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ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES-VIII

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§ 1. **INTRODUCTION AND NOTATION.** In the paper VII[4] of this series (for the earlier papers of the series see the list of references in the paper VII[4]) K. Ramachandra started a new problem "Let $s = \sigma + it, T \geq T_0$. For what values $\alpha = \alpha(T)$ the rectangle $(\sigma \geq \alpha(T), T \leq t \leq 2T)$ contains infinity of zeros of a generalised Dirichlet series of a certain type?" (In the earlier papers of this series he and R. Baslasubramanian, sometimes individually and sometimes jointly, considered the problem where $\alpha = \alpha(T)$ is independent of T). Since the series considered in that paper were too general the answer $\left(\alpha(T) = \frac{1}{2} - \frac{D}{loglog\ T}\right)$ was perhaps too weak. In the present paper we consider some of the Dirichlet series of the form $F(s) = \sum_{n=1}^{\infty} (a_n b_n \lambda_n^{-s})$ which were considered in the paper V[3] of this series. (The method of the present paper does not succeed for *all* the series considered in V[3] let alone those considered in VI[2]). Before we recall the general series of V[3], we record two neat results (the second being deeper than the first) as two theorems. In what follows T is the only variable and we assume that T exceeds a large positive constant.

THEOREM 1. *Let* $\{\chi(n)\}(n = 1, 2, 3, \cdots)$ *be any sequence of complex* *numbers with* $\sum_{n\leq x}\chi(n) = O(1)$. *Let, as usual,* $s = \sigma + it$. *Then the number of zeros of* $\zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})$ *in the rectangle*

$$\{\sigma \geq \frac{1}{2} - C_0(loglog\ T)^{\frac{3}{2}}(log\ T)^{-\frac{1}{2}}, T \leq t \leq 2T\}$$

is $\gg T(loglog\ T)^{-1}$ *for a suitable positive constant* C_0.

THEOREM 2. *Let* $1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ *be an infinite sequenvce of real numbers such that for* $n \geq n_0$ *(*n_0*, a constant),* λ_n *is the restriction to integers of a twice continuously differentiable function* $g(x)$ *of a real variable* x *with the following properties.*

(1) *As* $x \to \infty, x^{-1}g(x)$ *tends to a positive limit.*

(2) *There exist positive constants* a *and* b *such that for all* $x \geq n_0$, *we have,*

$$a \leq g'(x) \leq b$$

and

$$a \leq (g'(x))^2 - g(x)g''(x) \leq b.$$

Then the number of zeros of $F(s) = \sum_{n=1}^{\infty}((-1)^n\lambda_n^{-s})$ *in the rectangle*

$$\left\{\sigma \geq \frac{1}{2} - C_0(loglog\ T)^{\frac{3}{2}}(log\ T)^{-\frac{1}{2}}, T \leq t \leq 2T\right\}$$

is $\gg T(loglog\ T)^{-1}$ *for a suitable positive constant* C_0.

REMARK. For $n = 1, 2, 3, \cdots$, let $\beta_n = \beta_n^{(1)} + \beta_n^{(2)}$ where $\beta_n^{(1)}$ and $\beta_n^{(2)}$ are two bounded monotonic sequences of real numbers. Then for $n \geq n_0$ we can replace λ_n by $\lambda_n + \beta_n$ and the result is practically unchanged (i.e. except for a change of C_0).

The general theorem is too lengthy to state. We now proceed to state it. We consider series of the form $F(s) = \sum_{n=1}^{\infty}(a_n b_n \lambda_n^{-s})$ where λ_n has been introduced already (the change of λ_n to $\lambda_n + \beta_n$ mentioned in the remark below Theorem 2 is certainly permissible in what follows). Let $f(x)$ be a positive real valued function with the following properties.

(1) $f(x)x^{\eta}$ is increasing and $f(x)x^{-\eta}$ is decreasing for every $\eta > 0$ and all $x \geq x_0(\eta)$.

(2) For $n \geq n_0, a \leq | b_n | (f(n))^{-1} \leq b$.

