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A Chebychev's Type of Prime Number Theorem in a Short Interval-II

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§ 1. INTRODUCTION.

We shall investigate the number of primes in the interval $(x-y,x]$ for $y=x^\theta$ with $1/2<\theta\le 7/12$. In [1], we proved

Theorem A. *Suppose x be a large number, then*

$$1.01\frac{y}{\log x}\ge \pi(x)-\pi(x-y)\ge 0.99\frac{y}{\log x} \tag{1.1}$$

with $y=x^\theta$, uniformly for

$$\frac{11}{20}<\theta\le\frac{7}{12}. \tag{1.2}$$

Denote $p(d_i)$ the smallest prime factor of d_i. We write

$$S_k:=\{d:d=d_1\cdots d_k, d\in I^y, p(d_i)\ge z, 1\le i\le k\}. \tag{1.3}$$

Let

$$\sum_{n\in I^y}a_n(k)=\sum_{\substack{d_1\cdots d_k=n\\ p(d_i)\ge z,1\le i\le k\\ n\in I^y}}1=\sum_{s\in S_k}1. \tag{1.4}$$

Let the interval $I^y=(x-y,x]$ with

$$x^{1/2}<y\le\frac{1}{2}x$$

and the parameter z satisfying

$$x^c < z \leq x^{1/5}$$

where c is a positive integer that will be chosen later.

Let $I_j, 1 \leq j \leq r$, be a set of integers, and $I_j \subseteq [2, x]$ and H be the "Direct Product" of sets I_j, for $1 \leq j \leq r$, it means $d \in H$ if and only if $d = d_1 \cdots d_r$ with $d_j \in I_j, 1 \leq j \leq r$, and $d \in I^y$. (1.5)

Suppose θ be fixed in the interval $(1/2, 1), y \in [x^\theta, x\ exp(-(log\ x)^{1/6})]$. Define the conditions (A_1) and (A_2) as following :

(A_1). Let k' be an integer. If there exist some sets $\mathbf{H}_k, 1 \leq k < k'$, which are collections of direct products H's and constants c_H such that

$$\sum_{n \in I^y} a_n(k) = \sum_{H \in \mathbf{H}_k} C_H \sum_{d \in H} 1 + O\left(\frac{y}{log^2 x}\right), \tag{1.6}$$

then we call $\mathbf{H}_k, 1 \leq k < k'$, satisfy (A_1).

(A_2). If $\mathbf{H}_k, 1 \leq k < k'$, satisfy (A_1), there exists a subset $\mathbf{H}'_k$, and a function $E_k(H, z)$ independent of y such that

$$\sum_{d \in H_k} 1 = yE_k(H, z) + O(y\ exp(-(log\ x)^{1/7})), \tag{1.7}$$

uniformly for

$$x^\theta \leq y \leq x\ exp(-(log\ x)^{1/6}),$$

then we call $\mathbf{H}'_k, 1 \leq k < k'$, satisfy (A_2). We call $\mathbf{H}'_k$ a 'good set' and call $\mathbf{H}''_k = \mathbf{H}_k \setminus \mathbf{H}'_k$, a 'bad set', for $1 \leq k < k'$.

In [3], we proved :

THEOREM B. *Let x be a sufficient large number, θ be fixed with $1/2 < \theta < 1, x^\theta \leq y < (1/2)x, I^y = (x - y, x], k_0$ be an integer which is dependent on θ, and z be fixed in $(x^{1/k_0}, x^{1/5}]$. Let $\mathbf{H}_k, 1 \leq k < k'$, such that (A_1). If there exists a subset $\mathbf{H}'_k$ of $\mathbf{H}_k$ such that (A_2), and writing $\mathbf{H}''_k = \mathbf{H}_k \setminus \mathbf{H}'_k$, then we have*

$$\pi(x) - \pi(x - y) = yE(x, z) + R(y) + O(y\ exp(-(log\ x)^{1/7}) \tag{1.8}$$

uniformly for

$$x^{\theta} \le y \le x\ exp(-(log\ x)^{1/6}),$$

where $E(x,z)$ *independent of* y, *and*

$$R(y) = \sum_{1\le k\le k_0} (-1)^{k-1}k^{-1} \sum_{H\in \mathbf{H}_k'} c_H \sum_{d\in H} 1. \tag{1.9}$$

$$\pi(x) - \pi(x-y) = \sum_{1\le k\le k_0} (-1)^{k-1}k^{-1} \sum_{s\in S_k} 1 + O(yx^{-\frac{1}{3}}). \tag{1.10}$$

Theorem B is a generalization of Theorem 1 of Heath-Brown [2]. Take $k' = 7, S_1, \cdots, S_5$ as good sets and only S_6 as a bad set i.e. $H_1' = S_1, \cdots, \mathbf{H}_5' = S_5, \mathbf{H}_6' = \phi$; and $\mathbf{H}_1'' = \cdots = H_5'' = \phi, \mathbf{H}_6'' = S_6$; then Theorem B become Theorem 1 of [2]. We are not limited that the good set S_k' or that the bad set S_k'' should to be whole of S_k. In fact, $R(y)$ is the contribution of all bad sets. In [1], we also proved :

THEOREM C. *Suppose that* θ *is fixed in* $(1/2,1), y_0 = x\ exp(-(log\ x)^{\frac{1}{7}}).\mathcal{H}_k, 1$ $k < k'$, *satisfy* (A_1) *and* (A_2). *If there exists constants* e_1'', e_1', e_2'' *and* e_2' *such that*

$$\frac{(e_1'+\varepsilon)y_0}{log\ x} < \sum_{1\le k<k'} (-1)^{k-1}k^{-1}R_k(y_0) < \frac{(e''-\varepsilon)y_0}{log\ x}; \tag{1.11}$$

and

$$\frac{(e_2'+\varepsilon)y}{log\ x} < \sum_{1\le k<k'} (-1)^{k-1}k^{-1}R_k(y) < \frac{(e''-\varepsilon)y}{log\ x}; \tag{1.12}$$

where ε *is a small positive constant. Then*

$$\frac{(1-e_1''-e_2')y}{log\ x} < \pi(x) - \pi(x-y) < \frac{(1+e_1'+e_2'')y}{log\ x} \tag{1.13}$$

uniformly for $x^{\theta} \le y \le y_0$.

We will prove the main theorem of this paper :

Theorem 1. *Let* $y = x^{\theta}, \theta = 6/11 + \varepsilon$, *then*

$$\frac{0.969y}{log\ x} < \pi(x) - \pi(x-y) < \frac{1.031y}{log\ x}. \tag{1.14}$$

Using Theorem B and Theorem C, we need to find $\mathbf{H}_k$ and $\mathbf{H}'_k$. In [4] we gave some sufficient conditions that imply some kind of "direct product" be "good set". In § 2, we record those results from [4]. In § 3, we use those conditions to prove that $\mathbf{H}'_k$ which will be defined in § 6 below be "good set".

A criterion for good sets is extracted. However, the technical work needed to choose good sets and to make the size of the bad sets as small as possible, is precisely the main difference between our method and Heath-Brown's. The new Theorem 1 will enable us to improve the results of Heath-Brown and Iwaniec [5]. Moreover, we can improve (1.14) further but only at the cost of much arduous computation.

§ 2 "GOOD SET"

Let c_0 be a constant that will be defined later on. Let I_0 be an interval $[a_0, b_0]$ which contains in $[1, x]$ and $I_j (1 \leq j \leq r)$ be a subset of interval $[a_j, b_j]$ contains in $[x^{c_0}, x]$ also. Denote $D = I_0 \cdots I_r$ be a direct product of I_j. Let $i_j = log\ a_j/log\ x$ and $i'_j = log\ b_j/log\ x$ and let $d_j = x^{\theta_j}$ with $i_j \leq \theta_j \leq i'_j$ and $0 \leq j \leq r$. For convenience, we write $d = \{\theta_0, \theta_1, \cdots, \theta_r\} \in D$, and a set

$$\mathbf{D} = \{\{\theta_0, \theta_1, \cdots, \theta_r\} : 1/2 \geq 1 - \theta_1 - \cdots - \theta_r = \theta_0 \geq \theta_1 \geq \cdots \geq \theta_r\}. \quad (2.1)$$

For short, we denote $\{\theta_j\} = \{\theta_0, \theta_1, \cdots, \theta_r\}$.

Let $\mathbf{D} \cap \mathbf{I}^y$ be a set of integers, $d \in \mathbf{D} \cap \mathbf{I}^y$ if and only if $d \in \mathbf{D}$ and $d \in \mathbf{I}^y \cdot d = d'$ with $d, d' \in \mathbf{D} \cap \mathbf{I}^y$ means $d = d_0 \cdots d_r$ and $d' = d'_0 \cdots d'_r$ with $d_j = d'_j$ for $0 \leq j \leq r$. We shall show the sufficient conditions for $\mathbf{D} \cap \mathbf{I}^y$ be a "good set", i.e. for a fixed z with $x^{1/5} > z = x^c$, there exists a function $E_{\mathbf{D}}(x, z)$, independent of y, which satisfies that

$$\sum_{d \in \mathbf{D} \cap \mathbf{I}^y} 1 = y E_{\mathbf{D}}(x, z) + O(y\ exp(-log^{1/7} x)), \quad (2.2)$$

where $E_{\mathbf{D}}(x, z)$ and constant in "O" are uniformly for

$$x^{\theta} \leq y \leq x\ exp(-4(log\ x)^{\frac{1}{3}}(loglog\ x)^{-\frac{1}{3}}).$$

We discuss those sequences $d = \{\theta_0, \theta_1, \cdots, \theta_r\} = \{\theta_j\}$ in $\mathbf{D}$. For such $\{\theta_j\}$, we define a corresponding set Θ of all of sequences $\{\theta'_0, \theta_1, \cdots, \theta_r, \theta_{r+1}, \cdots, \theta_{r+r_1}\}$

with $\theta_0' \leq \theta_0, \theta_1 \geq \cdots \geq \theta_r \geq log\ z/log\ x > \theta_{r+1} \geq \cdots \geq \theta_{r+r_1}$ and

$$\theta_0' + \theta_1 + \cdots + \theta_{r+r_1} = 1. \tag{2.3}$$

By (2.3) and (2.1), we have that if $r_1 = 0$, then

$$\theta_0' = \theta_0 \geq \theta_1. \tag{2.4}$$

For short, write $\{\theta_j\}' = \{\theta_0', \theta_1, \cdots, \theta_r, \theta_{r+1}, \cdots, \theta_{r+r_1}\} = \{\theta_j\}$, and $\{\theta_j\}' \in \Theta$. Let $\theta_0' = log\ X/log\ x, \theta_j = log\ X_d(j)/log\ x(1 \leq j \leq r)$ and $\theta_{r+j} = log\ Z_j/log\ x(1 \leq j \leq r_1)$. For each $\{\theta_0', \theta_1, \cdots, \theta_r, \theta_{r+1}, \cdots, \theta_{r+r_1}\}$, we define a product of Dirichlet series :

$$W(s) = X(s)\prod_{j=1}^{r} X_d^{(j)}(s)\ Y(s)\prod_{j=1}^{r_1} Z_j(s) \tag{2.5}$$

where

$$X(s) = \sum_{X<n\leq 2X} n^{-s};$$

$$X_d^{(j)}(s) = \sum_{X_d^{(j)}<m\leq 2X_d^{(j)}} f_m^{(j)} m^{-s}, \mid f_m^{(j)} \mid \leq 1;$$

$$Z_j(s) = \sum_{Z_j<I\leq 2Z_j} c_1 I^{-s}, \mid c_1 \mid \leq 1;$$

$$Y(s) = \sum_{Y<t\leq 2Y} \mu(t) v_t t^{-s}, \mid v_t \mid \leq 1.$$

with $Y = O(x^\delta), \delta$ be a sufficient small number with $\delta \ll \varepsilon$. Each $\{\theta_j\} \in \mathbf{D}$ corresponds all of $W(s, \{\theta_j\}')$'s for which $\{\theta_j\}' \in \Theta$. Define that $\mathbf{W(D)}$ is a set of all of such $W(s, \{\theta_j\}')$. For short, we write $W(s, \{\theta_j\}') = W(s)$. In [4], we proved that

