Hardy-Ramanujan Journal
Vol.16 (1993) 21-43

THE NUMBER OF PRIMES IN A SHORT INTERVAL

SHITUO LOU AND QI YAO

§ 1. INTRODUCTION.

Let x be a sufficient large number.

We shall investigate the number of primes in the interval $(x-y,x]$ for $y=x^{\theta}$ with $1/2<\theta\leq 7/12$. Hoheisel [1] was the first to give a value of $\theta<1$ such that

$$\pi(x)-\pi(x-y)\sim\frac{y}{\log x}, y=x^{\theta}. \quad (1.1)$$

Ingham [2] connected the problem with zero density estimates for $\zeta(s)$, and Montgomery [3] showed how a method of Halász could be used to estimate $N(\sigma,T)$ (the number of zeros of $\zeta(s)$ in the range Re $s\geq\sigma, 0<Ims\leq T$). Huxley [4] proved that for

$$\frac{7}{12}<\theta\leq 1$$

(1.1) holds. His work built on foundations laid by the authors mentioned above.

Heath-Brown [5] has given an alternative proof of Huxley's result : Heath-Brown has actually proved more namely

THEOREM A [5] *Let $\varepsilon(x)\leq 1/12$ be a non-negative function of x. Then*

$$\pi(x)-\pi(x-y)=\frac{y}{\log x}\left\{1+O(\varepsilon^{4}(x))+O\left(\left(\frac{\log\log x}{\log x}\right)^{4}\right)\right\} \quad (1.2)$$

uniformly for

$$x^{7/12-\varepsilon(x)} \leq y \leq \frac{x}{(log\ x)^4}.$$

Thus (1.1) holds for such y, providing only that $\varepsilon(x) \to 0$ as $x \to \infty$. Moreover, he obtained

$$\pi(x) - \pi(x - x^{7/12}) = \frac{x^{7/12}}{log\ x}\left\{1 + O\left(\left(\frac{loglog\ x}{log\ x}\right)^4\right)\right\}. \qquad (1.3)$$

In [5], Heath-Brown has shown :

THEOREM B. *Let*

$$\sum(z) = \sum_{\substack{x-y<p_1\cdots p_6\leq x \\ p_i\geq z, i=1,\cdots,6}} 1 \qquad (1.4)$$

and

$$E(x,z) = \frac{1}{log\ x} + \frac{1}{6}\int\cdots\int_{\substack{t_i\geq z \\ t_1t_2t_3t_4t_5\geq x/z}} \left(log\ \frac{x}{t_1t_2t_3t_4t_5}\right)^{-1}$$

$$\frac{dt_1dt_2dt_3dt_4dt_5}{t_1t_2t_3t_4t_5\ log\ t_1\ log\ t_2\ log\ t_3\ log\ t_4\ log\ t_5} \qquad (1.5)$$

is independent of y, where z may take any value in the range

$$x^{1/7} < z \leq x^{1/6}exp(-(log\ x)^{43/44}), y \geq x^{7/12};$$

$$x^{1/7} < z \leq y^{50}x^{-29}exp(-(log\ x)^{43/44}), y < x^{7/12}.$$

Then

$$\pi(x) - \pi(x-y) = yE(x,z) - \frac{1}{6}\sum(z) + O(y\ exp(-(log\ x)^{1/7}) \qquad (1.6)$$

uniformly for

$$x^{7/12-1/6000} \leq y \leq x\ exp(-(log\ x)^{1/6}). \qquad (1.7)$$

In this paper, we shall give a generalization of Theorem B in § 2. Let the interval $I^y = (x-y, x]$ with

$$x^{1/2} < y \leq \frac{1}{2}x$$

and the parameter z satisfying

$$x^{1/k_0} < z \le x^{1/5}$$

where k_0 is a positive integer that will be chosen later. For example, with $y = x^{\theta}$, we shall choose $k_0 = 11$ if $\theta = 11/20 + \varepsilon$.

Denote $p(d_i)$ the smallest prime factor of d_i. We write

$$S_k := \{d_1 \cdots d_k : d_1 \cdots d_k = d \in |^y, p(d_i) \ge z, 1 \le i \le k\}.$$

$d_1 \cdots d_r = d'_1 \cdots d'_j \in S_k$ if and only if $r = j$ and $d_i = d'_i$ for $1 \le i \le r$.

Let

$$\sum_{n \in |^y} a_n(k) = \sum_{\substack{d_1 \cdots d_k = n \\ p(d_i) \ge z, 1 \le i \le k \\ n \in |^y}} 1 = \sum_{s \in S_k} 1$$

Let r be a positive integer, $I_j, 1 \le j \le r$, be a set of integers, and $I_j \subseteq [2, x]$ and H be the "Direct Product" of sets I_j, for $1 \le j \le r$, it means

$$d \in H \text{ if and only if } d = d_1 \cdots d_r \text{ with } d_j \in I_j, 1 \le j \le r, \text{ and } d \in |^y. \quad (1.8)$$

Suppose θ be fixed in the interval $(1/2, 1)$ and $y \in [x^{\theta}, x\ exp(-log\ x)^{1/6})]$. Define the conditions (A_1) and (A_2) as following :

(A_1). If there exist some sets $\mathbf{H}_k, 1 \le k < k_0$, which are collections of direct products H's and constants c_H such that

$$\sum_{n \in |^y} a_n(k) = \sum_{H \in \mathbf{H}_k} c_H \sum_{d \in H} 1 + O\left(\frac{y}{log^2 x}\right), \quad (1.9)$$

then we call $\mathbf{H}_k, 1 \le k < k_0$, satisfy (A_1).