(3) For all $x \geq 1$, $\sum_{x \leq n \leq 2x} | b_{n+1} - b_n | \leq bf(x)$. We next assume that $\{a_n\}$ and $\{b_n\}$ satisfy one at least of the following two conditions.

(4) **Monotonicity condition.** Let $a_n(n = 1, 2, 3, \cdots)$ be a bounded sequence of complex numbers such that $x^{-1}\sum_{n \leq x} a_n$ tends to a non-zero limit (which may be complex) and further $| b_n | \lambda_n^{-\eta}$ is monotonic decreasing for every $\eta > 0$ and all $n \geq n_0(\eta)$.

(5) **Real part condition.** There exists an infinite arithmetic progression J of positive integers such that

$$\liminf_{x \to \infty} \left(\frac{1}{x} \sum_{\substack{x \leq \lambda_n \leq 2x, Re\ a_n > 0 \\ n \in J}} Re\ a_n \right) > 0$$

and

$$\lim_{x \to \infty} \left(\frac{1}{x} \sum_{\substack{x \leq \lambda_n \leq 2x, Re\ a_n < 0 \\ n \in J}} \right) = 0.$$

We are now in a position to state our general theorem.

THEOREM 3. *Let* $F(s) = \sum_{n=1}^{\infty} (a_n b_n \lambda_n^{-s})$ *be as described above. Let* $Exp(-\sqrt{\log x}) \leq f(x)$ *for* $x \geq x_0$. *Let* β *be a positive constant* $< \frac{1}{2}$ *and that* $F(s)$ *can be continued analytically in* $(\sigma \geq \beta, \frac{1}{2}T \leq t \leq \frac{5}{2}T)$ *and here* $max | F(s) | \leq T^{A_1}$ *where* $A_1 \geq 2$ *is a positive constant. Finally let*

$$\frac{1}{T} \int_{\frac{1}{2}T}^{\frac{5}{2}T} | F(\frac{1}{2} + it) |^2 \, dt \leq (\log T)^{A_2}$$

where $A_2 \geq 2$ is a constant. Then the number of zeros of $F(s)$ in the rectangle

$$\{\sigma \geq \frac{1}{2} - C_0(\log\log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}}, T \leq t \leq 2T\}$$

is $\gg T(\log\log T)^{-1}$ where $C_0 \geq 0$ is a certain constant.

REMARK 1. The restriction of the theorem regarding the upper bound for the mean square of $| F(\frac{1}{2} + it) |$ is very strong. Practically (since the mean square can be proved to be $\gg (f(T))^2$) it forces us to consider the series of V[3], with the extra restriction $f(x) \leq (\log x)^A$ for some constant $A \geq 2$ and all $x \geq x_0(A)$. Further the restriction $f(x) \geq Exp(-\sqrt{\log x})$ forces us to consider only a sub-class of functions considered in V[3]. It may be remarked that the mean square hypothesis is satisfied for all functions considered in V[3] by imposing $f(x) \leq (\log x)^A$.

REMARK 2. A nice example of the functions covered by Theorem 3 is $\sum_{n=1}^{\infty}((-1)^n Exp(-\sqrt{\log n})n^{-s})$. It may be noted (as a special case of a very general Theorem [1]) that this is an entire function.

REMARK 3. In the theorem it is not difficult to relax the rectangle of analytic continuation to $(\sigma \geq \beta, T \leq t \leq 2T)$ and replace the mean-value condition by

$$\frac{1}{T}\int_T^{2T} | F(\frac{1}{2} + it) |^2 \leq (\log T)^{A_2}$$

where $A_2 \geq 2$ is a constant.

REMARK 4. It is possible to generalise our results further. As a simple example we can in Theorem 1 replace $\zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})$ by

$$K^{-s}(\zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})) + \sum_{n=1}^{\infty} d_n \lambda_n^{-s}$$

where $\sum_{n \leq x} d_n = O(1), K$ is a positive constant, $| \lambda_m - Kn | \geq (100)^{-1}$ for all $m, n, 1 \ll \lambda_{n+1} - \lambda_n$ and finally $\lambda_n = O(n)$.

