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ON THE ZEROS OF A CLASS OF GENERALISED
DIRICHLET SERIES-IX
BY
R. BALASUBRAMANIAN AND K. RAMACHANDRA

§ 1. INTRODUCTION. In paper VII[I] of this series K. Ramachandra
raised the question "Let $s = \sigma + it, T \geq T_0$ a large positive constant. For
what values $\alpha = \alpha(T)$ the rectangle ($\sigma \geq \alpha(T), T \leq t \leq 2T$) contains infinity
of zeros of a generalised Dirichlet series of a certain type?" (In the earlier
papers of this series he and R. Balasubramanian, sometimes individually and
sometimes jointly considered the problem where $\alpha = \alpha(T)$ does not depend
on $T$). Since the series considered in that paper were too general the answer
($\alpha(T) = \frac{1}{2} - \frac{D}{\log log T}$) was perhaps too weak. In paper VII[2] of the series
we considered somewhat restricted series and obtained the result that we
can take $\alpha(T) = \frac{1}{2} - C_0(\log log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}}$. In the present paper we
assume a serious restrictive condition namely an Euler product and show
that we can take $\alpha(T) = \frac{1}{2} - C_0(\log log T)(\log T)^{-1}$. (Actually we work
with shorter rectangles and also obtain a quantitative result on the number
of zeros). Accordingly we prove the following

THEOREM 1. Let $p$ run over primes and $\omega(p)$ complex numbers whose
absolute value is 1 except for a finite set of primes where we define $\omega(p)$ to
be zero. Let \( s = \sigma + it \) as usual, and let
\[
F(s) = \sum_{n=1}^{\infty} (a_n n^{-s}) = \prod_p \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1}, (\sigma > 1),
\]
(1.1)
be continuable analytically in
\[
\{ \sigma \geq \frac{1}{2} - \frac{C}{\log T} \log \log T, T \leq t \leq T + H \} (\log T)^{C'} \leq H \leq T, C' \text{ to be specified}
\]
(1.2)
and there \( \max |F(s)| \leq T^A \) where \( A(\geq 1) \) is a constant. Let \( C(\geq 1) \) be a large constant depending on \( A \) and let \( H \) exceed \((\log T)^{C'} \) where \( C'(\geq 1) \) is a large constant. Then \( F(s) \) has at least \( (C'')^{-1} H (\log \log T)^{-1} \) zeros in the rectangle (1.2), where \( C''(\geq 1) \) is a large constant.

**REMARK 1.** The condition on the Euler product (1.1) covers the Riemann zeta-function and the ordinary \( L \)-series. But it can be stated in such a way that it covers zeta and \( L \)-functions of algebraic number fields. The restriction (1.1) and the restriction \( |F(s)| \leq T^A \) practically force us to give these (and perhaps only these, with trivial changes) as examples and in these cases the series have a functional equation and so in fact we can take \( \alpha(T) = \frac{1}{2} \). Hence the main content of the present paper is the emphasis on results which can be proved without the assumption of a functional equation for \( F(s) \). Thus we stress that Theorem 1 does not depend on the assumption of a functional equation for \( F(s) \).

**REMARK 2.** In the case where \( F(s) \) is the Riemann zeta-function or ordinary \( L \)-series we can prove (without using the functional equation) that
\[
\frac{1}{T} \int_{T}^{2T} \left| F\left(\frac{1}{2} + it\right) \right|^2 \, dt \leq (\log T)^2
\]
(1.3)
and from this we can deduce that the number of zeros of \( F(s) \) in \((\sigma \geq \beta, T \leq t \leq 2T)\) is \( \leq T^\theta (\log T)^{A_1} \) where \( \frac{1}{2} \leq \beta \leq \frac{1}{2}, \theta = 4(1 - \beta)(3 - 2\beta)^{-1} \) and \( A_1 \geq 1 \) is a certain numerical constant. (For a proof of this fact see [5]). Thus in these cases and with the extra condition \( H = T \), the zeros of the theorem “already belong” to the rectangle
\[
\left( \sigma \leq \frac{1}{2} + C \frac{\log T}{\log T}, T \leq t \leq 2T \right)
\]
(1.4)
§ 2. PROOF OF THEOREM 1. In the proof we use the Ramachandra kernel \( \text{Exp}((\sin z)^2) \), a well-known theorem due to Montgomery and Vaughan and the well-known