Theorem A. *If* $\mathbf{D}$ *satisfies one of following conditions*

(1) $a_0 \geq x^{1/2}$;

(2) *all of* $W(s) \in \mathbf{W(D)}$ *such that*

$$\int_T^{2T} \mid W(\frac{1}{2}+it) \mid dt \ll x^{\frac{1}{2}} exp(-(log\ x)^{\frac{1}{3}}(loglog\ x)^{-\frac{1}{3}}) \tag{2.6}$$

for

$$T_1 \leq T \leq \frac{x^{1-\Delta}}{y},$$

where Δ *is any fixed positive constant, and*

$$T_1 = exp((log\ x)^{\frac{1}{3}}(loglog\ x)^{-\frac{1}{3}}).$$

Then (1.2) holds i.e. **D** *is a good set.*

Let $\theta_0, \theta_1, \cdots, \theta_k$ be positive numbers. In [4], we discussed the sequence $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with positive number k such that

$$\theta_0 + \theta_1 + \cdots + \theta_k = 1 \tag{2.7}$$

defined a set $E(\theta)$ of some $\{\theta_0, \theta_1, \cdots, \theta_k\}$'s and acutely proved that (4, § 5]).

Theorem B. *Let* $\{\theta_j\} \in$ **D**. *For each* $\{\theta_j\}' \in \Theta$ *define*

$$W'(s) = X(s)\prod_{j=1}^{r} X_d^{(j)}(s)\prod_{j=1}^{r_1} Z_j(s)$$

If $\{\theta_j\}' \in E(\theta)$, *then*

$$\int_T^{2T} \mid W'(\frac{1}{2}+it) \mid dt \ll x^{1/2-\epsilon}. \tag{2.8}$$

Moreover, (3.9) holds.

We now describe the set $E(\theta)$.

Suppose $\{a_1, a_2, \sigma\}$ or $\{a_1, a_2, a_3, \sigma\}$ be a complementary partial sum (it means that each θ_j belongs one and only one set and their sum in a set be σ or a_i) of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with $\sigma = \theta_0$ or $\sigma \leq t_0/2$, then

$$a_1 + a_2 + \sigma = 1. \tag{2.9}$$

or

$$a_1 + a_2 + a_3 + \sigma = 1. \tag{2.10}$$

Later on, we only define two of a_1, a_2, σ if (2.9) holds; or define three of a_1, a_2, a_3, σ if (2.10) holds.

Let $\theta = 6/11 + \varepsilon, t_0 = 1 - \theta + \varepsilon/2$ and $z = x^c$ with $c = t_0/10$. Define

$$\mathbf{D} = \{\{\theta_0, \theta_1, \cdots, \theta_r\} : \theta_1 \geq \cdots \geq \theta_r > c, \theta_0 + \theta_1 + \cdots + \theta_r = 1\}. \quad (2.11)$$

Define $\mathbf{D}_i^* (1 \leq i \leq 7)$ be the subsets of $\mathbf{D}$ and

$\mathbf{D}_1^* = \{\{\theta_0, \theta_1, \cdots, \theta_9\} : \theta_0 \geq \cdots \geq \theta_9 > t_0/5$ and $\theta_1 + \theta_2 + \theta_3 + \theta_4 \leq 8t_0/9\}$;

$\mathbf{D}_2^* = \{\{\theta_0, \theta_1, \cdots, \theta_7\} : 2t_0/7 \geq \theta_0 \geq \theta_1 \geq \cdots \geq \theta_7 \geq t_0/5\}$.

$\mathbf{D}_3^* = \{\{\theta_0, \theta_1, \cdots, \theta_5\} : 2t_0/5 \geq \theta_0 \geq \cdots \geq \theta_5 > t_0/5, \theta_3 + \theta_4 + \theta_5 \geq t_0\}$;

$\mathbf{D}_4^* = \{\{\theta_0, \theta_1, \cdots, \theta_5\} : \theta_0 \geq \theta_1 \geq \theta_2 \geq \theta_3 \geq 1 - 20t_0/11, t_0/3 \geq \theta_4 \geq t_0/4$,

$\theta_5 \geq t_0/5, \theta_0 + \theta_1 \leq 6t_0/7, \theta_2 + \theta_3 \geq 4 - 8t_0, \theta_0 \leq \theta_5/8 + 3t_0/8\}$;

$\mathbf{D}_5^* = \{\{\theta_0, \theta_1, \cdots, \theta_5\} : t_0/2 \geq \theta_0 \geq \theta_1 \geq \theta_2 \geq \theta_3 \geq \theta_4 \geq 1 - 20t_0/11, \theta_3 + \theta_4 \geq 4 - 8t_0, t_0/5 \geq \theta_5 \geq t_0/10, \theta_0 + \theta_1 \leq t_0/2\}$;

$D_6^* = \{\{\theta_0, \theta_1, \cdots, \theta_4\} : t_0/2 \geq \theta_1 \geq \theta_2 \geq \theta_3 \geq \theta_4 \geq 1 - 20t_0/11, \theta_3 + \theta_4 \geq 4 - 8t_0\}$

$\mathbf{D}_7^* = \{\{\theta_0, \theta_1, \cdots, \theta_6\} : 2t_0/5 \geq \theta_0 \geq \theta_1 \geq \theta_2 \geq \theta_3 \geq \theta_4 \geq 1 - 20t_0/11, t_0/5 \geq \theta_5 \geq \theta_6 \geq t_0/10, \theta_4 + \theta_6 > t_0/2\}$

and

$$\mathbf{D}^* = \cup_{j=1}^7 \mathbf{D}_j^*. \quad (2.12)$$

In § 4, we shall prove :

Theorem 1. *Suppose* $\mathbf{D}'$ *be a subset of* $\mathbf{D}$ *and*

$$\mathbf{D}' \cap \mathbf{D} = \emptyset, \quad (2.13)$$

then $\mathbf{D}'$ *satisfies (1.2), i.e.* $\mathbf{D}'$ *is a good set.*

In [4], we gave that some sufficient conditions which imply that D is a good set. In this paper, in § 2, we record those conditions from [4]. In § 3, we use them to show that $\mathbf{D} \setminus \mathbf{D}^*$ which is defined in (2.11) and (2.12) is "good".

§ 3. THEOREM 2.

We discuss those sequences $d = \{\theta_0, \theta_1, \cdots, \theta_r\} = \{\theta_j\}$ in $\mathbf{D}$. For such $\{\theta_j\}$, we define a corresponding set Θ of all sequences $\{\theta_0', \theta_1, \cdots, \theta_r, \theta_{r+1}, \cdots, \theta_{r+r_1}\}$ with $\theta_0' \le \theta_0$,

$$1/2 > \theta_0' = \theta_{r+1} \ge \cdots \ge \theta_{r+r_1} \ge \theta_1 \ge \cdots \ge \theta_r \ge \log z/\log x > \theta_{r+1} \ge \cdots \ge \theta_{r+r_1} \tag{3.1}$$

and

$$\theta_0' + \theta_1 + \cdots \theta_{r+r_1} = 1. \tag{3.2}$$

By (2.11) and (3.2), we have that if $r_1 = 0$, then

$$\theta_0' = \theta_0 \ge \theta_1. \tag{3.3}$$

For short, write $\{\theta_j\}' = \{\theta_0', \theta_1, \cdots, \theta_r, \theta_{r+1}, \cdots, \theta_{r+r_1}\}$, and $\{\theta_j\}' \in \Theta$. Write

$$r = r' + r'',$$

with

$$\theta_1 \ge \cdots \ge \theta_{r'} \ge t_0/5 > \theta_{r'+1} \ge \cdots \ge \theta_{r'+r''}. \tag{3.4}$$

We now describe the set $E(\theta)$. Suppose $\theta = 6/11 + \varepsilon$ and $t_0 = 5/11 + \varepsilon/2$. We define $E(\theta)$ be a set which contain all of sequence $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with (2.7) which satisfies one of following four properties :

(I) There exists at least one complementary partial sum $\{a_1, a_2, \sigma\}$ of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ which satisfies one of following conditions :

(3.5) $a_1 \le t_0, a_2 < 4 - 8t_0$ (see Lemma 4.4 of [4]);

(3.6) $1 - 20t_0/11 > \sigma > t_0/5$ and $t_0 \ge a_2 > 8t_0/9$ or $\sigma > t_0/5, a_1 \ge t_0$ and $a_2 > 8t_0/9$ (see (4.1.3) with $i = 3$ of [4]);

(3.7) $a_2 \le t_0, a_3 \le t_0$ and $\sigma < 1 - 20t_0/11$ (see (4.6.1) of [4]);

(3.8) $\sigma > t_0/2$ (see Lemma 4.3 of [4]);

(3.9) $a_1 \ge t_0, a_2 \ge t_0$ (see (4.1.1) of [4]);

(3.10) $a_1 \ge t_0, a_2 > 4t_0/5$, and $\sigma > t_0/3$ (see (4.1.3) with $i = 1$ of [4]);

(3.11) $t_0 \ge a_2 > 4t_0/5$, and $1 - 20t_0/11 > \sigma > t_0/3$ (see (3.7) and (3.10) above);

(3.12) $1 - 20t_0/11 > \sigma$ and $m_\sigma < a_1 < M_\sigma$ (see (4.4.4) and (4.6.1));

(3.13) $1/2 \ge a_1 \ge t_0$, and $\sigma < 1/2 - 8t_0/9$ (see (4.5.6) of [4]);

(3.14) $\sigma \ge t_0/4, a_1 \ge t_0$ and $a_2 > 6t_0/7$ (see (4.1.3) with $i = 2$ of [4]);

(II) There exists at least one complementary partial sum $\{a_1, a_2, a_3, \sigma\}$ of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ which satisfies

(3.15) $a_1 \ge t_0, a_2 \ge t_0/2, a_3 \ge t_0/4$ and $\sigma > 2t_0/7$ (see (4.2.1) of [4]);

(3.16) $a_1 \ge t_0, a_2 \ge t_0/3, a_3 \ge t_0/3$ and $\sigma > 2t_0/5$ (see (4.2.2) of [4]).

