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ON THE ZEROS OF A CLASS OF GENERALISED
DIRICHLET SERIES-IX

BY
R. BALASUBRAMANIAN AND K. RAMACHANDRA

§ 1. INTRODUCTION. In paper VII\textsuperscript{[4]} of this series K. Ramachandra raised the question "Let \( s = \sigma + it, T \geq T_0 \) a large positive constant. For what values \( \alpha = \alpha(T) \) the rectangle \( (\sigma \geq \alpha(T), T \leq t \leq 2T) \) contains infinity of zeros of a generalised Dirichlet series of a certain type?" (In the earlier papers of this series he and R. Balasubramanian, sometimes individually and sometimes jointly considered the problem where \( \alpha = \alpha(T) \) does not depend on \( T \)). Since the series considered in that paper were too general the answer \( (\alpha(T) = \frac{1}{2} - \frac{P}{\log \log T}) \) was perhaps too weak. In paper VIII\textsuperscript{[2]} of the series we considered somewhat restricted series and obtained the result that we can take \( \alpha(T) = \frac{1}{2} - C_0(\log \log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}} \). In the present paper we assume a serious restrictive condition namely an Euler product and show that we can take \( \alpha(T) = \frac{1}{2} - C_0(\log \log T)((\log T)^{-1}. \) (Actually we work with shorter rectangles and also obtain a quantitative result on the number of zeros). Accordingly we prove the following

THEOREM 1. Let \( p \) run over primes and \( \omega(p) \) complex numbers whose absolute value is 1 except for a finite set of primes where we define \( \omega(p) \) to
be zero. Let \( s = \sigma + it \) as usual, and let

\[
F(s) = \sum_{n=1}^{\infty} (a_n n^{-s}) = \prod_{p} \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1}, (\sigma > 1),
\]

(1.1)

be continuable analytically in

\[
\left\{ \sigma \geq \frac{1}{2} - \frac{C \log \log T}{\log H}, T \leq t \leq T + H \right\} ((\log T)^C' \leq H \leq T, C' \text{ to be specified})
\]

(1.2)

and there \( \max | F(s) | \leq T^A \) where \( A(\geq 1) \) is a constant. Let \( C(\geq 1) \) be a large constant depending on \( A \) and let \( H \) exceed \( (\log T)^C' \) where \( C'(\geq 1) \) is a large constant. Then \( F(s) \) has at least \( (C'')^{-1} H (\log \log \log T)^{-1} \) zeros in the rectangle (1.2), where \( C''(\geq 1) \) is a large constant.

**Remark 1.** The condition on the Euler product (1.1) covers the Riemann zeta-function and the ordinary L-series. But it can be stated in such a way that it covers zeta and L-functions of algebraic number fields. The restriction (1.1) and the restriction \( | F(s) | \leq T^A \) practically force us to give these (and perhaps only these) as examples and in these cases the series have a functional equation and so in fact we can take \( a(T) = \frac{1}{2} \). Hence the main content of the present paper is the emphasis on results which can be proved without the assumption of a functional equation for \( F(s) \). Thus we stress that Theorem 1 does not depend on the assumption of a functional equation for \( F(s) \).

**Remark 2.** In the case where \( F(s) \) is the Riemann zeta-function or ordinary L-series we can prove (without using the functional equation) that

\[
\frac{1}{T} \int_{t}^{2T} | F(\frac{1}{2} + it) |^2 \, dt \leq (\log T)^2
\]

(1.3)

and from this we can deduce that the number of zeros of \( F(s) \) in \( (\sigma \geq \beta, T \leq t \leq 2T) \) is \( \leq T^\theta (\log T)^{A_1} \) where \( \frac{1}{2} \leq \beta \leq 1, \theta = 4(1 - \beta)(3 - 2\beta)^{-1} \) and \( A_1 \geq 1 \) is a certain numerical constant. (For a proof of this fact see [5]). Thus in these cases and with the extra condition \( H = T \), the zeros of the theorem "already belong" to the rectangle

\[
\left( \sigma \leq \frac{1}{2} + C \frac{\log \log T}{\log T}, T \leq t \leq 2T \right)
\]

(1.4)
§ 2. PROOF OF THEOREM 1. In the proof we use the Ramachandra kernel $\text{Exp}((\sin z)^2)$, a well-known theorem due to Montgomery and Vaughan and the well-known

THEOREM 2 (Borel-Caratheodory Theorem). Suppose $G(z)$ is analytic in $|z - z_0| \leq R$ and on $|z - z_0| = R$ we have $\text{Re} G(z) \leq U$. Then in $|z - z_0| \leq r < R$, we have,

$$|G(z)| \leq \frac{2rU}{R - r} + \frac{R + r}{R - r} |G(z_0)|.$$  \hfill (2.1)

PROOF. See [6], page 174.