REMARK 5. We have imposed the restriction $f(x) \geq Exp(-\sqrt{\log x})$ for

$x \geq x_0$ to obtain some worthwhile results, but it is possible to obtain weaker results by relaxing this condition.

NOTATION. The letter A with or without subscripts will denote constants ≥ 2. The letter C with or without subscripts will denote positive constants.

§ 2. **A GENERAL LEMMA.** *Let* $1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ *be an infinite sequence of real numbers with* $1 \gg \lambda_{n+1} - \lambda_n \gg 1$ *and* $\{k_n\}(n = 1, 2, 3, \cdots)$ *be any sequence of complex numbers such that* $k_1 = 1$ *and the series* $\phi(s) = \sum_{n=1}^{\infty}(k_n \lambda_n^{-s})$ *is convergent in* $\sigma \geq A_1$ *and is continuable analytically in* $(\sigma \geq \beta, T - (log\ T)^2 \leq t \leq T + (log\ T)^2)$ *and there max* $|\ \phi(s)\ | \leq T^{A_2}$, *where* $\beta < \frac{1}{2}$ *is a positive constant. Let*

$$\frac{1}{T}\int_{T-(log\ T)^2}^{2T+(log\ T)^2} |\ \phi(\frac{1}{2} + it)\ |^2\ dt \leq (log\ T)^{A_3}.$$

Then, we have,

$$\frac{1}{T}\int_{\frac{1}{2}-(log\ T)^{-1}}^{A_1+2} \int_{T-1}^{2T+1} |\ \phi(\sigma + it)\ |^2\ dt d\sigma \leq (log\ T)^{A_4}.$$

REMARK. This lemma is well-known to experts in the subject and so its proof will be postponed to the last section. Also it is possible to replace $(log\ T)^2$ by a constant multiple of *loglog T*.

§ 3. **THE FUNCTION** $F_2(s)$. As in VI[2] we introduce the function (in VI[2] we have used the kernel $Exp(W^{4a+2})$ but we now use the kernel $Exp((Sin\ W)^2)$)

$$F_2(s) = \sum_{n=1}^{\infty} a_n b_n (\Delta(T) - \Delta(TD^{-1}))\lambda_n^{-s}$$

where D is a large positive constant and $\Delta(x)$ for $x > 0$ is defined by

$$\Delta(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(W)x^W Exp((Sin\ \frac{W}{1000})^2)\frac{dW}{W}.$$

As in VI[2] we have

LEMMA 1. *Let q be any real constant satisfying $\beta < q < \frac{1}{2}$. Then we have the inequalities*

$$\frac{1}{T}\int_T^{2T} | F_2(q+it) |^2 dt \ll T^{1-2q}(f(T))^2, \tag{1}$$

and

$$\frac{1}{T}\int_T^{2T} | F_2(q+it) | \gg T^{\frac{1}{2}-q} f(T). \tag{2}$$

PROOF. Similar to the proof of Lemma 10 of VI[2].

LEMMA 2. *Let T be an integer. Then the number of integers M in the range $T \leq M \leq 2T-1$ for which*

$$\int_M^{M+1} | F_2(q+it) | dt > C_1 T^{\frac{1}{2}-q} f(T)$$

exceeds $C_2 T$.

PROOF. Similar to that of Lemma 4 of VI[2].

LEMMA 3. *There exist at least $C_3 T(\text{loglog } T)^{-1}$ points t_j with*

$$| F_2(q+it_j) | > c_1 T^{\frac{1}{2}-q} f(T)$$

and such that any two points t_j and $t_{j'}$ with $j \neq j'$ differ by at least $C_4 \text{loglog } T$

REMARK. Here C_4 is arbitrary and C_3 depends on it.

PROOF. Follows from Lemma 2.

LEMMA 4. *Let r be a constant satisfying $\beta < r < q < \frac{1}{2}$. Put $C_5 = \frac{1}{100} C_4$ and $H = C_5 \text{loglog } T$. Then*

$$\int_{t_j-H}^{t_j+H} | F_2(r+it) | \geq C_6 V \text{ loglog } T$$

where $V = T^{\frac{1}{2}-r} f(T)$ for at most $C_7 C_6^{-1} T(\text{loglog } T)^{-1}$ points t_j.