(III) $\{\theta_0, \theta_1, \cdots, \theta_k\}$ satisfies one of the following conditions:

(3.17) $k_0 = 6, \sigma = \theta_0 \le t_0/2, t_0/5 < \theta_6 \le \cdots \le \theta_1 \le 2t_0/7$ (see (4.7.4) of [4]);

(3.18) There exists at least one complementary partial sum $\{a_1, a_2, a_3, \sigma\}$ of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ which satisfies $\sigma = \theta_0, a_1 < t_0, a_2 < 1/3$ and $a_3 < t_0/5$ (see Lemma 3.10 with $t_0 = 5/11 - \varepsilon$ of [4]).

(3.19) $\sigma = \theta_0, a_1 \le 8t_0/9, a_2 \le 4t_0/9, a_3 \le t_0/4$, and $a_4 = 1 - \sigma - a_1 - a_2 - a_3 \le t_0/4$ (see (4.7.3) of [4]).

(3.20) $\sigma = \theta_0, a_1 \le t_0/2, a_2 \le t_0/2, a_3 \le 4t_0/9, a_4 \le t_0/4$, and $a_5 = 1 - \sigma - a_1 - a_2 - a_3 \le t_0/4$ (see (4.7.7) of [4]).

(3.21) $\sigma = \theta_0, \theta_i < t_0/5$, and $\sigma > 3t_0/8 + \theta_i/8$ (see (4.7.9) of [4]).

For a fixed $\sigma < 1 - 20t_0/11$, in [4] we proved that there exists a pair of numbers (m_σ, M_σ) with the properties

$$M_\sigma - m_\sigma > t_0/5 \text{ if } \sigma \geq t_0/5; \tag{3.22}$$

$$M_\sigma - m_\sigma < \sigma \text{ if } \sigma < t_0/5; \tag{3.23}$$

$$M_\sigma > t_0 > m_\sigma; \tag{3.24}$$

and

$$M_\sigma + m_\sigma + \sigma = 1 \tag{3.25}$$

(IV) Suppose $\{a_1, a_2, \sigma\}$ or $\{a_1, a_2, a_3, \sigma\}$ be a complementary sum of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with

$$m_\sigma < a_i < M_\sigma, \quad (i = 1 \text{ or } 2), \tag{3.26}$$

(See Lemma 4.5 of [2]).

Moreover, for $t_0/3 \geq \theta \geq t_0/4$, we have that

$$3m_\theta/2 + 3\theta < 1. \tag{3.27}$$

Applying Theorem A and Theorem B, Theorem 1 follows from

Theorem 2. *Suppose $\theta = 6/11 + \varepsilon$, and $\mathbf{D}'$ such that*

$$\mathbf{D} \cap \mathbf{D}^* = \emptyset,$$

then for every $\{\theta_j\} \in \mathbf{D}'$, the all of corresponding $\{\theta_j\}' \in \Theta$ contain in $E(\theta)$.

§ 4. LEMMAS.

Let $\theta = 6/11 + \varepsilon$ and $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with (2.7), i.e.

$$\theta_0 + \theta_1 + \cdots + \theta_k = 1.$$

In this section, we shall show some sufficient conditions for $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$. By the definition of $E(\theta)$ we check that $\{\theta_j\}$ satisfies at least one of conditions (3.5) - (3.19), and (3.25). When $\theta_0' > t_0/2, \{\theta_j\}' \in E$ by (3.8).

When $t_0 \le \theta_1 \le 1/2$ and $\theta'_0 \le t_0/2$, we have that $r_1 \ne 0$ by (3.3) and $\theta_{r+r_1} < t_0/5$, let $a_1 = \theta_1, \sigma = \theta_{r+r_1}$ and $a_2 = 1 - a_1 - \sigma$, then $\{\theta_j\}' \in E$ by (3.13).

Lemma 4.1. *Suppose there exist two elements θ' and θ'' of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with $\theta' \le 1 - 20t_0/11$ and $\theta'' < t_0/5$. If there exists a partial sum s of $\{\theta_0, \theta_1, \cdots, \theta_k\} \backslash \{\theta', \theta''\}$ such that $s < t_0$ and $s + \theta' \ge t_0$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$.*

Proof. We discuss following three cases :

Case 1. $t_0 \le s + \theta' < M_{\theta''}$.

Let $\sigma = \theta''$ and $a_1 = s + \theta'$, we have that $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.24) and (3.26).

Case 2. $s + \theta' \ge M_{\theta''}$.

By (3.25), we have

$$1 - s - \theta' - \theta'' \le 1 - M_{\theta''} - \theta'' = m_{\theta''}$$

and, by (3.23),

$$1 - s - \theta' \le \theta'' + m_{\theta''} < M_{\theta''}.$$

Let $a_1 = 1 - s - \theta'$ and $\sigma = \theta''$, if $a_1 > m_{\theta''}$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.26). If $a_1 \le m_{\theta''} \le t_0$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.7) since $a_2 = 1 - a_1 - \sigma \le m_{\theta''} \le t_0$ and $\sigma = \theta'' < 1 - 20t_0/11$.

Lemma 4.2. *Suppose $\{a_1, a_2, \sigma\}$ be a complementary partial sum of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with $a_1 = \theta''_1 + \cdots + \theta''_k, a_2 = \theta'_1 + \cdots + \theta'_k, a_1 \ge a_2, 1 - 20t_0/11 > \sigma = 1 - a_1 - a_2 > 1/2 - 8t_0/9$ and*

$$max\{\theta''_1, \cdots, \theta''_k\} - max\{\theta'_1, \cdots, \theta'_k\} < t_0/5; \tag{4.1}$$

then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$.

Proof. If $a_1 \le t_0$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.7); if $a_2 \ge t_0$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.6); if $m_\sigma < a_1 < M_\sigma, \{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.26).

Now we suppose $a_1 \geq M_\sigma$.

By (3.23), we have

$$\theta_1'' + \cdots + \theta_{k-1}'' + \theta_k = \theta_1'' + \cdots + \theta_k'' + (\theta_k' - \theta_k'')$$
$$> M_\sigma - \frac{t_0}{5} \geq m_\sigma.$$

If

$$\theta_1'' + \cdots + \theta_{k-1}'' + \theta_k' < M_\sigma,$$

let $a_1 = \theta_1'' + \cdots + \theta_{k-1}'' + \theta_k'$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.25); if

$$\theta_1'' + \cdots + \theta_{k-1}'' + \theta_k' \geq M_\sigma,$$

and

$$\theta_1'' + \cdots + \theta_{k-2}'' + \theta_{k-1}' + \theta_k' < M_\sigma,$$

repeating above process, let $a_1 = \theta_1'' + \cdots + \theta_{k-2}'' + \theta_{k-1}' + \theta_k'$ we also have $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$. And repeat it again, we have that, in all cases, $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ since

$$\theta_1' + \cdots + \theta_k' < t_0 \leq M_\sigma.$$

LEMMA 4.3. *$k_0 = 3, \sigma > 2t_0/5, a_2 \geq t_0/3, a_3 \geq t_0/3, a_2 + a_3 < 4 - 8t_0$ and $a_1 = 1 - \sigma - a_2 - a_3$; then $\{\theta_j\} \in E$.*

Proof. If $a_1 \geq t_0, \{\theta_j\} \in E$ by (3.16). If $a_1 < t_0, \{\theta_j\} \in E$ by (3.5).

§ 5. PROOF OF THEOREM 2.

We will prove that : if $\{\theta_j\} \in \mathbf{D} \setminus \mathbf{D}^*$, then $\{\theta_j\} \in E$. By Theorem 1, it is enough to show those $\{\theta_j\}'$ with $\{\theta_0' + \theta_{r+1} + \cdots + \theta_{r+r_1}, \theta_1, \cdots \theta_r\} \notin \mathcal{D}^*$ belong to E.

Denote k_0 be the integer with

$$\sum_{1 \leq j \leq k_0 - 1} \theta_j < t_0 \tag{5.1}$$

and

$$\sum_{1 \leq j \leq k_0} \theta_j \geq t_0. \tag{5.2}$$

By (2.14), $\{\theta_j\} \in E$ if $\theta_0' > t_0/2$. By (2.3), we only need to discuss the cases with

$$t_0/2 \geq \theta_0'. \tag{5.3}$$

If $r_1 = 0$, then

$$t_0/2 \geq \theta_0' = \theta_0 \geq \cdots \geq \theta_r, \tag{5.4}$$

and

$$\{\theta_j\}' = \{\theta_j\}.$$

Lemma 5.1. *Suppose $k_0 \geq 4$ and $r'' + r_1 > 0$, then $\{\theta_j\} \in E$.*

Proof. By (5.1) and $k_0 \geq 4$, we have

$$\theta_3 \leq \frac{t_0}{3} < 1 - \frac{20}{11}t_0.$$

By Lemma 4.2 and $\theta_{r+r_1} < t_0/5$ if $r'' + r_1 > 0$, we have $\{\theta_j\} \in E$ if $k_0 < r + r_1$. If $k_0 = r + r_1$, let $a_1 + \cdots + \theta_{k_0-1} < t_0, a_2 = \theta_0' < t_0$ and $\sigma = \theta_{k_0} < 1 - 20t_0/11$, we have $\{\theta_j\} \in E$ by (3.7).

Lemma 5.2. *Suppose $r' \geq k_0 + 5$, then $\{\theta_j\} \in E$.*

Proof. Let $a_1 = \theta_1 + \cdots + \theta_{k_0} > t_0, a_2 = \theta_{k_0+1} + \cdots \theta_{r'} > t_0$ and $\sigma = \theta_0$, then $\{\theta_j\} \in E$ by (3.9).

Lemma 5.3. *Suppose that $k_0 > r'$, then $\{\theta_j\} \in E$.*

Proof. Let $\sigma = \theta_0, a_1 = \theta_1 + \cdots + \theta_{k_0-1}$ and $a_2 = \theta_{k_0} < t_0/2 < 4 - 8t_0$ if $k_0 = r + r_1$, then $\{\theta_j\} \in E$ by (3.5). If $k_0 < r + r_1$, then $\{\theta_j\} \in E$ by Lemma 4.1.

Now may suppose that

$$k_0 \leq r' \leq k_0 + 4. \tag{5.5}$$

By (3.4), $\theta_j \geq t_0/5$ for $j \leq k_0$ (since $k_0 \leq r'$); then by (5.1), we have that $k_0 \leq 5$. By (5.2) and $\theta_1 \leq t_0$, then we have that $k_0 \geq 2$. Now we may suppose that

$$2 \leq k_0 \leq 5. \tag{5.6}$$

We discuss the following cases :

Case 1. $k_0 = 5$.

By Lemma 4.1, we may suppose that $r'' + r_1 = 0$. Then $\{\theta_j\} = \{\theta_0, \cdots, \theta_r\}$. By (5.5) we may suppose that $5 \le r' \le 9$.

If $\underline{r' \le 7}$, let $a_1 = \theta_1 + \cdots + \theta_{k_0-1} < t_0, a_2 = \theta_{k_0} + \theta_{k_0+1} + \theta_{k_0+2} \le 3t_0/4 < 4 - 8t_0$ and $\sigma = \theta_0$, then $\{\theta_j\} \in E$ by (3.5).

If $\underline{r' = 8}$ let $s = \theta_0, a_1 = \theta_1 + \theta_2 < t_0, a_2 = \theta_3 + \theta_4 < t_0/2, a_3 = \theta_5 + \theta_6 < t_0/2, a_4 = \theta_7 \le t_0/4, a_5 = \theta_7 \le t_0/4$, then $\{\theta_j\} \in E$ by (3.20).

