |^2\, dt_0 \geq \frac{H}{33\times 17} + \frac{H}{16\times 17}e^{-8}\left(\sum_{2\leq n\leq M_1} | a_n |^2\right).$$

Starting from this we now deduce a lower bound for $\int_0^H | F(iv) |^2\, dv$. We do this in a series of Lemmas.

**LEMMA 3.5.** *Let* $| \sigma | \leq 2B, 0 \leq T_1 \leq \frac{H}{8}, \frac{7H}{8} \leq T_2 \leq H, H \geq 64B$. *Then, we have,*

$$\int_{\frac{H}{4}}^{\frac{2H}{4}} | F(\sigma + it) |^2\, dt$$

$$\leq \frac{1}{B}\int_{T_1}^{T_2} dv \left(\int_{\frac{H}{4}}^{\frac{3H}{4}} | F(\sigma+iv) || Exp\left(Sin^2\left(\frac{iv-it_0}{8B}\right)\right) | dt_0\right)$$

**PROOF.** Consider the contribution of the RHS from $\frac{H}{4} \le v \le \frac{3H}{4}$. The integral with respect to $t_0$ is

$\int_{\frac{H}{4}}^{\frac{3H}{4}} Exp\left(-sinh^2\left(\frac{v-t_0}{8B}\right)\right) dt_0 = \int_{v-\frac{3H}{4}}^{v-\frac{H}{4}} Exp\left(-sinh^2 \frac{u}{8B}\right) du$

$\ge \int_0^{\frac{H}{4}} Exp\left(-sinh^2 \frac{u}{8B}\right) du$ (since $v - \frac{3H}{4} \le 0$ and $v - \frac{H}{4} \ge 0$ and their difference is $\frac{1}{2}H$)

$= 8B \int_0^{\frac{H}{32B}} Exp(-sinh^2 u)du \ge 8B\ Exp(-sinh^2 1) \ge B$

(since $sinh^2 1 \le \frac{e^2-2+e^{-2}}{4} \le \frac{9-2+1}{4} = 2$ and $e^{-2} > \frac{1}{8}$). The lemma is completely proved.

We now apply the convexity theorem with $q = 2, f(z) = F^2(z) Exp\left(Sin^2\left(\frac{z-s_0}{8B}\right)\right)$ (where $s_0 = a + it_0, a = \frac{4}{log\ H}$) to the rectangle bounded by the lines $x = 0, x = (2^n+1)a, y = T_1, y = T_2$ and choose $n$ such that $B \le x \le 2B, (B = DH^{\varepsilon})$, i.e. $\frac{B}{a} - 1 \le 2^n \le \frac{2B}{a} - 1$ (observe that $\frac{2B}{a} - 1 > 2(\frac{B}{a} - 1)$). We need an upper bound for $2^{2(n+1)}a$ which is plainly $a \cdot 4 \cdot \left(\frac{2B}{a}\right)^2 \le 4(log\ H)D^2H^{2\varepsilon} \le H^4$ (if $H \ge 4D^2$).

Also $Q_{ab_n} \le K^2 max \mid Exp\left(Sin^2\left(\frac{z-s_0}{8B}\right)\right) \mid$, where the maximum is taken over $0 \le x \le 2B, y = T_1, y = T_2$ and hence (with the condition $\frac{H}{4} \le t_0 \le \frac{3H}{4}, \frac{H}{8} \cdot \frac{1}{8B} \ge 2$ i.e. $H \ge (16\ D)^E$) we have

$$2^{2(m+1)} a Q_{ab_m} \le K^2 H^4 \left(ExpExp\ \frac{H^{1-\varepsilon}}{64D}\right)^{-1} \le \frac{1}{H^2}$$

under the conditions imposed at the beginning of this step. Hence by our convexity theorem we obtain

$$I_a \le \left(I_0 + \frac{1}{H^2}\right)^{\frac{1}{2}} \left(I_a + \frac{1}{H^2}\right)^{\frac{1}{2} - \frac{1}{2^n+1}} \left(I^* + \frac{1}{H^2}\right)^{\frac{1}{2^n+1}}$$

where $I^*$ is the integral over $x = (2^n+1)a$ fixed already. All the integrals contain a parameter $t_0$. Now we integrate with respect to $t_0$ in $\frac{H}{4} \le t_0 \le \frac{3H}{4}$ and get by Hölder's inequality,

**LEMMA 3.6** *We have,*

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0 \le \left(\int_{\frac{H}{4}}^{\frac{3H}{4}} \left(I_0 + \frac{1}{H^2}\right) dt_0\right)^{\frac{1}{2}} \left(\int_{\frac{H}{4}}^{\frac{3H}{4}} \left(I_a + \frac{1}{H^2}\right) dt_0\right)^{\frac{1}{2} - \frac{1}{2^n+1}}$$

$$\times \left(\int_{\frac{H}{4}}^{\frac{3H}{4}} \left(I^* + \frac{1}{H^2}\right) dt_0\right)^{\frac{1}{2^n+1}}$$

**LEMMA 3.7.** *We have,*

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} I^* dt_0 \leq 100BH.$$

**PROOF.** LHS does not exceed (by Lemmas 2.1 and 2.2)

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} dt_0 \left(\int_{T_1}^{T_2} \left(\frac{101}{100}\right) \mid Exp\left(Sin^2\left(\frac{a^* + iy - a - it_0}{8B}\right)\right) \mid dy\right)$$

(where $a^* = (2^n + 1)a(\leq 2B)$)

$$\leq \int_{T_1}^{T_2} \tfrac{101}{100} \left(\int_{-\infty}^{\infty} \mid Exp\left(Sin^2\left(\tfrac{a^*+iy-a-it_0}{8B}\right)\right) \mid dt_0\right) dy$$

$$\leq \int_{T_1}^{T_2} \tfrac{101}{100}(64B + 32B) \leq 100BH,$$

by breaking the last but one integral into $\mid y - t_0 \mid \leq 16B$ (from which the contribution is $64B$) and using over the remaining portion

$$\int \cdots \leq 4 \int_0^{\infty} \left(ExpExp\left(\frac{u}{8B}\right)\right)^{-1} du \leq 32B.$$

This proves the lemma completely.

