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To cite this version:
R Balasubramanian, K Ramachandra. On the zeros of a class of generalised Dirichlet series-IX.
Hardy-Ramanujan Journal, Hardy-Ramanujan Society, 1991, 14, pp.34 - 43. hal-01104799

HAL Id: hal-01104799
https://hal.archives-ouvertes.fr/hal-01104799
Submitted on 22 Jan 2015

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ON THE ZEROS OF A CLASS OF GENERALISED
DIRICHLET SERIES-IX

BY
R. BALASUBRAMANIAN AND K. RAMACHANDRA

§ 1. INTRODUCTION. In paper VII[4] of this series K. Ramachandra
raised the question "Let $s = \sigma + it, T \geq T_0$ a large positive constant. For
what values $\alpha = \alpha(T)$ the rectangle $(\sigma \geq \alpha(T), T \leq t \leq 2T)$ contains infinity
of zeros of a generalised Dirichlet series of a certain type?" (In the earlier
papers of this series he and R. Balasubramanian, sometimes individually and
sometimes jointly considered the problem where $\alpha = \alpha(T)$ does not depend
on $T$). Since the series considered in that paper were too general the answer
$\alpha(T) = \frac{1}{2} - \frac{D}{\log \log T}$ was perhaps too weak. In paper VII[2] of the series
we considered somewhat restricted series and obtained the result that we
can take $\alpha(T) = \frac{1}{2} - C_0(\log \log T)(\log T)^{-\frac{1}{2}}$. In the present paper we
assume a serious restrictive condition namely an Euler product and show
that we can take $\alpha(T) = \frac{1}{2} - C_0(\log \log T)(\log T)^{-1}$. (Actually we work
with shorter rectangles and also obtain a quantitative result on the number
of zeros). Accordingly we prove the following

THEOREM 1. Let $p$ run over primes and $\omega(p)$ complex numbers whose
absolute value is 1 except for a finite set of primes where we define $\omega(p)$ to
be zero. Let \( s = \sigma + it \) as usual, and let
\[
F(s) = \sum_{n=1}^{\infty} (a_n n^{-s}) = \prod_{p} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}, (\sigma > 1),
\]
be continuable analytically in
\[
\left\{ \sigma \geq \frac{1}{2} - \frac{C \log \log T}{\log H}, T \leq t \leq T + H \right\} ((\log T)^{C'} \leq H \leq T, C' \text{ to be specified})
\]
and there \( \max |F(s)| \leq T^A \) where \( A(\geq 1) \) is a constant. Let \( C(\geq 1) \) be a large constant depending on \( A \) and let \( H \) exceed \((\log T)^{C'}\) where \( C'(\geq 1) \) is a large constant. Then \( F(s) \) has at least \((C'')^{-1}H(\log \log \log T)^{-1}\) zeros in the rectangle (1.2), where \( C''(\geq 1) \) is a large constant.

**REMARK 1.** The condition on the Euler product (1.1) covers the Riemann zeta-function and the ordinary \( L \)-series. But it can be stated in such a way that it covers zeta and \( L \)-functions of algebraic number fields. The restriction (1.1) and the restriction \( |F(s)| \leq T^A \) practically force us to give these (and perhaps only these) as examples and in these cases the series have a functional equation and so in fact we can take \( \alpha(T) = \frac{1}{2} \). Hence the main content of the present paper is the emphasis on results which can be proved without the assumption of a functional equation for \( F(s) \). Thus we stress that Theorem 1 does not depend on the assumption of a functional equation for \( F(s) \).

**REMARK 2.** In the case where \( F(s) \) is the Riemann zeta-function or ordinary \( L \)-series we can prove (without using the functional equation) that
\[
\frac{1}{T} \int_{T}^{2T} |F(\frac{1}{2} + it)|^2 \, dt \leq (\log T)^2
\]
and from this we can deduce that the number of zeros of \( F(s) \) in \((\sigma \geq \beta, T \leq t \leq 2T)\) is \( \leq T^\theta (\log T)^{A_1} \) where \( \frac{1}{2} \leq \beta \leq 1, \theta = 4(1 - \beta)(3 - 2\beta)^{-1} \) and \( A_1 \geq 1 \) is a certain numerical constant. (For a proof of this fact see [5]). Thus in these cases and with the extra condition \( H = T \), the zeros of the theorem "already belong" to the rectangle
\[
\left( \sigma = \frac{1}{2} + C \frac{\log \log T}{\log T}, T \leq t \leq 2T \right)
\]
§ 2. PROOF OF THEOREM 1. In the proof we use the Ramachandra kernel $Exp((\sin z)^2)$, a well-known theorem due to Montgomery and Vaughan and the well-known