If $\underline{r' = 9}$, we have

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 > 8t_0/9$$

since $\{\theta_j\} \notin D_1^*$. Let $a_1 = \theta_1 + \theta_2 + \theta_3 + \theta_4$ and $\sigma = \theta_5 > t_0/5$, we have that $\sigma < a_1/4 \le t_0/4 < 1 - 20t_0/11$, then $\{\theta_j\} \in E$ by (3.6).

Case 2. $k_0 = 4$.

By Lemma 4.1, we may suppose that $r'' + r_1 = 0$ again. By (5.4), we may suppose that $4 \le r' \le 8$.

If $\underline{r' \le 5}$, let $a_1 = \theta_1 + \theta_2 + \theta_3 < t_0, a_2 = \theta_4 + \theta_5 \le 2t_0/3 < 4 - 8t_0$, then $\{\theta_j\} \in E$ by (3.5).

$\underline{r' = 6.}$

When $\theta_0 > t_0/2$, then $\{\theta_j\} \in E$ by (3.8).

When $\theta_0 \le t_0/2$, we discuss the following three cases :

(1) $\theta_0 + \theta_5 + \theta_6 < t_0$.

Let $a_1 = \theta_0 + \theta_5 + \theta_6 < t_0, a_2 = \theta_1 + \theta_2 + \theta_3 < t_0$ by (5.1) and $k_0 = 4$ and $\sigma = \theta_4 \le \theta_3 < 1 - 20t_0/11$, then $\{\theta_j\} \in E$ by (3.7).

(2) $\theta_0 + \theta_5 + \theta_6 \ge t_0$.

If $\theta_1 > 2t_0/7$, by $\theta_0 \le t_0/2$, we have $\theta_5 \ge t_0/4$. Let $a_1 = \theta_0 + \theta_5 + \theta_6 \ge t_0, a_2 = \theta_3 + \theta_4 \ge t_0/2, a_3 = \theta_2 > t_0/4$ and $\sigma = \theta_0 > 2t_0/7$, then $\{\theta_j\} \in E$ by (3.15). If $\theta_1 \le 2t_0/7, \{\theta_j\} \in E$ by (3.17).

$\underline{r = r' = 7.}$

If $\theta_4 + \theta_5 + \theta_6 + \theta_7 \geq t_0$, we have

$$\theta_0 > 2t_0/7 \tag{5.7}$$

since $\{\theta_0, \cdots, \theta_7\} \notin D_2^*$.

Let $a_1 = \theta_4 + \theta_5 + \theta_6 + \theta_7 \geq t_0, a_2 = \theta_2 + \theta_3 \geq t_0/2, a_3 = \theta_1 > t_0/4$ and $\sigma = \theta_0 > 2t_0/7$, then $\{\theta_j\} \in E$ by (3.15).

If $\theta_4+\theta_5+\theta_6+\theta_7 < t_0$, by Lemma 4.2 and $\theta_1+\theta_2+\theta_3+\theta_4 \geq t_0, \{\theta_j\} \in E$ if $\theta_1 - \theta_7 < t_0/5$. If $\theta_1 - \theta_7 \geq t_0/5$, then $\theta_1 > 2t_0/5$, and, by (5.1) and $k_0 = 4$,

$$\theta_2 < t_0 - \theta_3 - \theta_1 < 2t_0/5,$$

i.e. $\theta_2 - \theta_7 < t_0/5$. By Lemma 4.2 again, we may suppose that $\theta_2 + \theta_3 + \theta_4 + \theta_5 \leq t_0$ then $\theta_4 + \theta_5 \leq t_0/2$, let $a_1 = \theta_1 < t_0/2, a_2 = \theta_2 + \theta_3 < t_0/2, a_3 = \theta_4 + \theta_5 < t_0/2, a_4 = \theta_6 < t_0/4$, and $a_5 = \theta_7 < t_0/4$, we have that $\{\theta_j\} \in E$ by (3.20).

$r' = 8.$

If $\theta_5+\theta_6+\theta_7+\theta_8 \geq t_0$, let $a_1 = \theta_1+\theta_2+\theta_3+\theta_4 \geq t_0, a_2 = \theta_5+\theta_6+\theta_7+\theta_8 \geq t_0$, and $\sigma = \theta_0$, then $\{\theta_j\} \in E$ by (3.9).

If $\theta_5 + \theta_6 + \theta_7 + \theta_8 < t_0$, by Lemma 4.1, $\{\theta_j\} \in E$ if $\theta_1 - \theta_8 < t_0/5$. Now may suppose $\theta_1 - \theta_8 \geq t_0/5$, then

$$\theta_3 + \cdots + \theta_8 < 1 - (\theta_0 + \theta_1 + \theta_2) \leq 1 - 5(t_0/5) < 1 - 8t_0/9;$$

let $a_1 = \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 \geq t_0, \sigma = \theta_3 > t_0/5, a_2 = 1 - a_1 - \sigma > 8t_0/9$ then $\{\theta_j\} \in E$ by (3.6).

Case 3. $k_0 = 3$.

By (5.1) and (5.2), we have that

$$\theta_1 + \theta_2 < t_0 \tag{5.8}$$

and

$$\theta_1 + \theta_2 + \theta_3 \geq t_0. \tag{5.9}$$

We discuss the following cases :

Case 3.1. $\theta_3 + \theta_4 + \theta_5 > t_0$.

By (3.2), (3.3) and (3.4), we have that, if $r \geq 5$

$$\theta_3 + \theta_4 + \theta_5 \leq 1/2;$$

and, if $r \leq 4$,

$$\theta_3 + \theta_4 + \theta_5 \leq 0.4 + t_0/5 < 1/2.$$

If $r'' + r_1 > 0$, let $\sigma = \theta_{r+r_1} < t_0/5, a_1 = \theta_3 + \theta_4 + \theta_5 \in [t_0, 1/2]$, and $a_2 = 1 - a_1 - \sigma$, then $\{\theta_j\}' \in E$ by (3.13).

If $r'' + r_1 = 0$, we have that $r' \geq 5$ since $\theta_3 + \theta_4 + \theta_5 > t_0$ and $\theta_4 \leq \theta_3 \leq t_0/2$.

If $r' = 5$, from $\{\theta_0, \theta_1, \cdots, \theta_5\} \notin D_3^*$, then $\theta_0 > 2t_0/5$. In this case we have $\theta_3 \geq t_0/3$. Let $a_1 = \theta_3 + \theta_4 + \theta_5 \geq t_0, a_2 = \theta_2 \geq t_0/3, a_3 = \theta_1 \geq t_0/3$, and $\sigma = 1 - a_1 - a_2 - a_3 = \theta_0$, then $\{\theta_j\} \in E$ by (3.22).

If $r' > 5$, $\sigma = 1 - a_1 - a_2 - a_3 > \theta_0 + \theta_6 > 2t_0/5$, then $\{\theta_j\} \in E$ by (3.16) again.

Case 3.2. $\theta_3 + \theta_4 + \theta_5 \leq t_0$.

By Lemma 4.2, we only need to discuss the cases with

$$\theta_1 - \theta_5 > t_0/5. \tag{5.10}$$

By (3.1), we have that

$$\theta_0 \geq \theta_1 > 2t_0/5. \tag{5.11}$$

We discuss the following cases :

Case 3.2.1. $\theta_2 < t_0/3$.

By Lemma 4.1, we only need to discuss those cases with $r'' + r_1 = 0$. If $r' \leq 5$, let $a_1 = \theta_0 + \theta_1 \leq t_0, \sigma = \theta_2 < 1 - 20t_0/11$, and $\sigma = \theta_3 + \theta_4 + \theta_5 < t_0$, then $\{\theta_j\} \in E$ by (3.7).

If $r' \geq 6$, by Lemma 4.2, we only need to discuss those cases with $\theta_1 + \theta_4 + \theta_5 \geq t_0$ since $\theta_1 + \theta_2 + \theta_3 \geq t_0$ and $\theta_2 - \theta_5 < t_0/5$. Let $a_1 = \theta_0 + \theta_2 + \theta_3 \geq t_0, a_2 = \theta_1 + \theta_4 + \theta_5 \geq t_0, \sigma = 1 - a_1 - a_2$, then $\{\theta_j\} \in E$ if $r' \geq 6$ by (3.9).

Case 3.2.2. $\theta_2 \geq t_0/3$ and $\theta_3 < 1 - 20t_0/11$.

Let $\sigma = \theta_0 > 2t_0/5, a_3 = \theta_2 \geq t_0/3, a_2 = \theta_1 \geq t_0/3$ and

$$a_1 = \theta_3 + \cdots + \theta_r,$$

then $\{\theta_j\} \in E$ by (3.16) if $\theta_3 + \cdots + \theta_r \geq t_0$. Now may suppose that

$$\theta_3 + \cdots + \theta_r < t_0.$$

Let $\sigma = \theta_0, a_1 = \theta_3 + \cdots + \theta_r$, and $a_2 = \theta_1 + \theta_2$, then $\{\theta_j\} \in E$ by (3.5) if $\theta_1 + \theta_2 < 4 - 8t_0$. Now may suppose that

$$\theta_1 + \theta_2 \geq 4 - 8t_0 \tag{5.12}$$

also. Let $a_1 = \theta_1 + \theta_2 \geq 4 - 8t_0 > 4t_0/5, \sigma = \theta_4$, then $\{\theta_j\} \in E$ if $t_0/3 \leq \theta_3 < 1 - 20t_0/11$ by (3.11).

By Lemma 4.1 and $\theta_3 < 1 - 20t_0/11$, we may suppose that $r'' + r_1 = 0$.

We discuss the following cases :

(1) $r' \geq 7$.

If $r' \geq 7$, let $a_1 = \theta_1 + \theta_2 + \theta_3 \geq t_0, a_2 = \theta_4 + \theta_5 + \theta_6 + \theta_7 + \cdots + \theta_r > 4(t_0/5)$ and $\sigma = \theta_0 \geq \theta_1 \geq (\theta_1 + \theta_2 + \theta_3)/3 \geq t_0/3$, then $\{\theta_j\} \in E$ by (3.10).

(2) $r' = 6$.

If $\theta_3 + \theta_4 \leq t_0/2$, in (3.20), take $a_1 = \theta_1, a_2 = \theta_2, a_3 = \theta_3 + \theta_4, a_4 = \theta_5, a_5 = \theta_6$ and $\sigma = \theta_0$, then $\{\theta_j\} \in E$. If $\theta_3 + \theta_4 > t_0/2$, let

$$\begin{aligned} a_1 &= \theta_1 + \theta_3 + \theta_4 \geq 2 - 4t_0 + t_0/2 > 8t_0/9, \\ a_2 &= \theta_0 + \theta_2 + \theta_5 \geq 4 - 8t_0 - t_0/5 \geq t_0, \end{aligned}$$

and $\sigma = \theta_6 \geq t_0/5$, then $\{\theta_j\} \in E$ by (3.6).

(3) $r' = 5$.

