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ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES-IX

BY

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§ 1. **INTRODUCTION.** In paper VII[4] of this series K. Ramachandra raised the question "Let $s = \sigma + it, T \geq T_0$ a large positive constant. For what values $\alpha = \alpha(T)$ the rectangle $(\sigma \geq \alpha(T), T \leq t \leq 2T)$ contains infinity of zeros of a generalised Dirichlet series of a certain type?" (In the earlier papers of this series he and R. Balasubramanian, sometimes individually and sometimes jointly considered the problem where $\alpha = \alpha(T)$ does not depend on T). Since the series considered in that paper were too general the answer $\left(\alpha(T) = \frac{1}{2} - \frac{D}{loglog\ T}\right)$ was perhaps too weak. In paper VIII[2] of the series we considered somewhat restricted series and obtained the result that we can take $\alpha(T) = \frac{1}{2} - C_0(loglog\ T)^{\frac{3}{2}}(log\ T)^{-\frac{1}{2}}$. In the present paper we assume a serious restrictive condition namely an Euler product and show that we can take $\alpha(T) = \frac{1}{2} - C_0(loglog\ T)(log\ T)^{-1}$. (Actually we work with shorter rectangles and also obtain a quantative result on the number of zeros). Accordingly we prove the following

THEOREM 1. *Let p run over primes and $\omega(p)$ complex numbers whose absolute value is 1 except for a finite set of primes where we define $\omega(p)$ to* *be zero. Let $s = \sigma + it$ as usual, and let*

$$F(s) = \sum_{n=1}^{\infty}(a_n n^{-s}) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}, (\sigma > 1), \tag{1.1}$$

be continuable analytically in

$$\left\{\sigma \geq \frac{1}{2} - \frac{C \; loglog \; T}{log \; H}, T \leq t \leq T + H\right\} ((log \; T)^{C'} \leq H \leq T, C' \text{ to be specified}) \tag{1.2}$$

and there max $| F(s) | \leq T^A$ where $A(\geq 1)$ is a constant. Let $C(\geq 1)$ be a large constant depending on A and let H exceed $(log \; T)^{C'}$ where $C'(\geq 1)$ is a large constant. Then $F(s)$ has at least $(C'')^{-1}H(logloglog \; T)^{-1}$ zeros in the rectangle (1.2), where $C''(\geq 1)$ is a large constant.

REMARK 1. The condition on the Euler product (1.1) covers the Riemann zeta-function and the ordinary L-series. But it can be stated in such a way that it covers zeta and L-functions of algebraic number fields. The restriction (1.1) and the restriction $| F(s) | \leq T^A$ *practically force us to give these (and perhaps only these,* with trivial changes) *as examples* and in these cases the series have a functional equation and so in fact we can take $\alpha(T) = \frac{1}{2}$. Hence the main content of the present paper is the emphasis on results which can be proved without the assumption of a functional equation for $F(s)$. Thus we stress that *Theorem 1 does not depend on the assumption of a functional equation for $F(s)$.*

REMARK 2. In the case where $F(s)$ is the Riemann zeta-function or ordinary L-series we can prove (*without using the functional equation*) that

$$\frac{1}{T}\int_T^{2T} | F(\frac{1}{2} + it) |^2 \, dt \leq (log \; T)^2 \tag{1.3}$$

and from this we can deduce that the number of zeros of $F(s)$ in $(\sigma \geq \beta, T \leq t \leq 2T)$ is $\leq T^{\theta}(log \; T)^{A_1}$ where $\frac{1}{2} \leq \beta \leq 1, \theta = 4(1-\beta)(3-2\beta)^{-1}$ and $A_1 \geq 1$ is a certain numerical constant. (For a proof of this fact see [5]). Thus in these cases and with the extra condition $H = T$, the zeros of the theorem "already belong" to the rectangle

$$\left(\sigma \leq \frac{1}{2} + C \frac{loglog \; T}{log \; T}, T \leq t \leq 2T\right) \tag{1.4}$$

§ 2. **PROOF OF THEOREM 1.** In the proof we use the Ramachandra kernel $Exp((Sin\ z)^2)$, a well-known theorem due to Montgomery and Vaughan and the well-known

THEOREM 2 (Borel-Caratheodory Theorem). *Suppose $G(z)$ is analytic in $| z - z_0 | \leq R$ and on $| z - z_0 | = R$ we have $Re\ G(z) \leq U$. Then in $| z - z_0 | \leq r < R$, we have,*

$$| G(z) | \leq \frac{2rU}{R-r} + \frac{R+r}{R-r} | G(z_0) | . \tag{2.1}$$

PROOF. See [6], page 174.