**LEMMA 3.8.** *We have,*

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} I_a^* dt_0 \leq 56100 \int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0$$

*and*

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0 \geq B \int_{\frac{H}{4}}^{\frac{3H}{4}} \mid F(\sigma + it_0) \mid^2 dt_0 \geq \frac{BH}{561}.$$

**PROOF.** The first part of the second inequality follows from Lemma 3.5. Its second part follows from our assumption preceeding Lemma 3.5. By Lemma 3.7, LHS of the first part is $\leq 100BH \leq 56100 \int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0$. This completes the proof of the lemma.

Now $\int_{\frac{H}{4}}^{\frac{3H}{4}}\left(I^* + \frac{1}{H^2}\right)dt_0 \leq 56100\int_{\frac{H}{4}}^{\frac{3H}{4}}\left(I_a + \frac{1}{H^2}\right)dt$ as is shown by Lemma 3.8 and so by Lemma 3.6

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0 \leq \left(\int_{\frac{H}{4}}^{\frac{3H}{4}}\left(I_0 + \frac{1}{H^2}\right)dt_0\right)^{\frac{1}{2}}\left(\int_{\frac{H}{4}}^{\frac{3H}{4}}\left(I_a + \frac{1}{H^2}\right)dt_0\right)^{\frac{1}{2}}(56100)^{\frac{1}{2^{n+T}}}.$$

Also by the second part of Lemma 3.8, we have

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} \frac{1}{H^2}dt_0 \leq H^{-1} = \frac{561}{BH^2}\cdot\frac{BH}{561} \leq \frac{561}{BH^2}\int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0 \leq 10^{-3}\int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0$$

under the conditions imposed on $H$. Note that $(56100)^{\frac{1}{2^{n+T}}} \leq (56100)^{\frac{1}{D}} \leq 2^{\frac{1}{4}}$ since $D \geq 2560$. Thus since $(1+10^{-3})^{\frac{1}{2}}2^{\frac{1}{4}} \leq \sqrt{2}$ we obtain

**LEMMA 3.9.** *We have,*

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0 \leq 2\left(\int_{\frac{H}{4}}^{\frac{3H}{4}} I_0 dt_0 + \frac{1}{H}\right) \leq 192B\int_0^H \mid F(iv)\mid^2 dv + \frac{2}{H}.$$

**PROOF.** The second part of the inequality follows exactly as in the proof of Lemma 3.7.

From Lemma 3.8 it follows that

$$\int_{\frac{H}{4}}^{\frac{3H}{4}} I_a dt_0 \geq B\int_{\frac{H}{4}}^{\frac{3H}{4}} \mid F(\sigma+it_0)\mid^2 dt_0 \geq \frac{BH}{33\times 17}+\frac{BH}{16\times 17}e^{-8}\sum_{2\leq n\leq M_1}\mid a_n\mid^2.$$

Thus by Lemma 3.9 we obtain

$$\begin{aligned}\int_0^H \mid F(iv)\mid^2 dt &\geq \frac{H}{192\times 17}\left\{\frac{1}{33}+\frac{e^{-8}}{16}\sum_{2\leq n\leq M_1}\mid a_n\mid^2\right\} - \frac{1}{96DH}\\ &\geq \frac{H}{192\times 17}\left\{\frac{1}{34}+\frac{e^{-8}}{16}\sum_{2\leq n\leq M_1}\mid a_n\mid^2\right\}\end{aligned}$$

provided $96H^2 \geq 192\times 17\times 33\times 34$ i.e. $H^2 \geq 2\times 17\times 33\times 34$. This is satisfied if $H \geq 34\times 6$ which is clearly satisfied by the conditions imposed on $H$.

Collecting we obtain

**LEMMA 3.10.** *Under the conditions on $H, \varepsilon$ and $D$ imposed already, we have,*

$$\int_0^H |F(iv)|^2 dv \geq \frac{He^{-8}}{16 \times 17 \times 192} \sum_{n \leq M_1} |a_n|^2 .$$

*where* $M_1 = \left[\frac{H^{1-\varepsilon}}{2000C^6D}\right]$. *Note* $\frac{e^{-8}}{16\times 17\times 192} > 10^{-8}$. *All the required conditions on $H, \varepsilon, C, D$ are satisfied by*

$$D \geq 2560C^2, H \geq max\left\{(256D \; loglog \; K)^E, (24000 \; C^6DE)^{3E}\right\}.$$

This proves the seventh main theorem completely.

**ACKNOWLEDGEMENT.** We would like to acknowledge here some difficult copying help given to us by A. Sankaranarayanan and K. Soundararajan.

**FINAL REMARK.**

## PROVE OR DISPROVE THE FOLLOWING CONJECTURE

*For all $N$-tuples of complex numbers $a_1, a_2, \cdots, a_N$ with $a_1 = 1$ and for all $N \geq H \geq 10000$,*

$$\frac{1}{H}\int_0^H \Big| \sum_{n \leq N} a_n n^{it} \Big|^2 dt \geq (log \; H)^{-10000} \sum_{n \leq H^{\frac{1}{10}}} |a_n|^2 .$$

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(MANUSCRIPT COMPLETED ON 19.10.1990).