Let $\sigma = \theta_3, a_2 = \theta_1 + \theta_2 \geq 4 - 8t_0 > 4t_0/5$ and $a_1 = 1 - \sigma - a_2$, then $\{\theta_j\} \in E$

by (3.10) if $\theta_3 \geq t_0/3$ and by (3.26) if $\theta_1 + \theta_2 > m_{\theta_3}$. Now may suppose that $\theta_3 < t_0/3$ and

$$\theta_1 + \theta_2 \leq m_{\theta_3}.$$

By (3.27), $\theta_3 \geq t_0/4$, thus

$$\theta_0 + \theta_1 = 1 - \theta_2 - \cdots - \theta_5 \geq 1 - m_{\theta_3}/2 - 3\theta_3 > m_{\theta_3}.$$

Let $\sigma = \theta_3, a_2 = \theta_0 + \theta_1$ and $a_1 = 1 - \sigma - a_2$, then $\{\theta_j\} \in E$ by (3.26) since $\theta_0 + \theta_1 < t_0 < M_{\theta_3}$.

(4) $r' \leq 4$, let $a_1 = \theta_1 + \theta_2 < t_0$ by (5.3) $a_2 = \theta_3 + \theta_4 < 2(1 - 20t_0/11) < 4 - 8t_0$, then $\{\theta_j\} \in E$ by (2.5).

Case 3.2.3. $\theta_3 \geq 1-20t_0/11$ and $\theta_4 < 1-20t_0/11$. By Lemma 4.2, $\{\theta_j\}' \in E$ if $r'' + r_1 > 0$ and

$$a_1 = \theta_1 + \theta_2 + \theta_4 + \cdots \theta_{r+r_1-1} \geq t_0.$$

Now we discuss the following cases

(1) $r'' + r_1 = 0$.

We know that $r' \geq 3$.

When $r' = 3$, let $a_1 = \theta_1 + \theta_2 < t_0, a_2 = \theta_3 < 1/3 < 4 - 8t_0$, and $\sigma = \theta_0$, then $\{\theta_j\} \in E$ by (3.5).

When $r' = 4$, let $a_1 = \theta_0+\theta_1 < t_0, a_2 = \theta_2+\theta_3 < t_0$, and $\sigma = 1-a_1-a_2 = \theta_4 < 1 - 20t_0/11$, then $\{\theta_j\} \in E$ by (3.7).

When $r' \geq 5$,

$$\theta_1 + \theta_2 + \theta_4 > 2(1 - 20t_0/11) + t_0/5 > 8t_0/9.$$

Let $a_1 = \theta_1 + \theta_2 + \theta_4 > 8t_0/9$ and $\sigma = \theta_5 > t_0/5$, then $\{\theta_j\} \in E$ if $\theta_1 + \theta_2 + \theta_4 < M_{\theta_5}$ by (3.25). Now we may suppose that

$$\theta_1 + \theta_2 + \theta_4 \geq M_{\theta_5}$$

When $r' \geq 6$, we have that

$$\theta_0 + \theta_3 + \theta_6 > 2(1 - 20t_0/11) + t_0/5 > 8t_0/9$$

also. Let $a_1 = \theta_1 + \theta_2 + \theta_4 > t_0, a_2 = \theta_0 + \theta_3 + \theta_6 > 8t_0/9$ and $\sigma = \theta_5 > t_0/5$, then $\{\theta_j\} \in E$ by (3.6). Now we discuss those cases with $r' = 5$. When $\theta_0 + \theta_1 > 6t_0/7$ and $\theta_4 > t_0/4$, let $a_2 = \theta_0 + \theta_3, \sigma = \theta_4$ and $a_1 = 1 - a_2 - \sigma$, then $\{\theta_j\} \in E$ by (3.14). Now we may suppose that

$$\theta_0 + \theta_1 \leq 6t_0/7.$$

If $\theta_2 + \theta_3 < 4 - 8t_0$. Since $\theta_1 > 2t_0/5$ (see (5.11) above), let $\sigma = \theta_1, a_2 = \theta_2 > t_0/3, a_3 = \theta_3 > t_0/3$, and $a_1 = \theta_0 + \theta_4 + \theta_5$ if $\theta_0 + \theta_4 + \theta_5 > t_0$ by (3.6); let $\sigma = \theta_1, a_2 = \theta_2 + \theta_3$, and $a_1 = \theta_0 + \theta_4 + \theta_5$ if $\theta_0 + \theta_4 + \theta_5 \leq t_0$ by (3.5). Now may suppose that

$$\theta_2 + \theta_3 \geq 4 - 8t_0.$$

Let $a_1 = \theta_1 + \theta_2 \geq 4 - 8t_0 > 4t_0/5, \sigma = \theta_4$, then $\{\theta_j\} \in E$ by (3.11) if $t_0/3 \leq \theta_4 < 1 - 20t_0/11$. Now may suppose that

$$\theta_4 < t_0/3$$

also. If $\theta_0 > \theta_5/8 + 3t_0/8$, then $\{\theta_j\} \in E$ by (3.18). Thus $\{\theta_j\} \notin \mathrm{D}_5^*$ implies $\theta_4 < t_0/4$, we have that $\{\theta_j\} \in E$ by (3.20).

(2) $r' + r_1 > 0$ and $\theta_1 + \theta_2 + \theta_4 + \cdots + \theta_{r+r_1-1} < t_0$.

By (3.4), $r' + r_1 > 0$ implies $\theta_{r+r_1} \leq \frac{t_0}{5}$. By (3.4) $\theta_4 + \theta_{r+r_1} < \frac{t_0}{2} + \frac{t_0}{5} < 4 - 8t_0$.

Let $\sigma = \theta_0, a_2 = \theta_4 + \theta_{r+r_1}$ and $a_1 = \theta_1 + \theta_2 + \theta_4 + \cdots + \theta_{r+r_1-1} < t_0$, then $\{\theta_j\}' \in E$ by (3.5).

Case 3.2.4. $\theta_4 \geq 1 - 20t_0/11$.

If $\theta_1 > t_0/2$, by (3.1), we have that

$$\theta_0' + \theta_{r+1} + \cdots + \theta_{r+r_1} \geq \theta_1 \geq \theta_2 \geq \theta_3 \geq \theta_4,$$

and

$$\theta_0' + \sum_{j=1}^{r+r_1} \theta_j = 1.$$

Thus

$$\theta_2 + \theta_3 + \theta_4 \leq 1 - 2\theta_1 < 1 - t_0,$$

and

$$\theta_3 + \theta_4 < \frac{2}{3}(1 - t_0) < 4 - 8t_0.$$

By Lemma 4.3, we have that then $\{\theta_j\}' \in E$ if there exists a partial sum of $\{\theta_0', \theta_1, \theta_2, \theta_5, \cdots \theta_{r+r_1}\}$ belong to $(2t_0/5, t_0/2]$. Now we only need to discuss those cases with $\theta_0' \leq 2t_0/5$. By (3.1), we have that

$$\theta_2 + \theta_{r+1} + \cdots + \theta_{r+r_1} \geq 1 - 20t_0/11 + \theta_1 - \theta_0' > 2t_0/5.$$

Then there exists a partial sum of $\{\theta_2, \theta_{r+1}, \cdots, \theta_{r+r_1}\}$ belong to $(2t_0/5, t_0/2]$ since $\theta_2 < t_0/5$ and $\theta_{r+1} < t_0/10 = t_0/2 - 2t_0/5$. Thus $\{\theta_j\}' \in E$ in this case.

Now we discuss the case : $\theta_1 \leq t_0/2$.

When $\theta_3 + \theta_4 \geq 4 - 8t_0$,

Since we only need to discuss those $\{\theta_j\}'$ which corresponding $\{\theta_j\} \notin \mathbf{D}_5^*$, we may suppose $r \geq 5$. Thus

$$\theta_5 < t_0 - \theta_3 - \theta_4 < t_0/5.$$

Since we only need to discuss those $\{\theta_j\}'$ which corresponding $\{\theta_j\} \notin \mathbf{D}_5^*$, we may suppose $r \geq 6$. Let $a_1 = \theta_3 + \theta_4 + \theta_5 + \theta_6 \geq 4 - 8t_0 + t_0/5 > t_0, a_2 = \theta_2 > t_0/3, a_3 = \theta_2 > t_0/3, \sigma = 1 - a_1 - a_2 - a_3 \geq \theta_1 > 2t_0/5$, then $\{\theta_j\} \in E$ by (3.24).

When $\theta_3 + \theta_4 < 4 - 8t_0$, by Lemma 4.3, we only need to discuss those cases with

$$\theta_2 \leq \theta_1 \leq 2t_0/5,$$

$$\theta_0' \leq 2t_0/5,$$

and

$$\theta_0' + \theta_{r+1} + \cdots + \theta_{r+r_1} \leq 2t_0/5$$

since $\theta_{r+1} \leq t_0/10 = t_0/2 - 2t_0/5$. Thus

$$\theta_3 + \cdots + \theta_r = 1 - \theta_0' - \theta_1 - \theta_2 - \theta_{r+1} - \cdots - \theta_{r+r_1} > t_0,$$

and $r \geq 6$ since $\theta_3 + \theta_4 + \theta_5 < t_0$. By (3.1),

$$\theta_5 = 1 - \theta_0' - \cdots - \theta_4 - \theta_6 - \cdots - \theta_{r+r_1} < 1 - 5(1 - 20t_0/11) - t_0/10 < t_0/5.$$

Since we only need to discuss those $\{\theta_j\}'$ which corresponding $\{\theta_j\} \notin \mathbf{D}_6^*$, we may suppose that $\theta_4 + \theta_6 < t_0/2$. By Lemma 4.3 again, we have that then $\{\theta_j\} \in E$.

Case 4. $k_0 = 2$.

We have $\theta_1 \geq (\theta_1 + \theta_2)/2 > t_0/2$.

If $\theta_0' > t_0/2$, then $\{\theta_j\} \in E$ by (3.4).

Now we may suppose that $\theta_0' \leq t_0/2$, thus by $\theta_0' < \theta_1$, we have that $r_1 > 0$. Let $a_1 = \theta_1 + \theta_2, \sigma = \theta_{r+r_1}$ and $a_2 = 1 - a_1 - \sigma$, then $\{\theta_j\}' \in E$ by (3.13) if $\theta_1 + \theta_2 < 1/2$. Now may suppose that

$$\theta_1 + \theta_2 \geq 1/2.$$

By (3.2), we have that $\theta_2 < 1/3$. Let $\sigma = \theta_0, a_2 = \theta_2$ and $a_1 = \theta_1 + \theta_3 + \cdots + \theta_{r+r_1}$, then $\{\theta_j\}' \in E$ by (3.5) if $a_1 < t_0$. Now may suppose that

$$\theta_1 + \theta_3 + \cdots + \theta_{r+r_1} > t_0. \tag{5.14}$$

If $\theta_3 < 1 - 20t_0/11$, by Lemma 4.1, it is sufficient to discuss that case with

$$\theta_1 + \theta_3 + \cdots + \theta_{r+r_1-1} < t_0. \tag{5.15}$$

Let $a_1 = \theta_1 + \theta_3 + \cdots + \theta_{r+r_1-1}, a_2 = \theta_2, a_3 = \theta_{r+r_1}$ and $\sigma = \theta_0$, by (3.18) we have that $\{\theta_j\}' \in E$.