We break up the proof into a few lemmas. The rough idea of the proof is as follows. We first prove that on the line $\sigma = \frac{1}{2} - 2\delta$, where $\delta = \frac{1}{4}C(loglog\ T)(log\ H)^{-1}$, we have (between T and $T + H$) $\gg H$ well-spaced points which are all heavy for $log\ F(s)$ in a certain sense. Out of these we select $\gg H(logloglog\ T)^{-1}$ points with the property that any two (distinct) points are at a distance $\geq L = D\ logloglog\ T$, where $D(\geq 1)$ is a large constant. We then prove that if $\frac{1}{2} - 2\delta + it_j$ is any such point the rectangle $(\sigma \geq \frac{1}{2} - 4\delta, | t_j - t | \leq \frac{1}{3}L)$ contains a zero of $F(s)$. This will prove the theorem completely. We begin with

LEMMA 1. *For $u > 0$ let*

$$\Delta(u) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} u^w Exp\left(\left(Sin\ \frac{w}{1000}\right)^2\right) \frac{dw}{w}. \tag{2.2}$$

Then

$$\Delta(u) = O(u^2) \text{ and also } \Delta(u) = 1 + O(u^{-2})$$

where the implied constants are absolute.

PROOF. By moving the line of integration to $Re\ w = 2$ (resp. $Re\ w = -2$) the lemma follows by easy estimations.

LEMMA 2. *Let $\sigma = \frac{1}{2} - 2\delta, X = H^{\frac{1}{2}}$ and*

$$F_1(s) = \sum_p \omega(p) p^{-s} \Delta\left(\frac{X}{p}\right). \tag{2.3}$$

Then
$$\frac{1}{H}\int_{T}^{T+H} | F_1(s) |^2 \, dt \gg H^{2\delta}(\delta \log H)^{-1} \tag{2.4}$$
and
$$\frac{1}{H}\int_{T}^{T+H} | F_1(s) |^4 \, dt \ll H^{4\delta}(\delta \log H)^{-2}. \tag{2.5}$$

PROOF. By a well-known theorem of H.L. Montgomery and R.C. Vaughan (see [3] for a simpler proof of a special case of their result which we need here) LHS of (2.4) is

$$\sum_{p}\frac{| \omega(p) |^2}{p^{2\sigma}} \left| \Delta\left(\frac{X}{p}\right) \right|^2 \left(1+O\left(\frac{p}{H}\right)\right) \tag{2.6}$$

which is $\gg H^{2\delta}(\delta \log H)^{-1}$ by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.4).

Also LHS of (2.5) is

$$\leq \sum_{p_1,p_2}(p_1p_2)^{-2\sigma} \left| \Delta\left(\frac{X}{p_1}\right)\Delta\left(\frac{X}{p_2}\right) \right|^2 \left(1+O\left(\frac{p_1p_2}{H}\right)\right) \tag{2.7}$$

which is $\ll H^{4\delta}(\delta \log H)^{-2}$ by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.5).

Thus Lemma 2 is completely proved.

LEMMA 3. *Let T and H be positive integers (this can be assumed without loss of generality in Theorem 1). Then there exist $\gg H$ integers M satisfying $T \leq M \leq T+H-1$ for which*

$$\int_{M}^{M+1} | F_1(s) |^2 \, dt \gg H^{2\delta}(\delta \log H)^{-1}. \tag{2.8}$$