We break up the proof into a few lemmas. The rough idea of the proof is as follows. We first prove that on the line $\sigma = \frac{1}{2} - 2\delta$, where $\delta = \frac{1}{4} C(\log \log T)(\log H)^{-1}$, we have (between $T$ and $T + H$) $\gg H$ well-spaced points which are all heavy for $\log F(s)$ in a certain sense. Out of these we select $\gg H(\log \log \log T)^{-1}$ points with the property that any two (distinct) points are at a distance $\geq L = D \log \log \log T$, where $D(\geq 1)$ is a large constant. We then prove that if $\frac{1}{2} - 2\delta + it_j$ is any such point the rectangle $(\sigma \geq \frac{1}{2} - 4\delta, |t_j - t| \leq \frac{1}{3} L)$ contains a zero of $F(s)$. This will prove the theorem completely. We begin with

LEMMA 1. For $u > 0$ let

$$\Delta(u) = \frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} u^w \text{Exp} \left( \left( \frac{\sin \frac{w}{1000}}{w} \right)^2 \right) \frac{dw}{w}. \hfill (2.2)$$

Then

$$\Delta(u) = O(u^2) \text{ and also } \Delta(u) = 1 + O(u^{-2})$$

where the implied constants are absolute.

PROOF. By moving the line of integration to $\text{Re} w = 2$ (resp. $\text{Re} w = -2$) the lemma follows by easy estimations.

LEMMA 2. Let $\sigma = \frac{1}{2} - 2\delta, X = H^\frac{1}{3}$ and

$$F_1(s) = \sum_p \omega(p)p^{-s} \Delta \left( \frac{X}{p} \right). \hfill (2.3)$$
Then
\[
\frac{1}{H} \int_T^{T+H} | F_1(s) |^2 \, dt \gg H^{2\delta} (\delta \log H)^{-1} \tag{2.4}
\]
and
\[
\frac{1}{H} \int_T^{T+H} | F_1(s) |^4 \, dt \ll H^{4\delta} (\delta \log H)^{-2} \tag{2.5}
\]

**PROOF.** By a well-known theorem of H.L. Montgomery and R.C. Vaughan (see [3] for a simpler proof of a special case of their result which we need here) LHS of (2.4) is
\[
\sum_p \left\{ \frac{\omega(p)}{p^{2\sigma}} \right\}^2 \Delta \left( \frac{X}{p} \right)^2 (1 + O \left( \frac{p}{H} \right)) \tag{2.6}
\]
which is \( \gg H^{2\delta} (\delta \log H)^{-1} \) by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.4).

Also LHS of (2.5) is
\[
\leq \sum_{p_1, p_2} (p_1 p_2)^{-2\sigma} \Delta \left( \frac{X}{p_1} \right) \Delta \left( \frac{X}{p_2} \right)^2 (1 + O \left( \frac{p_1 p_2}{H} \right)) \tag{2.7}
\]
which is \( \ll H^{4\delta} (\delta \log H)^{-2} \) by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.5).

Thus Lemma 2 is completely proved.

**LEMMA 3.** Let \( T \) and \( H \) be positive integers (this can be assumed without loss of generality in Theorem 1). Then there exist \( \gg H \) integers \( M \) satisfying \( T \leq M \leq T + H - 1 \) for which
\[
\int_M^{M+1} | F_1(s) |^2 \, dt \gg H^{2\delta} (\delta \log H)^{-1} \tag{2.8}
\]