**THEOREM 2** (Borel-Caratheodory Theorem). Suppose \( G(z) \) is analytic in \( |z - z_0| \leq R \) and on \( |z - z_0| = R \) we have \( \text{Re} G(z) \leq U \). Then in \( |z - z_0| \leq r < R \), we have,

\[
G(z) \leq \frac{2rU}{R - r} + \frac{R + r}{R - r} |G(z_0)|.
\]

**PROOF.** See [6], page 174.

We break up the proof into a few lemmas. The rough idea of the proof is as follows. We first prove that on the line \( \sigma = \frac{1}{2} - 2\delta \), where \( \delta = \frac{1}{4}C(\log \log T)(\log H)^{-1} \), we have (between \( T \) and \( T + H \)) \( \gg H \) well-spaced points which are all heavy for \( \log F(s) \) in a certain sense. Out of these we select \( \gg H(\log \log \log T)^{-1} \) points with the property that any two (distinct) points are at a distance \( \geq L = D \log \log T \), where \( D \geq 1 \) is a large constant. We then prove that if \( \sigma \geq \frac{1}{2} - 4\delta \) is any such point the rectangle \( (\sigma \geq \frac{1}{2} - 4\delta, |t_j - t| \leq \frac{1}{3}L) \) contains a zero of \( F(s) \). This will prove the theorem completely. We begin with

**LEMMA 1.** For \( u > 0 \) let

\[
\Delta(u) = \frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} u^w \text{Exp} \left( \left( \frac{\sin \frac{w}{1000}}{1000} \right)^2 \right) \frac{dw}{w}.
\]

Then

\[
\Delta(u) = O(u^2) \text{ and also } \Delta(u) = 1 + O(u^{-2})
\]

where the implied constants are absolute.

**PROOF.** By moving the line of integration to \( \text{Re} w = 2 \) (resp. \( \text{Re} w = -2 \)) the lemma follows by easy estimations.

**LEMMA 2.** Let \( \sigma = \frac{1}{2} - 2\delta, X = H^\frac{1}{3} \) and

\[
F_1(s) = \sum_p \omega(p)p^{-s} \Delta \left( \frac{X}{p} \right).
\]
Then
\[ \frac{1}{H} \int_T^{T+H} |F_1(s)|^2 \, dt \gg H^{26}(\delta \log H)^{-1} \]  
and
\[ \frac{1}{H} \int_T^{T+H} |F_1(s)|^4 \, dt \ll H^{45}(\delta \log H)^{-2}. \]

**PROOF.** By a well-known theorem of H.L. Montgomery and R.C. Vaughan (see [3] for a simpler proof of a special case of their result which we need here) LHS of (2.4) is
\[ \sum_p \frac{\omega(p)^2}{p^{2\sigma}} \Delta \left( \frac{X}{p} \right)^2 \left( 1 + O \left( \frac{p}{H} \right) \right) \]
which is \( \gg H^{26}(\delta \log H)^{-1} \) by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to §3). This proves (2.4).

Also LHS of (2.5) is
\[ \leq \sum_{p_1, p_2} (p_1 p_2)^{-2\sigma} \Delta \left( \frac{X}{p_1} \right) \Delta \left( \frac{X}{p_2} \right)^2 \left( 1 + O \left( \frac{p_1 p_2}{H} \right) \right) \]
which is \( \ll H^{45}(\delta \log H)^{-2} \) by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to §3). This proves (2.5).

Thus Lemma 2 is completely proved.

**LEMMA 3.** Let \( T \) and \( H \) be positive integers (this can be assumed without loss of generality in Theorem 1). Then there exist \( \gg H \) integers \( M \) satisfying \( T \leq M \leq T + H - 1 \) for which
\[ \int_M^{M+1} |F_1(s)|^2 \, dt \gg H^{26}(\delta \log H)^{-1}. \]

**PROOF.** By (2.4) we have
\[ \sum_M \int_M^{M+1} |F(s)|^2 \, dt \geq C_1 H \psi \]
where \( C_1 \) is a positive constant and \( \psi = H^{26}(\delta \log H)^{-1} \). Here in the sum we drop those \( M \) for which the integral from \( M \) to \( M + 1 \) does not exceed \( \frac{1}{2} C_1 \psi \) and obtain
\[ \sum_M' \int_M^{M+1} |F_1(s)|^2 \, dt \geq \frac{1}{2} C_1 H \psi \]
where the slash indicates that we restrict only to those integrals which exceed \( \frac{1}{2} C_1 \psi \). By applying Hölder’s inequality we obtain

\[
\left( \sum_{M}^{M+1} \left( \sum_{M}^{M+1} |F_1(s)|^2 \, dt \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \geq \frac{1}{2} C_1 H \psi.
\]

The inequality

\[
\int_{M}^{M+1} |F_1(s)|^2 \, dt \leq \left( \int_{M}^{M+1} |F_1(s)|^4 \, dt \right)^{\frac{1}{2}}
\]

completes the proof of the lemma on using (2.5).