REMARK. Here C_6 is arbitrary and C_7 is independent of C_6.

PROOF. By (1) of Lemma 1, the sum over j of the quantity on the LHS does not exceed C_7VT and this gives Lemma 4.

LEMMA 5. *There are at least $\frac{1}{2}C_3T(\log\log T)^{-1}$ points t_j seperated by (distances) at least $C_4 \log\log T$ such that if $H = \frac{1}{100}C_4 \log\log T$ then with $V = T^{\frac{1}{2}-r}f(T)$, we have,*

$$\int_{t_j-H}^{t_j+H} | F_2(r+it) | \, dt \leq C_6 V \log\log T.$$

REMARK. Here C_4 is arbitrary and C_3 depends on it.

PROOF. The lemma follows by choosing a large C_6 in Lemma 4.

LEMMA 6. *Uniformly in σ with $q < \sigma_0 \leq \sigma < \frac{1}{2}$, we have, for the points t_j of Lemma 5,*

$$\int_{t_j-2H}^{t_j+2H} | F_2(\sigma+iv) Exp((Sin \frac{W}{1000})^2) \frac{dW}{W} | > C_8 T^{\frac{1}{2}-\sigma} f(T) (\log\log T)^{-\theta}$$

where σ_0 is a constant $W = \sigma - q + iv$, and $\theta = \frac{1}{2(q-r)}$.

PROOF. Put $s_0 = q + it_j$, we have

$$F_2(s_0) = \frac{1}{2\pi i} \int F_2(s_0 + W) X^W Exp((Sin \frac{W}{1000})^2) \frac{dW}{W}$$

where the integral is taken over the (anticlockwise) boundary of the rectangle bounded by the lines $Re\ W = r - q$, $Re\ W = \sigma - q$, $Im\ W = \pm H$. We take the absolute values (using Lemma 3) of the integrand on the RHS and choose $X = C_8 T(\log\log T)^{(q-r)^{-1}}$, where C_8 is a large positive constant. This leads to Lemma 6.

LEMMA 7. *Given any σ in $\sigma_0 \leq \sigma < \frac{1}{2}$, there exist points v_j satisfying $t_j - 2H \leq v_j \leq t_j + 2H$, such that uniformly in σ there holds*

$$| F_2(\sigma + iv_j) | > C_9 T^{\frac{1}{2}-\sigma} f(T) (\log\log T)^{-\theta}$$

where $\theta = (2(q-r))^{-1}$.

REMARK. Note that v_j are seperated by (distances) at least $\frac{24}{25}C_4 \log\log T$

where C_4 is at our disposal.

PROOF. Follows from Lemma 6.

LEMMA 8. *Given any σ in $\sigma_0 \leq \sigma < \frac{1}{2}$ there exist points p_j satisfying $v_j - H \leq p_j \leq v_j + H$ such that uniformly in σ, there holds,*

$$| F(\sigma + ip_j) | > C_{10} T^{\frac{1}{2}-\sigma} f(T)(loglog\ T)^{-\theta}$$

where θ is the constant defined before.

REMARK 1. Note that p_j are seperated by (distances) at least $\frac{1}{2}C_4\ loglog\ T$. Also the number of points p_j is at least $\frac{1}{2}C_3 T(loglog\ T)^{-1}$. Here C_4 is arbitrary and C_3 depends on it. (Both are independent of σ).

REMARK 2. We can refine the lower bound for $| F(\sigma + ip_j) |$ but we do not do it since it does not have an application.

PROOF. We start with

$$F_2(\sigma + iv_j) = \frac{1}{2\pi i} \int F(\sigma + iv_j + W) T^W (1 - D^{-W}) Exp((Sin\ \frac{W}{1000})^2) \frac{dW}{W}$$

where the integration is over $Re\ W = 2$. We break off the portion $| v | \geq C_{11}\ loglog\ T$ with a small error and move the line of integration in the rest to $Re\ W = 0$. Here C_{11} is a specific constant and not arbitrary. We now use Lemma 7 and majorise the integrand. This leads to the lemma.