**PROOF.** By (2.4) we have
\[
\sum_M \int_M^{M+1} | F(s) |^2 \, dt \geq C_1 H \psi
\]
where \( C_1 \) is a positive constant and \( \psi = H^{2\delta} (\delta \log H)^{-1} \). Here in the sum we drop those \( M \) for which the integral from \( M \) to \( M + 1 \) does not exceed \( \frac{1}{2} C_1 \psi \) and obtain
\[
\sum_M' \int_M^{M+1} | F_1(s) |^2 \, dt \geq \frac{1}{2} C_1 H \psi
\]
where the slash indicates that we restrict only to those integrals which exceed \( \frac{1}{2} C_1 \psi \). By applying Hölder’s inequality we obtain
\[
\left( \sum_{M}^{1} \right)^{\frac{1}{2}} \left( \sum_{M}^{1} \left( \int_{M}^{M+1} | F_1(s) |^2 \, dt \right)^{2} \right)^{\frac{1}{2}} \geq \frac{1}{2} C_1 H \psi.
\]

The inequality
\[
\int_{M}^{M+1} | F_1(s) |^2 \, dt \leq \left( \int_{M}^{M+1} | F_1(s) |^4 \, dt \right)^{\frac{1}{2}}
\]
completes the proof of the lemma on using (2.5).

**REMARK.** Actually the deduction of Lemma 3 from Lemma 2 involves a general principle which was first observed and applied for discussing the zeros of Dirichlet series by R. Balasubramanian and K. Ramachandra in paper III\(^{[1]}\) of this series.

**LEMMA 4.** In the interval \([T, T + H]\) there are \( R \gg H (\log \log \log T)^{-1} \) points \( t_1, t_2, \ldots, t_R \) with the properties \( | t_j - t_{j'} | \geq L \) for all \( j \neq j' \) where \( L = D \log \log \log T \) and further
\[
| F_1 \left( \frac{1}{2} - 2\delta + t_j \right) | \asymp H^{\delta} (\delta \log H)^{-\frac{1}{2}}.
\]
As has been said before \( D \) is a large positive constant.

**PROOF.** The proof follows from Lemma 3.

**LEMMA 5.** Consider any of the points \( t_j (= t_0 \text{ say}) \) given by Lemma 4 excluding the upper and the lower extreme points. Assume that the region \( \sigma \geq \frac{1}{2} - 4\delta, \, | t - t_0 | \leq \frac{1}{3} L \) is zero-free for \( F(s) \). Then in \( \sigma \geq \frac{1}{2} - 3\delta, \, | t - t_0 | \leq \frac{1}{4} L \) we have,
\[
| \log F(s) | \leq (\log T)^3.
\]

**PROOF.** Apply Borel-Caratheodory theorem as follows. Take \( z_0 = 2 + it \) at which definitely \( | \log F(z_0) | \leq 10 \), whatever be \( t \). Choose \( R \) such that the circle \( | z - z_0 | = R \) touches \( \sigma = \frac{1}{2} - 4\delta \) (we mean the line \( \text{Re} \, z = \frac{1}{2} - 4\delta \))
and \( r \) to be \( R - \delta \). This proves the lemma.

**Lemma 6.** Put \( s_0 = \frac{1}{2} - 2\delta + it_0 \). Then

\[
F_1(s_0) = \log F(s_0) + O(1).
\]

**Proof.** Since the powers of primes higher than the first contribute \( O(1) \), LHS of (2.11) is

\[
\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \log F(s_0 + w)X^w \exp \left( \left( \sin \frac{w}{1000} \right)^2 \right) \frac{dw}{w} + O(1).
\]

We now break off the portion \( |\text{Im } w| \geq \frac{1}{4} L \) of the integral with an error \( O(1) \) and in the rest of the integral move the line of integration to \( \text{Re } w = -\delta \). This involves an error \( O(X^{-\delta}(\log T)^4) = O(1) \) if \( C \) is large. The pole at \( w = 0 \) contributes \( \log F(s_0) \). This proves the lemma.

**Lemma 7.** The equations (2.9), (2.10) and (2.11) are contradictory and hence the assumption of Lemma 5 is not true.

**Proof.** The equations referred to in the lemma give

\[
H^\delta (\delta \log H)^{-\frac{1}{4}} \ll (\log T)^3.
\]

This is false if \( \delta = \frac{1}{4} C (\log \log T) (\log H)^{-1} \) (as specified already) and \( C \) is a large positive constant.