If $\theta_3 \geq 1 - 20t_0/11$, by (3.2), $\theta_1 + \theta_3 \leq 1/2$. By (3.19), $\{\theta_j\}' \in E$ if $\theta_1 + \theta_3 \geq t_0$. Now may suppose that $\theta_1 + \theta_3 < t_0$. If $r \geq 4$,

$$\begin{aligned}\theta_4 &< 1 - (\theta_0' + \theta_{r+1} + \cdots + \theta_{r+r_1}) - \theta_1 - \theta_2 - \theta_3 \\ &< 1 - 3/4 - (1 - 20t_0/11) < 1 - 20t_0/11.\end{aligned}$$

By (5.14) and Lemma 4.2, we only need to discuss those case with (5.15) again. Let $a_1 = \theta_1 + \theta_3 + \cdots + \theta_{r+r_1-1}, a_2 = \theta_2, a_3 = \theta_{r+r_1}$ and $\sigma = \theta_0$, by (3.18) we have that $\{\theta_j\}' \in E$ again.

Theorem is complete already.

§ 6. THE PROOF OF THEOREM 1.

Take $c = t_0/10$.

Define a direct product $I = \{\theta_0, \cdots, \theta_r\}$ be

$$I = \{d : d = d_0 \cdots d_r, d_i \in I_i, I_i = [x^{\theta_i}, 2x^{\theta_i}) \text{ and } p(d_i) \geq x^c 1 \leq i \leq r\}. \tag{6.1}$$

For I, we define, for $1 \leq k \leq r$,

$$I(k) = \{d : d = d_0 \cdots d_k p_{k+1} \cdots p_r, d_i \in I_i (1 \leq i \leq k), p_j \in I_j, (k+1 \leq j \leq r) \text{ and } d \in I\}. \tag{6.2}$$

In this section we will choose $\mathbf{H}'_k$ and to make it as large as possible.

First we write S'_k to be a sum of some disjoint direct product of I which is defined in (6.1). Set

$$\mathbf{H}'_{k,1} = \cup_{I \cap \mathbf{D} = \emptyset} \{I\}.$$

By Theorem 3, $H'_{k,1}$ satisfies (2.2) write

$$\mathbf{H}''_{k,1} = S'_k \setminus \mathbf{H}'_{k,1}.$$

Thus

$$\mathbf{H}''_{k,1} \subseteq \mathbf{D}^*.$$

We want to choose some "good set" from D^*. We write

$$\mathbf{D}^*_{1,1} = \{I(9) : I \in \mathbf{D}^*_1\}.$$

Then we have

$$\mathbf{D}^*_1 = \mathbf{D}^*_{1,1} \cup (\mathbf{D}'_1 \setminus \mathbf{D}''_1),$$

where $\mathbf{D}'_1$ is a collection of direct product I's with $I = I_0 \cdots I_r, r \geq 10$,

$$I = \{\theta_0, \cdots, \theta_r\},$$

where $(\theta_0, \cdots, \theta_8)$ satisfy the conditions same as in $\mathbf{D}^*_1, \theta_9 + \cdots + \theta_r \in (t_0/5, \theta_8)$, and $d \in I$ if and only if $d = d_0 \cdots d_8 p_9 \cdots p_r, d_i \in Ii$ and $p_j \in I_j$ and

$$\mathbf{D}''_1 = \mathbf{D}'_1 \setminus \mathbf{D}^*_1.$$

By Theorem 3, we have that I in $\mathbf{D}_1'$ satisfies (2.2).

By the method that we will use in § 7 (see (7.4)), we have

$$|\mathbf{D}_1''| = O\left(\frac{y}{\log^2 x}\right).$$

Thus we can replace $\mathbf{D}_1^*$ by $\mathbf{D}_{1,1}^*$ in $\mathbf{H}_k''$. Repeat it again, we might change D_1^* to

$$\mathbf{D}_1^{**} = \{d : d = d_0 p_1 \cdots p_9 \in \mathbf{D}_1^*\};$$

For $\mathbf{D}_i^*(1 \le i \le 2, \text{ or } r \le i \le 7)$, we can change $\mathbf{D}_i^*$ to $\mathbf{D}_i^{**}$ as well. We only need change 9 to 8 $(i = 2), 5(4 \le i \le 5), 4(i = 6)$ or $6(i = 7)$. For $\mathbf{D}_3^*$, we only can change it to

$$\mathbf{D}_3^{**} \cup \mathbf{D}_7^{**}.$$

Now take

$$\mathbf{H}_{10}'' = \mathbf{D}_1^*, \mathbf{H}_8^* = \mathbf{D}_2^*, \mathbf{H}_6^* = \mathbf{D}_3^* \cup \mathbf{D}_4^* \cup \mathbf{D}_5^* \cup \mathbf{D}_7^*, \mathbf{H}_5'' = \mathbf{D}_6^* \text{ and } \mathbf{H}_7'' = \mathbf{D}_7^*.$$

Otherwise

$$\mathbf{H}_k'' = \emptyset.$$

In (1.9), take $c_H = 1$, then

$$\pi(x)-\pi(x-y) = yE(x,z) - \sum_{1\le i\le 5} (f(i)-1)! \sum_{D\in\mathbf{D}_i^*}\sum_{d\in D} 1 + 4! \sum_{D\in\mathbf{D}_6^*}\sum_{d\in D} 1 + (6!-5!) \sum_{D\in\mathbf{D}_7^*}\sum_{d\in D} 1.$$

where $f(1) = 10, f(2) = 8$, and $f(3) = f(4) = f(5) = 6$.

Suppose $e_i(1 \le i \le 7)$ and e_0 be constants which satisfy :

$$(f(i)-1)! \sum_{D\in\mathbf{D}_i^*}\sum_{d\in D} 1 \le (f(i)-1)! \frac{e_i y}{\log x}, 1 \le i \le 5;$$

$$4! \sum_{D\in\mathbf{D}_6^*}\sum_{d\in D} 1 \le \frac{e_6 y}{\log x};$$

$$(6!-5!) \sum_{D\in\mathbf{D}_7^*}\sum_{d\in D} 1 \le \frac{e_7 y}{\log x},$$

and

$$e_0 = \sum_{1\le i\le 7} e_i.$$

By (1.13), we have that

$$\frac{(1-e_0)y}{\log x} < \pi(x) - \pi(x-y) < \frac{(1+e_0)y}{\log x}.$$

We now estimate $e_i (1 \le i \le 7)$. First, we estimate e_3. We have $d \in \mathbf{D}_3^*$ implies $d = d_0 p_1 \cdots p_5$ with $d_0 \ge p_1 \ge \cdots \ge p_5, x - y \le d_0 p_1 \cdots p_5 \le x, p(d_0) \ge x^{t_0/10}$, and

$$x^{\frac{2t_0}{5}} \ge d_0 \ge p_1 \ge \cdots \ge p_5 \ge x^{\frac{t_0}{5}}.$$

Define

$$\Delta_3 = \{(p_1, \cdots, p_5) : \left(\frac{x}{x^{(\frac{2t_0}{5})}}\right)^{\frac{1}{5}} \le p_1 \le x^{\frac{2t_0}{5}}, \left(\frac{x}{p_1^2}\right)^{\frac{1}{4}} \le p_2 \le p_1, \left(\frac{x}{p_2^3}\right)^{\frac{1}{3}} \le p_3 \le p_2,$$
$$\left(\frac{x}{p_3^4}\right)^{\frac{1}{2}} \le p_4 \le p_3, \left(\frac{x}{p_4^5}\right) \le p_5 \le p_4\}.$$

Then

$$|\mathbf{D}_3^*| \le \sum_{(p_1,\cdots,p_5)\in\Delta_3} \frac{2y}{p_1p_2p_3p_4p_5 \log \frac{y}{p_1p_2p_3p_4p_5}}$$
$$\le \frac{2y}{\log x} \int \cdots \int_{\Delta'} \frac{dt_1dt_2dt_3dt_4dt_5}{t_1t_2t_3t_4t_5(1-t_1-t_2-t_3-t_4-t_5)}, \qquad (6.2)$$

where

$$\Delta' = \{(t_1, \cdots, t_5) : \frac{1-\frac{2t_0}{5}}{5} \le t_1 \le \frac{2t_0}{5}, \frac{1-2t_1}{4} \le t_2 \le t_1, \frac{1-3t_2}{3} \le t_3 \le t_2,$$
$$\frac{1-4t_3}{2} \le t_4 \le t_3, 1-5t_4 \le t_5 \le t_4\}.$$

Estimate the integration of right hand-side of (6.2) (see Appendix), we have

$$5! \, |\mathbf{D}_3^*| \le \frac{e_3 y}{\log x},$$

where

$$e_3 < 0.00625.$$

Define :

Δ^1 is a set of $(t_1, \cdots t_9)$ with the following conditions :

(1) $\frac{1}{2}\left(\frac{8t_0}{9} - \frac{t_0}{5}\right) \geq \theta_1 \geq \cdots \geq \theta_9 \geq \frac{t_0}{5}$;

(2) $1 - \theta_1 - \cdots - \theta_9 \geq \frac{1}{10}$;

Δ^2 is a set of $(t_1, \cdots, t_7)$ with the following conditions:

(1) $\frac{2}{7}t_0 \geq \theta_1 \geq \cdots \geq \theta_4 \geq \frac{t_0}{4}$;

(2) $\theta_4 \geq \cdots \geq \theta_7 \geq \frac{t_0}{5}$;

(3) $1 - \theta_1 - \cdots - \theta_7 \geq \frac{1}{8}$;

Δ^4 is a set of $(t_1, \cdots, t_5)$ with the following conditions :

(1) $\frac{3}{7}t_0 \geq t_1 \geq 2 - 4t_0$;

(2) $t_1 \geq t_2 \geq 2 - 4t_0$;

(3) $t_2 \geq t_3 \geq max\left\{1 - \frac{20t_0}{11}, 4 - 8t_0 - t_2\right\}$;

(4) $min\left\{\frac{t_0}{3}, \frac{9t_0}{10} - 2t_1 - t_2 - t_3\right\} \geq t_4 \geq max\left\{\frac{t_0}{4}, \frac{1}{2}\left(1 - \frac{6t_0}{7} - t_2 - t_3\right)\right\}$

(5) $min\{t_4, 1 - 2t_1 - t_2 - t_3 - t_4\} \geq t_5 \geq max\left\{1 - \frac{6t_0}{7} - t_2 - t_3 - t_4\right\}$;

(6) $t_1 \leq t_5/8 - 3t_0/8$;

Δ^5 is a set of $(t_1, \cdots, t_5)$ with the following conditions :

(1) $1 - \frac{3}{2}(4 - 8t_0) - \frac{t_0}{10} \geq t_1 \geq 2 - 4t_0$;

(2) $min\left\{1 - 4 + 8t_0 - \frac{t_0}{10} - 2t_1, t_1\right\} \geq t_2 \geq 2 - 4t_0$;

(3) $min\left\{1 - \left(1 - \frac{20t_0}{11}\right) - \frac{t_0}{10} - 2t_1 - t_2, t_2\right\} \geq t_3 \geq 2 - 4t_0$;

(4) $min\left\{1 - \frac{t_0}{10} - 2t_1 - t_2 - t_3, t_3\right\} \geq t_4 \geq max\left\{4 - 8t_0 - t_3, 1 - \frac{20t_0}{11}\right\}$;