PROOF. By (2.4) we have

$$\sum_{M}\int_{M}^{M+1} | F(s) |^2 \, dt \geq C_1 H \, \psi$$

where C_1 is a positive constant and $\psi = H^{2\delta}(\delta \log H)^{-1}$. Here in the sum we drop those M for which the integral from M to $M+1$ does not exceed $\frac{1}{2}C_1\psi$ and obtain

$$\sum_{M}{}' \int_{M}^{M+1} | F_1(s) |^2 \, dt \geq \frac{1}{2}C_1 H \psi$$

where the slash indicates that we restrict only to those integrals which exceed $\frac{1}{2}C_1\psi$. By applying Hölder's inequality we obtain

$$\left(\sum_M{}' 1\right)^{\frac{1}{2}} \left(\sum_M{}' \left(\int_M^{M+1} \mid F_1(s) \mid^2 dt\right)^2\right)^{\frac{1}{2}} \geq \frac{1}{2}C_1 H\psi.$$

The inequality

$$\int_M^{M+1} \mid F_1(s) \mid^2 dt \leq \left(\int_M^{M+1} \mid F_1(s) \mid^4 dt\right)^{\frac{1}{2}}$$

completes the proof of the lemma on using (2.5).

REMARK. Actually the deduction of Lemma 3 from Lemma 2 involves a general principle which was first observed and applied for discussing the zeros of Dirichlet series by R. Balasubramanian and K. Ramachandra in paper III[1] of this series.

LEMMA 4. *In the interval $[T, T+H]$ there are $R \gg H(logloglog\ T)^{-1}$ points $t_1, t_2, \cdots, t_R$ with the properties $\mid t_j - t_{j'} \mid \geq L$ for all $j \neq j'$ where $L = D\ logloglog\ T$ and further*

$$\mid F_1\left(\frac{1}{2} - 2\delta + t_j\right) \mid \gg H^{\delta}(\delta\ log\ H)^{-\frac{1}{2}}. \tag{2.9}$$

As has been said before D is a large positive constant.

PROOF. The proof follows from Lemma 3.

LEMMA 5. *Consider any of the points $t_j (= t_0$ say) given by Lemma 4 excluding the upper and the lower extreme points. Assume that the region $(\sigma \geq \frac{1}{2} - 4\delta, \mid t - t_0 \mid \leq \frac{1}{3}L)$ is zero-free for $F(s)$. Then in $(\sigma \geq \frac{1}{2} - 3\delta,$ $\mid t - t_0 \mid \leq \frac{1}{4}L)$ we have,*

$$\mid log\ F(s) \mid \leq (log\ T)^3. \tag{2.10}$$

PROOF. Apply Borel-Caratheodory theorem as follows. Take $z_0 = 2 + it$ at which definitely $\mid log\ F(z_0) \mid \leq 10$, whatever be t. Choose R such that the circle $\mid z - z_0 \mid = R$ touches $\sigma = \frac{1}{2} - 4\delta$ (we mean the line $Re\ z = \frac{1}{2} - 4\delta$)

and r to be $R - \delta$. This proves the lemma.

LEMMA 6. *Put* $s_0 = \frac{1}{2} - 2\delta + it_0$. *Then*

$$F_1(s_0) = \log F(s_0) + O(1). \tag{2.11}$$

PROOF. Since the powers of primes higher than the first contribute $O(1)$, LHS of (2.11) is

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \log F(s_0 + w) X^w Exp\left(\left(Sin \frac{w}{1000}\right)^2\right)\frac{dw}{w} + O(1).$$

We now break off the portion $| Im\ w | \geq \frac{1}{4}L$ of the integral with an error $O(1)$ and in the rest of the integral move the line of integration to $Re\ w = -\delta$. This involves an error $O(X^{-\delta}(\log T)^4) = O(1)$ if C is large. The pole at $w = 0$ contributes $\log F(s_0)$. This proves the lemma.

LEMMA 7. *The equations (2.9), (2.10) and (2.11) are contradictory and hence the assumption of Lemma 5 is not true.*

PROOF. The equations referred to in the lemma give

$$H^{\delta}(\delta \log H)^{-\frac{1}{2}} \ll (\log T)^3.$$

This is false if $\delta = \frac{1}{4}C(\log\log T)(\log H)^{-1}$ (as specified already) and C is a large positive constant.

This completes the proof of Theorem 1.

§ 3. EXPLANATIONS REGARDING (2.6) AND (2.7).