**REMARK.** Actually the deduction of Lemma 3 from Lemma 2 involves a general principle which was first observed and applied for discussing the zeros of Dirichlet series by R. Balasubramanian and K. Ramachandra in paper III[1] of this series.

**LEMMA 4.** In the interval \([T, T + H]\) there are \( R \gg H (\log \log \log T)^{-1} \) points \( t_1, t_2, \ldots, t_R \) with the properties \( |t_j - t_{j'}| \geq L \) for all \( j \neq j' \) where \( L = D \log \log \log T \) and further

\[
|F_1 \left( \frac{1}{2} - 2\delta + i t_j \right) | \gg H^6 (\delta \log H)^{-\frac{1}{2}}.
\]  

(2.9)

As has been said before \( D \) is a large positive constant.

**PROOF.** The proof follows from Lemma 3.

**LEMMA 5.** Consider any of the points \( t_j (= t_0 \text{ say}) \) given by Lemma 4 excluding the upper and the lower extreme points. Assume that the region \( (\sigma \geq \frac{1}{2} - 4\delta, |t - t_0| \leq \frac{1}{3} L) \) is zero-free for \( F(s) \). Then in \( (\sigma \geq \frac{1}{2} - 3\delta, |t - t_0| \leq \frac{1}{4} L) \) we have,

\[
|\log F(s)| \leq (\log T)^3.
\]  

(2.10)

**PROOF.** Apply Borel-Caratheodory theorem as follows. Take \( z_0 = 2 + it \) at which definitely \( |\log F(z_0)| \leq 10 \), whatever be \( t \). Choose \( R \) such that the circle \( |z - z_0| = R \) touches \( \sigma = \frac{1}{2} - 4\delta \) (we mean the line \( Re \, z = \frac{1}{2} - 4\delta \))
and \( r \) to be \( R - \delta \). This proves the lemma.

**Lemma 6.** Put \( s_0 = \frac{1}{2} - 2\delta + it_0 \). Then

\[
F_1(s_0) = \log F(s_0) + O(1). \tag{2.11}
\]

**Proof.** Since the powers of primes higher than the first contribute \( O(1) \), LHS of (2.11) is

\[
\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \log F(s_0 + w)X^w \text{Exp} \left( \left( \frac{\sin \frac{w}{1000}}{w} \right)^2 \right) \frac{dw}{w} + O(1).
\]

We now break off the portion \( |Im\ w| \geq \frac{1}{4} L \) of the integral with an error \( O(1) \) and in the rest of the integral move the line of integration to \( Re\ w = -\delta \). This involves an error \( O(X^{-\delta}(\log T)^4) = O(1) \) if \( C \) is large. The pole at \( w = 0 \) contributes \( \log F(s_0) \). This proves the lemma.

**Lemma 7.** The equations (2.9), (2.10) and (2.11) are contradictory and hence the assumption of Lemma 5 is not true.

**Proof.** The equations referred to in the lemma give

\[
H^\delta (\delta \log H)^{-\frac{1}{4}} \ll (\log T)^3.
\]

This is false if \( \delta = \frac{1}{4} C (\log \log T)(\log H)^{-1} \) (as specified already) and \( C \) is a large positive constant.

This completes the proof of Theorem 1.