(5) $min\left\{1 - 2t_1 - t_2 - t_3 - t_4, \frac{t_0}{5}\right\} \geq t_5 \geq \frac{t_0}{10}$;

Δ^6 is a set of $(t_1, \cdots, t_4)$ with the following conditions :

(1) $\frac{t_0}{3} \geq \theta_1 \geq 2 - 4t_0$;

(2) $min\{\theta_1, 1 - (4 - 8t_0) - 2\theta_1\} \geq \theta_2 \geq 2 - 4t_0$;

(3) $min\left\{\theta_2, 1 - \left(1 - \frac{20t_0}{11}\right) - 2\theta_1 - \theta_2\right\} \geq \theta_3 \geq 2 - 4t_0$;

(4) $min\{\theta_3, 1 - 2\theta_1 - \theta_2 - \theta_3\} \geq \theta_4 \geq max\left\{1 - \frac{20t_0}{11}, 4 - 8t_0 - \theta_3\right\}$;

Δ^7 is a set of $(t_1, \cdots, t_6)$ with the following conditions :

(1) $\frac{2t_0}{5} \geq t_1 \geq t_2 \geq t_3 \geq t_4 \geq 1 - \frac{20t_0}{11}$;

(2) $min\left\{1 - \frac{t_0}{2} - 2t_1 - t_2 - t_3, \frac{t_0}{5}\right\} \geq t_5 \geq \frac{t_0}{2} - t_4$;

(3) $min\{1 - 2t_1 - t_2 - t_3 - t_4 - t_5, t_5\} \geq t_6 \geq \frac{t_0}{2} - t_4$;

With same reason we have

$$e_i < (f(i) - 1)!(2) \int \cdots \int_{\Delta^i} \frac{dt_1 \cdots dt_r}{t_1 \cdots t_r(1 - t_1 - \cdots - t_r)},$$

where $r = r(i), r(1) = 9, r(2) = 7, r(3) = r(5) = r(6) = 5$ and $r(7) = 6$. We have that (see Appendix)

$$e_1 \leq 2\left(\ln \frac{\frac{1}{2}\left(\frac{8t_0}{9} - \frac{2t_0}{5}\right)}{\frac{t_0}{5}}\right)^9 (10) < 1.1(10)^{-5};$$

$$e_2 < 2\left(\left(\ln \frac{8}{7}\right)^7 (8) + \left(\ln \frac{8}{7}\right)^4 \left(\ln \frac{5}{4}\right)^3 \frac{7!}{3!(4!)}\right) < 0.002.$$

$$e_4 < 0.00276;$$

$$e_5 < 0.007817;$$

$$e_6 < 0.01351$$

and

$$e_7 < 0.0002071.$$

In Theorem C, take

$$e_1'' = e_2'' = \sum_{i=1}^{5} e_i \leq 0.0171.$$

and

$$\epsilon_1' = \epsilon_2' = e_6 + e_7 < 0.01372$$

then we have that

$$\frac{0.969y}{\log x} < \pi(x) - \pi(x-y) < \frac{1.031y}{\log x}.$$

Theorem 3 follows.

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APPENDIX

Estimate of e_1 :

Define

$$\Delta = \left\{(t_1,\cdots,t_9) : \frac{1}{2}\left(\frac{4t_0}{9} - \frac{2t_0}{5}\right) \geq t_1 \geq \cdots \geq t_9 \geq \frac{t_0}{5}\right\}.$$

Then

$$e_1 \leq \frac{9!(2)}{9!} \int \cdots \int_\Delta \frac{dt_1 \cdots dt_9}{t_1 \cdots t_9(1 - t_1 - \cdots - t_9)} \leq 20\left(\ln\left(\frac{\frac{1}{2}\left(\frac{40}{99} - \frac{2}{11}\right)}{\frac{1}{11}}\right)\right)^9 < 1.1(10)^{-5}.$$

Estimate of e_2 :

Define

$$\Delta_2 = \left\{(t_1,\cdots,t_7) : \frac{2t_0}{7} \geq t_1 \geq \cdots \geq t_7 \geq \frac{t_0}{5}\right\}.$$

Then

$$\begin{aligned} e_2 &\leq 2(7!) \int \cdots \int_{\Delta_2} \frac{dt_1 \cdots dt_7}{t_1 \cdots t_7(1 - t_1 - \cdots - t_7)} \\ &\leq 7!(2)\left(\left(\ln \tfrac{8}{7}\right)^7 \tfrac{8}{7!} + \tfrac{7!}{3!4!}\left(\ln \tfrac{8}{7}\right)^4 \left(\ln \tfrac{5}{4}\right)^3\right) < 2.6(10)^{-4}. \end{aligned}$$

Estimate of e_3 :

Define

$$\Delta_3 = \{(t_1,\cdots,t_5) : \frac{1 - \frac{2t_0}{5}}{5} \leq t_1 \leq \frac{2t_0}{5}, \frac{1 - 2t_1}{4} \leq t_2 \leq t_1, \frac{1 - 3t_2}{3} \leq t_3 \leq t_2,$$

$$\frac{1 - 4t_3}{2} \leq t_4 \leq t_3, 1 - 5t_4 \leq t_5 \leq t_4\}.$$

Then

$$\begin{aligned}
e_3 &\le 5!(2)(6) \int \cdots \int_{\Delta_3} \frac{dt_1 \cdots dt_5}{t_1 \cdots t_5} \\
&\le 1440 \int_{\frac{1.8}{11}}^{\frac{2}{11}} dt_1 \int_{\frac{9/11-t_1}{4}}^{t_1} dt_2 \int_{\frac{9/11-t_1-t_2}{3}}^{t_2} dt_3 \int_{\frac{9/11-t_1-t_2-t_3}{2}}^{t_3} dt_4 \int_{9/11-t_1-t_2-t_3-t_4}^{t_4} \frac{dt_5}{t_1 \cdots t_5} \\
&\le 1440 \int_{\frac{1}{5}}^{\frac{2}{9}} da_1 \int_{\frac{1}{4}}^{\frac{a_1}{1-a_1}} da_2 \int_{\frac{1}{3}}^{\frac{a_2}{1-a_2}} da_3 \int_{\frac{1}{2}}^{\frac{a_3}{1-a_3}} \frac{1}{a_1 a_2 a_3 a_4} \ln \frac{a_4}{1-a_4}\, da_4 \\
&\le 1440 \int_{\frac{1}{5}}^{\frac{2}{9}} da_1 \int_{\frac{1}{4}}^{\frac{a_1}{1-a_1}} da_2 \int_{\frac{1}{3}}^{\frac{a_2}{1-a_2}} da_3 \int_{\frac{1}{2}}^{\frac{a_3}{1-a_3}} \frac{1}{a_1 a_2 a_3 a_4} \left(\frac{a_4}{1-a_4} - 1\right) da_4 \\
&\le 1440 \int_{\frac{1}{5}}^{\frac{2}{9}} da_1 \int_{\frac{1}{4}}^{\frac{a_1}{1-a_1}} da_2 \int_{\frac{1}{3}}^{\frac{a_2}{1-a_2}} \frac{1}{a_1 a_2 a_3} \left(\frac{1}{\left(\frac{a_3}{1-a_3}\right)\left(1-\frac{a_3}{1-a_3}\right)}\right) \frac{1}{4} \left(\frac{2a_3}{1-a_3} - 1\right)^2 da_3 \\
&\le 360 \int_{\frac{1}{5}}^{\frac{2}{9}} da_1 \int_{\frac{1}{4}}^{\frac{a_1}{1-a_1}} \frac{1}{a_1 a_2} \left(\frac{1}{\left(\frac{a_2}{1-a_2}\right)\left(1-\frac{a_2}{1-a_2}\right)}\right) \left(\frac{1}{9}\right) \left(\frac{3a_2}{1-a_2} - 1\right) da_2 \\
&\le 40 \int_{\frac{1}{5}}^{\frac{2}{9}} \frac{1}{a_1} \left(\frac{a_1}{1-a_1}\right)^{-3} \left(1 - \frac{3a_1}{1-a_1}\right)^{-1} \left(\frac{1}{16}\right) \left(\frac{4a_1}{1-a_1} - 1\right)^4 da_1 \\
&\le \left(\tfrac{5}{2}\right)\left(\tfrac{9}{2}\right)^4\left(\tfrac{9}{25}\right)\left(\tfrac{1}{9}\right)^5 = 0.00625.
\end{aligned}$$

Estimation of e_4.

We have, for $\{\theta_j\} \in \mathbf{D}_4^*$

$$1 = \theta_0 + \cdots + \theta_5 \le 4\left(\frac{3t_0}{8} + \frac{\theta_5}{8}\right) + \theta_4 + \theta_5.$$

Thus

$$\theta_4 \ge \frac{2}{5} - \frac{3t_0}{5},$$

and

$$\theta_4 + \theta_5 \ge \frac{4}{5} - \frac{6t_0}{5}.$$

Moreover

$$\theta_0 \le 1 - 2\left(\frac{2t_0}{5} - \frac{3t_0}{5}\right) - 3(2 - 4t_0) = \frac{66t_0}{5} - \frac{29}{5} < \frac{1}{5},$$

and

$$\theta_1 \le \frac{1}{2}\left(1 - 2\left(\frac{2}{5} - \frac{3t_0}{5}\right) - 2(2 - 4t_0)\right) \le \frac{23t_0}{5} - \frac{19}{10} < \frac{21}{110}.$$

Then

$$\begin{aligned} e_4 &\le (5!)(5)(2) \int_{\frac{2}{11}}^{\frac{21}{110}} d\theta_1 \int_{\frac{2}{11}}^{\theta_1} d\theta_2 \int_{\frac{21}{121}}^{\theta_2} d\theta_3 \int_{\frac{7}{56}}^{\frac{5}{33}} d\theta_4 \int_{\frac{7}{33}-\frac{2\theta_4}{3}}^{\theta_4} \frac{d\theta_5}{\theta_1 \cdots \theta_5} \\ &\le 1200 \int_{\frac{2}{11}}^{\frac{21}{110}} d\theta_1 \int_{\frac{2}{11}}^{\theta_1} d\theta_2 \int_{\frac{21}{121}}^{\theta_2} d\theta_3 \int_{\frac{7}{55}}^{\frac{5}{33}} \frac{\left(\frac{5\theta_4}{3} - \frac{7}{33}\right) d\theta_4}{\left(\frac{7}{33} - \frac{2\theta_4}{3}\right)\theta_4} \\ &\le 1200 \left(\tfrac{9}{275}\right) \left((5.5)^3 \frac{1}{2(3)} \left(\tfrac{21}{110} - \tfrac{2}{11}\right)^3 + \left(\ln \frac{2(121)}{11(21)}\right) \frac{(5.5)^2}{2} \left(\tfrac{1}{110}\right)^2 \right) \\ &< 0.00276. \end{aligned}$$