(A) In (2.6) the main term is

$$= \sum_{p\leq X} p^{-2\sigma}\left(1 + O\left(\tfrac{p^2}{X^2}\right)\right)\left(1 + O\left(\tfrac{p}{X^2}\right)\right) + O(1)$$

$$= \sum_{p\leq X} p^{-2\sigma}\left(1 + O\left(\tfrac{p^2}{X^2}\right)\right) + O(1)$$

$$= \sum_{p\leq X} p^{-2\sigma} + O(X^{1-2\sigma}(\log H)^{-1})$$

$$= \sum_{p \le X} p^{-2\sigma} + O(H^{2\delta}(\log H)^{-1}).$$

The error term is

$$\ll \sum_{X \le p \le H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 (1 + \tfrac{p}{H}) + \sum_{p > H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 \left(\frac{p}{X^2}\right)$$

$$\ll \sum_{X \le p \le H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 + \sum_{p > H} p^{-1-2\sigma}$$

$$\ll X^2(\log H)^{-1} X^{-1-2\sigma} + (\log H)^{-1} H^{-2\sigma}$$

$$\ll (\log H)^{-1}(XH^{-\sigma} + H^{-2\sigma}) \ll H^{2\delta}(\log H)^{-1}.$$

Thus (2.6) is $\sum_{p \le X} p^{-2\sigma} + O(H^{2\delta}(\log H)^{-1})$. Now the first term here is $\asymp$ $\sum_{U \le X} U^{4\delta} (\log U)^{-1}$ (U runs over powers of 2) which is

$$\asymp \sum_{X^{\frac{1}{2}} \le U \le X} U^{4\delta} (\log H)^{-1} + O(H^{\delta} \log\log H)$$

$$\asymp H^{2\delta}(\log H)^{-1} \left(\sum_{\nu=0}^{Q} 2^{-4\nu\delta} + O\left(2^{-4Q\delta}\delta^{-1}\right) \right) + O(H^{\delta} \log\log H)$$ (where Q $\left[\frac{1}{4} \frac{\log H}{\log 2}\right]$),

$$\asymp H^{2\delta}(\log H)^{-1} \left(\tfrac{1}{\delta} + O\left(H^{-\delta}\delta^{-1}\right)\right) + O(H^{\delta} \log\log H)$$

$$= H^{2\delta}(\delta \log H)^{-1}(1 + O(H^{-\delta} + H^{-\delta}(\delta \log H)\log\log H))$$

$$= H^{2\delta}(\delta \log H)^{-1}(1 + O(H^{-\delta} + H^{-\delta/2}\log\log H)),$$ (since $H^{\delta/2} \ge \frac{\delta}{2} \log H$),

$\asymp H^{2\delta}(\delta \log H)^{-1}$ provided $H^{\delta} \ge (\log\log H)^3$ (which is certainly satisfied since $\delta = \frac{1}{4}C(\log\log T)(\log H)^{-1}$ and C is assumed to be large).

(B) The quantity in (2.7) is

$$\ll \left(\sum_{p} | \Delta\left(\frac{X}{p}\right) |^2 p^{-2\sigma} \right)^2 + \frac{1}{H} \left(\sum_{p} | \Delta\left(\frac{X}{p}\right) |^2 p^{1-2\sigma} \right)^2$$

$$\ll \left(\sum_{p\le X} p^{-2\sigma}\right)^2 + \left(\sum_{p>X} \frac{X^4}{p^{4+2\sigma}}\right)^2 + \frac{1}{X^2}\left(\sum_{p\le X} p^{1-2\sigma}\right)^2 + \frac{1}{X^2}\left(\sum_{p>X} \frac{X^4}{p^{3+2\sigma}}\right)^2$$

$$\ll H^{4\delta}(\delta \log H)^{-2} + \left(\frac{X^4}{X^{3+2\sigma}\log H}\right)^2 + \frac{1}{X^2}\left(\frac{X^{2-2\sigma}}{\log H}\right)^2$$

$$+\frac{1}{X^2}\left(\frac{X^4}{X^{2+2\sigma}\log H}\right)^2$$ (if $H^\delta \ge (\log\log H)^3$ which is satisfied by our assumption on H.

$$\ll H^{4\delta}(\delta \log H)^{-2} + H^{4\delta}(\log H)^{-2}$$

$$\ll H^{4\delta}(\delta \log H)^{-2} \text{ if } \delta \text{ is small.}$$

This completes explanations regarding (2.6) and (2.7).

Thus Theorem 1 is completely proved.

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