**THEOREM 2** (Borel-Caratheodory Theorem). Suppose $G(z)$ is analytic in $|z - z_0| \leq R$ and on $|z - z_0| = R$ we have $\text{Re} \, G(z) \leq U$. Then in $|z - z_0| \leq r < R$, we have,

$$|G(z)| \leq \frac{2rU}{R - r} + \frac{R + r}{R - r} |G(z_0)|.$$  \hfill (2.1)

**PROOF.** See [6], page 174.

We break up the proof into a few lemmas. The rough idea of the proof is as follows. We first prove that on the line $\sigma = \frac{1}{2} - 2\delta$, where $\delta = \frac{1}{4}C(\log \log T)(\log H)^{-1}$, we have (between $T$ and $T + H$) $\gg H$ well-spaced points which are all heavy for $\log F(s)$ in a certain sense. Out of these we select $\gg H(\log \log \log T)^{-1}$ points with the property that any two (distinct) points are at a distance $\geq L = D \log \log \log T$, where $D(\geq 1)$ is a large constant. We then prove that if $\frac{1}{2} - 2\delta + it$ is any such point the rectangle $(\sigma \geq \frac{1}{2} - 4\delta, |t_j - t| \leq \frac{1}{3}L)$ contains a zero of $F(s)$. This will prove the theorem completely. We begin with

**LEMMA 1.** For $u > 0$ let

$$\Delta(u) = \frac{1}{2\pi i} \int_{2 - \infty}^{2 + \infty} u^w \exp \left( \left( \frac{\sin \frac{w}{1000}}{2} \right)^2 \right) \frac{dw}{w}. \hfill (2.2)$$

Then

$$\Delta(u) = O(u^2) \text{ and also } \Delta(u) = 1 + O(u^{-2})$$

where the implied constants are absolute.

**PROOF.** By moving the line of integration to $\text{Re} \, w = 2$ (resp. $\text{Re} \, w = -2$) the lemma follows by easy estimations.

**LEMMA 2.** Let $\sigma = \frac{1}{2} - 2\delta, X = H^{\frac{1}{4}}$ and

$$F_1(s) = \sum_{p} \omega(p)p^{-s} \Delta \left( \frac{X}{p} \right). \hfill (2.3)$$
Then

\[ \frac{1}{H} \int_{T}^{T+H} |F_1(s)|^2 \, dt \gg H^{26}(\delta \log H)^{-1} \]  

(2.4)

and

\[ \frac{1}{H} \int_{T}^{T+H} |F_1(s)|^4 \, dt \ll H^{46}(\delta \log H)^{-2}. \]  

(2.5)

**PROOF.** By a well-known theorem of H.L. Montgomery and R.C. Vaughan (see [3] for a simpler proof of a special case of their result which we need here) LHS of (2.4) is

\[ \sum_p \frac{|\omega(p)|^2}{p^{2\sigma}} |\Delta \left( \frac{X}{p} \right)|^2 \left( 1 + O \left( \frac{p}{H} \right) \right) \]  

(2.6)

which is \( \gg H^{26}(\delta \log H)^{-1} \) by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.4).

Also LHS of (2.5) is

\[ \leq \sum_{p_1, p_2} (p_1 p_2)^{-2\sigma} |\Delta \left( \frac{X}{p_1} \right) \Delta \left( \frac{X}{p_2} \right)|^2 \left( 1 + O \left( \frac{p_1 p_2}{H} \right) \right) \]  

(2.7)

which is \( \ll H^{46}(\delta \log H)^{-2} \) by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.5).

Thus Lemma 2 is completely proved.

**LEMMA 3.** Let \( T \) and \( H \) be positive integers (this can be assumed without loss of generality in Theorem 1). Then there exist \( H \) integers \( M \) satisfying \( T \leq M \leq T + H - 1 \) for which

\[ \int_{M}^{M+1} |F_1(s)|^2 \, dt \gg H^{26}(\delta \log H)^{-1}. \]  

(2.8)