§ 3. **Explanations Regarding (2.6) and (2.7).**

(A) In (2.6) the main term is

\[
= \sum_{p \leq X} p^{-2\sigma} \left( 1 + O \left( \frac{p^2}{X^2} \right) \right) \left( 1 + O \left( \frac{p}{X^2} \right) \right) + O(1)
\]

\[
= \sum_{p \leq X} p^{-2\sigma} \left( 1 + O \left( \frac{p^2}{X^2} \right) \right) + O(1)
\]

\[
= \sum_{p \leq X} p^{-2\sigma} + O(X^{1-2\sigma}(\log H)^{-1})
\]
\[ = \sum_{p \leq X} p^{-2\sigma} + O(H^{26}(\log H)^{-1}). \]

The error term is
\[ \ll \sum_{X \leq p \leq H} p^{-2\sigma} \left( \frac{X}{p} \right)^2 (1 + \frac{p}{H}) + \sum_{p > H} p^{-2\sigma} \left( \frac{X}{p} \right)^2 \left( \frac{p}{X^2} \right), \]
\[ \ll \sum_{X \leq p \leq H} p^{-2\sigma} \left( \frac{X}{p} \right)^2 + \sum_{p > H} p^{-1 - 2\sigma} \]
\[ \ll X^2(\log H)^{-1}X^{-1 - 2\sigma} + (\log H)^{-1}H^{-2\sigma} \]
\[ \ll (\log H)^{-1}(XH^{-\sigma} + H^{-2\sigma}) \ll H^{26}(\log H)^{-1}. \]

Thus (2.6) is \[ \sum_{p \leq X} p^{-2\sigma} + O(H^{26}(\log H)^{-1}). \] Now the first term here is \[ \sum_{U \leq X} \mathcal{U}^{46}(\log U)^{-1} \] (\(U \) runs over powers of 2) which is
\[ \asy \sum_{X^{\frac{1}{2}} \leq U \leq X} \mathcal{U}^{46}(\log H)^{-1} + O(H^6 \log \log H) \]
\[ \asy H^{26}(\log H)^{-1} \left( \sum_{\nu = 0}^{Q} 2^{-4\nu \delta} + O(2^{-4Q\delta \delta^{-1}}) \right) + O(H^6 \log \log H) \] (where \( Q = \left\lfloor \frac{\log H}{\log 2} \right\rfloor \)),
\[ \asy \asy H^{26}(\log H)^{-1} \left( \frac{1}{\delta} + O \left( H^{-\delta \delta^{-1}} \right) \right) + O(H^6 \log \log H) \]
\[ = H^{26}(\delta \log H)^{-1}(1 + O(H^{-\delta} + H^{-\delta}(\delta \log H) \log \log H)) \]
\[ = H^{26}(\delta \log H)^{-1}(1 + O(H^{-\delta} + H^{-\delta/2} \log \log H)), \text{ (since } H^{\delta/2} \geq \frac{\delta}{2} \log H), \]
\[ \asy H^{26}(\delta \log H)^{-1} \text{ provided } H^\delta \geq (\log \log H)^3 \] (which is certainly satisfied since \( \delta = \frac{1}{4}C(\log \log T)(\log H)^{-1} \) and \( C \) is assumed to be large).

(B) The quantity in (2.7) is
\[ \ll \left( \sum_{p} |\Delta \left( \frac{X}{p} \right)^2 p^{-2\sigma} \right)^2 + \frac{1}{H} \left( \sum_{p} |\Delta \left( \frac{X}{p} \right)^2 p^{1-2\sigma} \right)^2 \]
\[ \ll \left( \sum_{p \leq x} p^{-2\sigma} \right)^2 + \left( \sum_{p > x} \frac{x^4}{p^{1+2\sigma}} \right)^2 + \frac{1}{x^2} \left( \sum_{p \leq x} p^{1-2\sigma} \right)^2 + \frac{1}{x^2} \left( \sum_{p > x} \frac{x^4}{p^{1+2\sigma}} \right)^2 \]
\[ \ll H^{4\delta} (\delta \log H)^{-2} + \left( \frac{x^4}{x^{3+2\sigma} \log H} \right)^2 + \frac{1}{x^2} \left( \frac{x^2}{\log H} \right)^2 \]
\[ + \frac{1}{x^2} \left( \frac{x^4}{x^{3+2\sigma} \log H} \right)^2 \quad \text{(if } H^\delta \geq (\log \log H)^3 \text{ which is satisfied by our assumption on } H \text{)} \]
\[ \ll H^{4\delta} (\delta \log H)^{-2} + H^{4\delta} (\log H)^{-2} \]
\[ \ll H^{4\delta} (\delta \log H)^{-2} \text{ if } \delta \text{ is small.} \]

This completes explanations regarding (2.6) and (2.7).

Thus Theorem 1 is completely proved.

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