Estimation of e_5

$$\mathbf{D}_5^* = \{(\theta_0, \cdots, \theta_5) : t_0/2 \ge \theta_0 \ge \theta_1 \ge \cdots \ge \theta_3 \ge 1 - 20t_0/11, \theta_3 + \theta_4 \ge 4 - 8t_0,$$

$$t_0/5 \ge \theta_5 \ge t_0/10, \theta_0 + \theta_5 \le t_0/2\}.$$

We have that

$$\theta_0 < 5/22 - \theta_5/2 < 9/44,$$

and

$$1/22 < \theta_5 < 5/11 - 2\theta_0 \le 5/11 - 2\theta_1.$$

Thus

$$e_5 \le (5!)(2)(5.5) \int \cdots \int_{\Delta_5} \frac{dt_1 \cdots dt_5}{t_1 \cdots t_5}$$

where

$$\begin{aligned} \Delta_5 = \{ & (t_1, \cdots, t_5) : \tfrac{2}{11} \le t_1 \le \tfrac{9}{44}, \tfrac{2}{11} \le t_2 \le \min\left\{\tfrac{13}{22} - 2t_1, t_1\right\}, \\ & \tfrac{2}{11} \le t_3 \le \min\left\{t_2, \tfrac{189}{242} - 2t_1 - t_2\right\}, \max\left\{\tfrac{4}{11} - t_3, \tfrac{21}{121}\right\} \le t_4 \le t_3, \\ & \tfrac{1}{22} \le t_5 \le \min\left\{\tfrac{5}{11} - 2t_1, \tfrac{1}{11}\right\}\}. \end{aligned}$$

We have

$$
\begin{aligned}
e_5 &\leq (5!)(2)(5.5)\int_{\frac{2}{11}}^{\frac{9}{44}} dt_1 \int_{\frac{2}{11}}^{min\left\{\frac{13}{22}-2t_1, t_1\right\}} dt_2 \int_{\frac{2}{11}}^{t_3} dt_3 \int_{\frac{21}{121}}^{t_3} \frac{\mathrm{In}(10-44t_1)}{t_1 t_2 t_3 t_4} dt_4 \\
&\leq 1320 \int_{\frac{2}{11}}^{\frac{9}{44}} dt_1 \int_{\frac{2}{11}}^{min\left\{\frac{13}{22}-2t_1, t_1\right\}} dt_2 \int_{\frac{2}{11}}^{t_2} \frac{1}{t_1 t_2 t_3}\left((5.5)\left(t_3-\frac{2}{11}\right)+\mathrm{In}\frac{22}{21}\right)(9-44t_1)dt_3 \\
&\leq 1320(5.5)^3 \int_{\frac{2}{11}}^{\frac{9}{44}} dt_1 \int_{\frac{2}{11}}^{min\left\{\frac{13}{22}-2t_1, t_1\right\}} \left(\frac{5.5}{2}\left(t_2-\frac{2}{11}\right)^2+\left(\mathrm{In}\frac{22}{21}\right)\left(t_2-\frac{2}{11}\right)\right)(9-44t_1)dt_2 \\
&\leq 1320(5.5)^3\left[\int_{\frac{2}{11}}^{\frac{13}{66}}\left(\frac{5.5}{6}\left(t_1-\frac{2}{11}\right)^3+\frac{\mathrm{In}\frac{22}{21}}{2}\left(t_1-\frac{2}{11}\right)^2\right)(9-44t_1)dt_1\right. \\
&+ \left.\int_{\frac{13}{66}}^{\frac{9}{44}}\left(\frac{5.5}{6}\left(\frac{9}{22}-2t_1\right)^3+\frac{\mathrm{In}\frac{22}{21}}{2}\left(\frac{9}{22}-2t_1\right)^2\right)(9-44t_1)dt_1\right] \\
&\leq 219625\left[\frac{5.5}{6}\left(\frac{1}{12}\left(\frac{1}{66}\right)^4+\frac{11}{5}\left(\frac{1}{66}\right)^5\right)+\frac{\mathrm{In}\frac{22}{21}}{2}\left(\frac{1}{9}\left(\frac{1}{66}\right)^3+\frac{11}{3}\left(\frac{1}{66}\right)^4\right)\right. \\
&+ \left.\frac{5.5}{6}\left(\frac{1}{24}\left(\frac{1}{66}\right)^4+\frac{44}{10}\left(\frac{1}{66}\right)^5\right)+\frac{\mathrm{In}\frac{22}{21}}{2}\left(\frac{1}{18}\left(\frac{1}{66}\right)^3+\frac{44}{8}\left(\frac{1}{66}\right)^4\right)\right] \\
&\leq 0.007817.
\end{aligned}
$$

We estimate e_6 now.

We have

$$\mathbf{D}_6^* = \left\{(\theta_0, \cdots, \theta_4) : \frac{t_0}{2} \geq \theta_0 \geq \cdots \geq \theta_4 \geq 1 - \frac{20}{11}t_0, \theta_3 + \theta_4 \geq 4 - 8t_0\right\}.$$

Then $1 - \theta_1 - \cdots - \theta_4 = \theta_0 \geq 1/5$, and

$$e_6 \leq (4!)(2)(5) \int \cdots \int_{\Delta_6} \frac{dt_1 dt_2 dt_3 dt_4}{t_1 t_2 t_3 t_4},$$

where

$$\Delta_6 = \{(t_1, \cdots, t_4) : 5/22 \geq t_1 \geq \cdots \geq t_4 \geq 21/121, t_3 + t_4 \geq 4/11, 2t_1 + t_2 + t_3 + t_4 \leq 1\}.$$

Thus

$$\begin{aligned}
e_6 &\le 240\int_{\frac{17}{88}}^{\frac{5}{22}} dt_1 \int_{\frac{17}{66}-\frac{t_1}{3}}^{min\{\frac{7}{11}-2t_1,t_1\}} dt_2 \int_{\frac{2}{11}}^{t_2} dt_3 \int_{\frac{21}{121}}^{t_3} \frac{dt_4}{t_1t_2t_3t_4} \\
&\le 240\int_{\frac{17}{88}}^{\frac{5}{22}} dt_1 \int_{\frac{17}{66}-\frac{t_1}{3}}^{min\{\frac{7}{11}-2t_1,t_1\}} dt_2 \int_{\frac{2}{11}}^{t_3} \frac{1}{t_1t_2t_3}\left((\text{In}\frac{11t_3}{2}) + \text{In}\frac{22}{21}\right) dt_3 \\
&\le 240(5.5)\int_{\frac{17}{88}}^{\frac{5}{22}} dt_1 \int_{\frac{17}{66}-\frac{t_1}{3}}^{min\{\frac{7}{11}-2t_1,t_1\}} dt_2 \int_{\frac{2}{11}}^{t_2} \frac{1}{t_1t_2t_3}\left(\frac{11t_3-2}{2} + \text{In}\frac{22}{21}\right) dt_3 \\
&\le 240(5.5)\int_{\frac{17}{88}.}^{\frac{5}{22}} dt_1 \int_{\frac{17}{66}-\frac{t_1}{3}}^{min\{\frac{7}{11}-2t_1,t_1\}} \frac{1}{t_1t_2}\left(\frac{5.5}{2}\left(t_2-\frac{2}{11}\right)^2 + \left(\text{In}\frac{22}{21}\right)\left(t_2-\frac{2}{11}\right)\right) dt_2 \\
&\le 1320\int_{\frac{7}{33}}^{\frac{5}{22}} dt_1 \int_{\frac{17}{66}-\frac{t_1}{3}}^{\frac{7}{11}-2t_1} \frac{1}{t_1\left(\frac{17}{66}-\frac{t_1}{3}\right)}\left(\frac{5.5}{2}\left(t_2-\frac{2}{11}\right)^2 + \left(\text{In}\frac{22}{21}\right)\left(t_2-\frac{2}{11}\right)\right) dt_2 \\
&\quad +1320\int_{\frac{17}{88}}^{\frac{7}{33}} dt_1 \int_{\frac{17}{66}-\frac{t_1}{3}}^{\frac{7}{11}-2t_1} \frac{1}{t_1\left(\frac{17}{66}-\frac{t_1}{3}\right)}\left(\frac{5.5}{2}\left(t_2-\frac{2}{11}\right)^2 + \left(\text{In}\frac{22}{21}\right)\left(t_2-\frac{2}{11}\right)\right) dt_2 \\
&\le 1320\left(\frac{22}{5}\right)\left(\frac{66}{12}\right)\int_{\frac{7}{33}}^{\frac{5}{22}} [\frac{5.5}{6}\left(\frac{5}{11}-2t_1\right)^3 - \frac{5.5}{6}\left(\frac{5}{66}-\frac{t_1}{3}\right)^3 \\
&\quad +\frac{1}{2}\left(\frac{5}{11}-2t_1\right)^2\left(\text{In}\frac{22}{21}\right) - \frac{\text{In}\frac{22}{21}}{2}\left(\frac{5}{66}-\frac{t_1}{3}\right)]dt_1 \\
&+ 1320\left(\frac{178}{37}\right)\left(\frac{33}{7}\right)\int_{\frac{17}{88}}^{\frac{7}{33}}\left(\frac{5.5}{6}\left(t_1-\frac{2}{11}\right)^3 - \frac{5.5}{6}\left(\frac{5}{66}-\frac{t_1}{3}\right)^3 + \left(\text{In}\frac{22}{21}\right)\frac{(t_1-\frac{2}{11})^2}{2} - \frac{\text{In}\frac{22}{21}}{2}\left(\frac{5}{66}-\frac{t_1}{3}\right)^2\right) dt_1 \\
&\le 31944\left(\frac{5.5}{48}\left(\frac{1}{33}\right)^4 - \frac{5.5}{6}\left(\frac{3}{4}\right)\left(\frac{1}{198}\right)^4 + \frac{1}{12}\left(\text{In}\frac{22}{21}\right)\left(\frac{1}{33}\right)^3 - \frac{\text{In}\frac{22}{21}}{2}\left(\frac{1}{198}\right)^3\right) \\
&+ 29937[\frac{5.5}{24}\left(\frac{1}{33}\right)^4 - \frac{5.5}{24}\left(\frac{1}{88}\right)^4 + \frac{16.5}{24}\left(\frac{1}{198}\right)^4 - \frac{16.5}{24}\left(\frac{1}{88}\right)^4 + \frac{\text{In}\frac{22}{21}}{6}\left(\frac{1}{33}\right)^3 - \frac{\text{In}\frac{22}{21}}{6}\left(\frac{1}{88}\right)^3 \\
&+ \frac{\text{In}\frac{22}{21}}{2}\left(\frac{1}{198}\right)^3 - \frac{\text{In}\frac{22}{21}}{2}\left(\frac{1}{88}\right)^3] < 0.01351.
\end{aligned}$$

Finally, we estimate e_7. We have

$$e_7 \le (5!)(5)(2)\left(\frac{121}{21}\right)\int\cdots\int_{\Delta_7}\frac{dt_1\cdots dt_6}{t_1\cdots t_6}$$

where

$$\Delta_7 = \{(t_1,\cdots,t_6) : 2/11 \ge t_1 \ge \cdots \ge t_4 \ge 21/121, 1/11 \ge t_5 \ge t_6 \ge 1/22\}.$$

Thus

$$e_7 \le \frac{1}{4!}(\text{In}\frac{22}{21})^4(1-\text{In}2) < 0.0002071.$$

EDITORIAL NOTE. Editors came to know (by private communication) that Theorem A of this paper was proved independently by Professor D.R. HEATH-BROWN long ago. He had a lot of unpublished material dating back to 1983 regarding Theorem A.