**PROOF.** By (2.4) we have

\[ \sum_M \int_{M}^{M+1} |F(s)|^2 \, dt \geq C_1 H \psi \]

where \( C_1 \) is a positive constant and \( \psi = H^{26}(\delta \log H)^{-1} \). Here in the sum we drop those \( M \) for which the integral from \( M \) to \( M + 1 \) does not exceed \( \frac{1}{2} C_1 \psi \) and obtain

\[ \sum_M' \int_{M}^{M+1} |F_1(s)|^2 \, dt \geq \frac{1}{2} C_1 H \psi \]
where the slash indicates that we restrict only to those integrals which exceed $\frac{1}{2}C_1\psi$. By applying Hölder's inequality we obtain
\[
\left(\sum_{M}^{1}\right)^{\frac{1}{2}} \left(\sum_{M}^{M+1} \left| F_1(s) \right|^2 dt \right)^{\frac{1}{2}} \geq \frac{1}{2}C_1H\psi.
\]

The inequality
\[
\int_{M}^{M+1} \left| F_1(s) \right|^2 dt \leq \left(\int_{M}^{M+1} \left| F_1(s) \right|^4 dt \right)^{\frac{1}{2}}
\]
completes the proof of the lemma on using (2.5).

**REMARK.** Actually the deduction of Lemma 3 from Lemma 2 involves a general principle which was first observed and applied for discussing the zeros of Dirichlet series by R. Balasubramanian and K. Ramachandra in paper III\([1]\) of this series.

**LEMMA 4.** In the interval \([T, T+H]\) there are \(R \gg H(\log\log\log T)^{-1}\) points \(t_1, t_2, \ldots, t_R\) with the properties \(|t_j - t_{j'}| \geq L\) for all \(j \neq j'\) where \(L = D \log\log\log T\) and further
\[
|F_1\left(\frac{1}{2} - 2\delta + t_j\right)| \gg H^6(\delta \log H)^{-\frac{1}{2}}.
\]

As has been said before \(D\) is a large positive constant.

**PROOF.** The proof follows from Lemma 3.

**LEMMA 5.** Consider any of the points \(t_j (= t_0\ say)\) given by Lemma 4 excluding the upper and the lower extreme points. Assume that the region \((\sigma \geq \frac{1}{2} - 4\delta, |t - t_0| \leq \frac{1}{3}L)\) is zero-free for \(F(s)\). Then in \((\sigma \geq \frac{1}{2} - 3\delta, |t - t_0| \leq \frac{1}{4}L)\) we have,
\[
|\log F(s)| \leq (\log T)^3.
\]

**PROOF.** Apply Borel-Caratheodory theorem as follows. Take \(z_0 = 2 + it\) at which definitely \(|\log F(z_0)| \leq 10\), whatever be \(t\). Choose \(R\) such that the circle \(|z - z_0| = R\) touches \(\sigma = \frac{1}{2} - 4\delta\) (we mean the line \(Re z = \frac{1}{2} - 4\delta\))
and $r$ to be $R - \delta$. This proves the lemma.

**Lemma 6.** Put $s_0 = \frac{1}{2} - 2\delta + it_0$. Then

$$F_1(s_0) = \log F(s_0) + O(1). \tag{2.11}$$

**Proof.** Since the powers of primes higher than the first contribute $O(1)$, LHS of (2.11) is

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \log F(s_0 + w) X^w \exp \left( \left( \sin \frac{w}{1000} \right)^2 \right) \frac{dw}{w} + O(1).$$

We now break off the portion $|Im w| \geq \frac{1}{4} L$ of the integral with an error $O(1)$ and in the rest of the integral move the line of integration to $Re w = -\delta$. This involves an error $O(X^{-\delta}(\log T)\delta) = O(1)$ if $C$ is large. The pole at $w = 0$ contributes $\log F(s_0)$. This proves the lemma.

**Lemma 7.** The equations (2.9), (2.10) and (2.11) are contradictory and hence the assumption of Lemma 5 is not true.

**Proof.** The equations referred to in the lemma give

$$H^\delta (\delta \log H)^{-\frac{1}{2}} \ll (\log T)^3.$$

This is false if $\delta = \frac{1}{4} C (\log \log T)(\log H)^{-1}$ (as specified already) and $C$ is a large positive constant.

This completes the proof of Theorem 1.

§ 3. EXPLANATIONS REGARDING (2.6) AND (2.7).