The rest of the proof consists in proving that at least $\frac{1}{3}C_3 T(loglog\ T)^{-1}$ of the rectangles

$$\left\{\sigma \geq \frac{1}{2} - C_0 (loglog\ T)^{\frac{3}{2}} (log\ T)^{-\frac{1}{2}}, p_j - H \leq t \leq p_j + H\right\}$$

contain a zero of $F(s)$ if C_0 is a large positive constant. This would complete the proof of Theorem 3.

§ 4. TWO APPLICATIONS OF BOREL-CARATHÉODORY THEOREM.

Suppose that the rectangle

$$\{\sigma \geq \frac{1}{2} - K\delta, p_j - H \leq t \leq p_j + H\}$$

is zero free for $F(s)$, where δ and K are positive quantities to be chosen in the next section. (The quantity δ will be chosen to be small and K to be large).

LEMMA 1. (Borel-Carathéodory Theorem. See [5] page 174). *Suppose $G(z)$ is analytic in $|z-z_0| \leq R$ and on $|z-z_0| = R$ we have $Re\ G(z) \leq U$. Then in $|z-z_0| \leq r < R$, we have,*

$$|G(z)| \leq \frac{2rU}{R-r} + \frac{R+r}{R-r}|G(z_0)|.$$

REMARK. The r of this lemma is not to be confused with that of the preceeding section.

LEMMA 2. *In the rectangle*

$$\{\sigma \geq \frac{1}{2} - (K-1)\delta, p_j - H + C_{12} \leq t \leq p_j + H - C_{12}\}$$

we have,

$$|\log F(s)| \leq C_{13}\delta^{-1}\log T.$$

PROOF. Choose z_0 to be a point in

$$\{\sigma \geq 2, p_j - H + C_{12} \leq t \leq p_j + H - C_{12}\}$$

where $\log F(s)$ is bounded and then take R to be such that the circle with centre z_0 and radius R touches $\sigma = \frac{1}{2} - K\delta$ and lies within the rectangle $\{\sigma \geq \frac{1}{2} - K\delta, p_j - H \leq t \leq p_j + H\}$. Next choose $r = R - \delta$. This proves Lemma 2.

LEMMA 3. *Let M_j denote the maximum of $|F(s)|$ in $\{\sigma \geq \frac{1}{2}, p_j - H \leq t \leq p_j + H\}$. Then, we have,*

$$\sum_j M_j^2 \leq T(\log T)^{A_5}.$$

PROOF. Let M_j be attained at s_j say. Then M_j^2 is majorised by the mean of $|F(s)|^2$ over a disc of radius $(\log T)^{-1}$ with centre s_j. The lemma now follows from the general result of § 2.

LEMMA 4. *We have,*

$$M_j^2 \geq (\log T)^{11A_5}$$

for at most $T(\log T)^{-10}$ *values of* j. *Hence we are still left with at least* $\frac{1}{3}\ C_3 T(\log\log T)^{-1}$ *values of* j *for which*

$$M_j^2 \leq (\log T)^{11A_5}.$$

REMARK. From now on we restrict j only to these values.

PROOF. Follows from Lemma 3.

LEMMA 5. *In the rectangle*

$$\{\sigma \geq \frac{1}{2} - \delta, p_j - H + C_{12} \leq t \leq p_j + H - C_{12}\}$$

we have,

$$|\log F(s)| \leq C_{14}\delta^{-1} \log\log T.$$

PROOF. Choose z_0 to be a point in

$$\{\sigma \geq 2, p_j - H + C_{12} \leq t \leq p_j + H - C_{12}\}$$

and then take R to be such that the circle with centre z_0 and radius R touches $\sigma = \frac{1}{2}$ and lies within the rectangle $\{\sigma \geq \frac{1}{2}, p_j - H \leq t \leq p_j + H\}$. Next choose $r = R - \delta$. The lemma now follows from Lemma 4.