This completes the proof of Theorem 1.

\[\text{§ 3. EXPLANATIONS REGARDING (2.6) AND (2.7).}\]

(A) In (2.6) the main term is

\[
= \sum_{p \leq X} p^{-2\sigma} \left( 1 + O \left( \frac{p}{X^T} \right) \right) \left( 1 + O \left( \frac{p}{X^T} \right) \right) + O(1)
\]

\[
= \sum_{p \leq X} p^{-2\sigma} + O \left( \frac{X^{1-2\sigma}(\log H)^{-1}}{X} \right).
\]
\[= \sum_{p \leq X} p^{-2\sigma} + O(H^{26}(\log H)^{-1}).\]

The error term is
\[
\ll \sum_{X \leq p \leq H} p^{-2\sigma} \left(\frac{X^2}{p^2}\right) (1 + \frac{p}{H}) + \sum_{p > H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 \left(\frac{p}{p^2}\right)
\]
\[
\ll \sum_{X \leq p \leq H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 + \sum_{p > H} p^{-1-2\sigma}
\]
\[
\ll X^2(\log H)^{-1}X^{-1-2\sigma} + (\log H)^{-1}H^{-2\sigma}
\]
\[
\ll (\log H)^{-1}(XH^{-\sigma} + H^{-2\sigma}) \ll H^{26}(\log H)^{-1}.
\]

Thus (2.6) is \[\sum_{p \leq X} p^{-2\sigma} + O(H^{26}(\log H)^{-1}).\] Now the first term here is \[\sum_{U \leq X} U^{4\delta} (\log U)^{-1} \quad (U \text{ runs over powers of } 2) \text{ which is}
\]
\[
\ll \sum_{X^{\frac{1}{2}} \leq U \leq X} U^{4\delta} (\log H)^{-1} + O(H^{\delta}\log\log H)
\]
\[
\ll H^{26}(\log H)^{-1} \left(\sum_{\nu=0}^{Q} 2^{-4\nu\delta} + O\left(2^{-4Q\delta}\delta^{-1}\right)\right) + O(H^{\delta}\log\log H) \quad (\text{where } Q = \left[\frac{\log H}{\log 2}\right]),
\]
\[
\ll H^{26}(\log H)^{-1}\left(\frac{1}{\delta} + O\left(H^{-\delta}\delta^{-1}\right)\right) + O(H^{\delta}\log\log H)
\]
\[
= H^{26}(\delta \log H)^{-1}(1 + O(H^{-\delta} + H^{-\delta}(\delta \log H)\log\log H))
\]
\[
= H^{26}(\delta \log H)^{-1}(1 + O(H^{-\delta} + H^{-\delta/2}\log\log H)), \quad (\text{since } H^{\delta/2} \geq \frac{\delta}{2} \log H),
\]
\[
\ll H^{26}(\delta \log H)^{-1} \text{ provided } H^{\delta} \geq (\log\log H)^3 \text{ (which is certainly satisfied since } \delta = \frac{1}{4}\left(\log\log T\right)(\log H)^{-1} \text{ and } C \text{ is assumed to be large}).
\]

(B) The quantity in (2.7) is
\[
\ll \left(\sum_{p} |\Delta\left(\frac{X}{p}\right)|^2 p^{-2\sigma}\right)^2 + \frac{1}{H} \left(\sum_{p} |\Delta\left(\frac{X}{p}\right)|^2 p^{1-2\sigma}\right)^2.
\]
\[
\ll \left( \sum_{p \leq X} p^{-2\sigma} \right)^2 + \left( \sum_{p > X} \frac{X^4}{p^{3+2\sigma}} \right)^2 + \frac{1}{X^2} \left( \sum_{p \leq X} p^{1-2\sigma} \right)^2 + \frac{1}{X^2} \left( \sum_{p > X} \frac{X^4}{p^{3+2\sigma}} \right)^2 \\
\ll H^{46}(\delta \log H)^{-2} + \left( \frac{X^4}{X^{3+2\sigma} \log H} \right)^2 + \frac{1}{X^2} \left( \frac{X^{2-2\sigma}}{\log H} \right)^2 \\
+ \frac{1}{X^2} \left( \frac{X^4}{X^{3+2\sigma} \log H} \right)^2 \text{ (if } H^\delta \geq (\log \log H)^3 \text{ which is satisfied by our assumption on } H^\delta \text{)} \\
\ll H^{46}(\delta \log H)^{-2} + H^{45}(\log H)^{-2} \\
\ll H^{46}(\delta \log H)^{-2} \text{ if } \delta \text{ is small.}
\]

This completes explanations regarding (2.6) and (2.7).
Thus Theorem 1 is completely proved.

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