(A) In (2.6) the main term is

$$= \sum_{p \leq X} p^{-2\sigma} \left( 1 + O \left( \frac{p^2}{X^2} \right) \right) \left( 1 + O \left( \frac{p^2}{X^2} \right) \right) + O(1)$$

$$= \sum_{p \leq X} p^{-2\sigma} \left( 1 + O \left( \frac{p^2}{X^2} \right) \right) + O(1)$$

$$= \sum_{p \leq X} p^{-2\sigma} + O(X^{1-2\sigma}(\log H)^{-1})$$
\[ = \sum_{p \leq X} p^{-2\sigma} + O(H^{26} \log H)^{-1} \]

The error term is:

\[ \ll \sum_{X < p \leq H} p^{-2\sigma} \left( \frac{X}{p} \right)^2 (1 + \frac{p}{H}) + \sum_{p > H} p^{-2\sigma} \left( \frac{X}{p} \right)^2 \left( \frac{p}{X} \right)^2 \]

\[ \ll \sum_{X < p \leq H} p^{-2\sigma} \left( \frac{X}{p} \right)^2 + \sum_{p > H} p^{-1-2\sigma} \]

\[ \ll X^2 (\log H)^{-1} X^{-1-2\sigma} + (\log H)^{-1} H^{-2\sigma} \]

\[ \ll (\log H)^{-1} (X H^{-\sigma} + H^{-2\sigma}) \ll H^{26} (\log H)^{-1} \]

Thus (2.6) is \( \sum_{p \leq X} p^{-2\sigma} + O(H^{26} (\log H)^{-1}) \). Now the first term here is

\[ \sum_{p \leq X} \cup^{4\delta} (\log U)^{-1} (\cup \text{ runs over powers of 2}) \]

which is

\[ \ll \sum_{X^{\frac{1}{2}} < \cup \leq X} \cup^{4\delta} (\log H)^{-1} + O(H^6 \log \log H) \]

\[ \ll H^{26} (\log H)^{-1} \left( \sum_{\nu=0}^Q 2^{-4\nu \delta} + O \left( 2^{-4Q\delta \delta^{-1}} \right) \right) + O(H^6 \log \log H) \]

where \( Q \left[ \frac{\log H}{\log \frac{H}{2}} \right] \),

\[ \ll H^{26} (\log H)^{-1} \left( \frac{1}{\delta} + O \left( H^{-\delta \delta^{-1}} \right) \right) + O(H^6 \log \log H) \]

\[ = H^{26} (\delta \log H)^{-1} (1 + O(H^{-\delta} + H^{-\delta} (\delta \log H) \log \log H)) \]

\[ = H^{26} (\delta \log H)^{-1} (1 + O(H^{-\delta} + H^{-\delta/2} \log \log H)), \text{ (since } H^{\delta/2} \geq \frac{\delta}{2} \text{ log } H), \]

\[ \ll H^{26} (\delta \log H)^{-1} \text{ provided } H^\delta \geq (\log \log H)^3 \text{ (which is certainly satisfied since } \delta = \frac{1}{4} C (\log \log T)(\log H)^{-1} \text{ and } C \text{ is assumed to be large).} \]

(B) The quantity in (2.7) is

\[ \ll \left( \sum_p \left| \Delta \left( \frac{X}{p} \right) \right|^2 p^{-2\sigma} \right)^2 + \frac{1}{H} \left( \sum_p \left| \Delta \left( \frac{X}{p} \right) \right|^2 p^{1-2\sigma} \right)^2 \]
\[
\ll \left( \sum_{p < X} p^{-2\sigma} \right)^2 + \left( \sum_{p > X} \frac{X^4}{p^{1+2\sigma}} \right)^2 + \frac{1}{X^2} \left( \sum_{p < X} p^{1-2\sigma} \right)^2 + \frac{1}{X^2} \left( \sum_{p > X} \frac{X^4}{p^{1+2\sigma}} \right)^2 \\
\ll H^{45} (\delta \log H)^{-2} + \left( \frac{X^4}{X^{3+2\sigma} \log H} \right)^2 + \frac{1}{X^2} \left( \frac{X^{2-2\sigma}}{\log H} \right)^2 \\
+ \frac{1}{X^2} \left( \frac{X^4}{X^{3+2\sigma} \log H} \right)^2 \text{ (if } H^\delta \geq (\log \log H)^3 \text{ which is satisfied by our assumption on } H) \\
\ll H^{45} (\delta \log H)^{-2} + H^{45} (\log H)^{-2} \\
\ll H^{46} (\delta \log H)^{-2} \text{ if } \delta \text{ is small.}
\]

This completes explanations regarding (2.6) and (2.7).

Thus Theorem 1 is completely proved.

ACKNOWLEDGEMENT. The authors are grateful to Professor M. JUTILA for encouragement.
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MANUSCRIPT COMPLETED ON 23 FEBRUARY 1991