§ 5. **COMPLETION OF THE PROOF.** Suppose that for a certain j, the rectangle $\{\sigma \geq \frac{1}{2} - K\delta, |p_j - t| \leq H\}$ does not contain a zero of $F(s)$. We obtain a contradiction in the following way. Put $s_0 = \sigma + ip_j$ where $\sigma = \frac{1}{2} - \delta$, and also let $\sigma_1 = \frac{1}{2} - (K-1)\delta, \sigma_2 = \sigma$, and $\sigma_3 = \frac{1}{2} + \delta$. We apply maximum modulus principle to

$$\psi(W) = \log F(s_0 + W) X^W Exp((Sin \frac{W}{1000})^2)$$

according to which

$$|\psi(0)| \leq max\ |\psi(W)|$$

maximum being taken over the boundary of the rectangle bounded by $Re\ W = -(K-2)\delta, Re\ W = 2\delta, Im\ W = \pm\frac{1}{2}H$. If $\delta \geq 6(log\ T)^{-\frac{1}{2}}$ we have (by a suitable choice of X and C_4)

$$\frac{\delta}{2}\ log\ T \leq \delta\ log\ T - 3\sqrt{log\ T}\ \leq |\ \psi(0)\ |$$

$$\leq C_{15}(\delta^{-1} log\ T)^{\frac{2}{K}}(\delta^{-1} loglog\ T)^{\frac{K-2}{K}}.$$

We now choose $K = loglog\ T$ and obtain

$$\frac{\delta}{2}\ log\ T \leq C_{16}\delta^{-1} loglog\ T.$$

This is a contradiction if we choose $\delta = C_{17}(loglog\ T)^{\frac{1}{2}}(log\ T)^{-\frac{1}{2}}$ and $C_{17}^2 > 2C_{16}$. This proves Theorem 3 provided we prove the general lemma of § 2.

§ 6. **PROOF OF THE GENERAL LEMMA.** Let $\varepsilon > 0$ be arbitrary but fixed. Then in $\{\sigma \geq \beta + \varepsilon, T \leq t \leq 2T\}$, we have, by Cauchy's theorem $|\ \phi'(s)\ | \leq T^{A_2+1}$ and so in $\{|\ \sigma - \frac{1}{2}\ | \leq T^{-4A_2}, T \leq t \leq 2T\}$ we have

$$|\ \phi^2(\frac{1}{2}+it) - \phi^2(\sigma+it)\ | \leq 1.$$

Hence it suffices to consider in this rectangle the portion $|\ \sigma - \frac{1}{2}\ | \geq T^{-4A_2}$. If now $\frac{1}{2} - (log\ T)^{-1} \leq \sigma \leq \frac{1}{2} - T^{-4A_2}$ we have

$$|\ \phi^2(s) = \frac{1}{2\pi i}\int \phi^2(s+W)X^W Exp(W^2)\frac{dW}{W}$$

the contour being the (anticlockwise) boundary of the rectangle bounded by $Re\ W = \beta - \sigma, Re\ W = \frac{1}{2} - \sigma, Im\ W = \pm log\ T$. We choose X to be a large power of T so that the integral over the left boundary is negligible. Clearly the integrals over the horizontal boundaries are together negligible. We take absolute values and integrate with respect to t from $t = T$ to $t = 2T$. This leads to the result since on the right boundary $|\ X^W\ | \leq 1$ and $\int |\ \frac{dW}{W}\ | \ll log\ T$.

If now $\sigma \geq \frac{1}{2} + T^{-4A_2}$ we start with

$$\phi^2(s) = \frac{1}{2\pi i}\int \phi^2(s+W)Exp(W^2)\frac{dW}{W}$$

the contour being the (anticlockwise) boundary of the rectangle bounded by $Re\ W = \frac{1}{2} - \sigma, Re\ W = 3A_1 - \sigma, Im\ W = \pm log\ T$. The proof proceeds as before using $\phi(s+W) = O(1)$ on the right boundary and negligible on the horizontal boundaries and the fact $\int \mid \frac{dW}{W} \mid \ll log\ T$ on the left boundary. This completes the proof of the general lemma.

Theorem 3 is now completely proved.

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