(A_2). If $\mathbf{H}_k, 1 \le k < k_0$, satisfy (A_1), There exists a subset H'_k and for each $H \in H'_k$ there exists a function $E_k(H, z)$ independent of y such that

$$\sum_{d \in H} 1 = yE_k(H, z) + O(y\ exp(-(log\ x)^{1/7})), \quad (1.10)$$

uniformly for

$$x^{\theta} \le y \le x\ exp(-(log\ x)^{1/6}),$$

then we call $\mathbf{H}'_k, 1 \le k < k_0$, satisfy (A_2).

We now state our Theorem here :

THEOREM 1. *Let x be a sufficient large number, θ be fixed in $(1/2,1), x^\theta \le y < (1/2)x$, $|^y = (x-y,x], k_0$ be an integer which is dependent on θ, and z be fixed in $(x^{1/k_0}, x^{1/5}]$. Let $\mathbf{H}_k, 1 \le k < k_0$, such that (A_1). If there exists a subset $\mathbf{H}'_k$ of $\mathbf{H}_k$ such that (A_2), and writing $\mathbf{H}''_k = \mathbf{H}_k \setminus \mathbf{H}'_k$, then we have*

$$\pi(x) - \pi(x-y) = yE(x,z) + R(y) + O(y \exp(-(log\ x)^{1/7}) \tag{1.11}$$

uniformly for

$$x^\theta \le y \le x\ exp(-(log\ x)^{1/6}),$$

where $E(x,z)$ independent of y, and

$$R(y) = \sum_{1 \le k \le k_0} (-1)^{k-1} k^{-1} \sum_{H \in \mathbf{H}''_k} c_H \sum_{d \in H} 1. \tag{1.12}$$

We call $\mathbf{H}'_k$ a 'good set' and call $\mathbf{H}''_k$ a 'bad set', for $1 \le k < k_0$. Heath-Brown [5] prove that

$$\pi(x) - \pi(x-y) = \sum_{1 \le k < k_0} (-1)^{k-1} k^{-1} \sum_{s \in S_k} 1 + O(yx^{-\frac{1}{3}}) \tag{1.13}$$

Comparing (1.13) and (1.6) with (1.5), Heath-Brown took $k_0 = 7, S_1, \cdots S_5$ as good sets and only S_6 as a bad set i.e. $\mathbf{H}'_1 = S_1, \cdots, \mathbf{H}'_5 = S_5, \mathbf{H}'_6 = \oslash$; and $\mathbf{H}''_1 = \cdots = \mathbf{H}''_5 = \oslash, \mathbf{H}''_6 = S_6$. In Theorem 1, we are not limited that the good set or that the bad set should to be whole of S_k. In fact, $R(y)$ is the contribution of all bad sets. He proved that the contribution of his bad sets is $\sum(z)$ in (1.5). Heath-Brown applied Theorem B to improve (1.3). He obtained that if x is sufficient large,

$$\pi(x) - \pi(x-y) \ge \frac{4y}{5\ log\ x}, \tag{1.14}$$

where

$$x^{\frac{7}{12} - \frac{1}{6000}} \le y \le x. \tag{1.15}$$

In § 2 we shall prove Theorem 1. In § 2, we shall prove the following theorem also :

THEOREM 2. *Suppose that θ is fixed in $(1/2, 1)$, $y_0 = x\ exp(-(log\ x)^{1/6})$, $\mathbf{H}_k$, $1 \leq k < k_0$, satisfy (A_1) and (A_2). If there exist constants e_1, e_1', e_2 and e_2' such that*

$$\frac{(-e_1' + \varepsilon)y_0}{log\ x} < \sum_{1 \leq k < k_0} (-1)^{k-1} k^{-1} R_k(y_0) < \frac{(e_1 - \varepsilon)y_0}{log\ x} \tag{1.16}$$

and

$$\frac{(-e_2' + \varepsilon)y}{log\ x} < \sum_{1 \leq k < k_0} (-1)^{k-1} k^{-1} R_k(y) < \frac{(e_2 - \varepsilon)y}{log\ x} \tag{1.17}$$

where ε is a small positive constant. Then

$$\frac{(1 - e_1 - e_2')y}{log\ x} < \pi(x) - \pi(x - y) < \frac{(1 + e_1' + e_2)y}{log\ x} \tag{1.18}$$

uniformly for $x^\theta \leq y \leq y_0$.

Take an applicable form $\mathbf{H}_k$ with condition (A_1), which makes it possible to extend the range of validity of

$$(1 - c)\frac{y}{log\ x} < \pi(x) - \pi(x - y) < (1 + c')\frac{y}{log\ x}, \tag{1.19}$$

where c and c' are constants. In this paper, we prove that (1.19) holds with $y = x^\theta, \theta = 11/20 + \varepsilon$ and $c = c' = 0.01$ in § 6.

In [7], we gave some sufficient conditions that imply some kind of "direct product" be "good set". In § 3 and § 4 below, we use those conditions to prove that $\mathbf{H}_k', 1 \leq k \leq k_0$, which will be defined in (3.3) and (3.4) below be "good set".

In § 6 we will prove

Theorem 3. *Suppose x be a large number, then*

$$1.01\frac{y}{log\ x} \geq \pi(x) - \pi(x - y) \geq 0.99\frac{y}{log\ x} \tag{1.20}$$

with $y = x^\theta$, uniformly for

$$\frac{11}{20} < \theta \leq \frac{7}{12}. \tag{1.21}$$

A criterion for good sets is extracted. However, the technical work needed to choose good sets and to make the size of the bad sets as small as possible, is precisely the main difference between our method and that Heath-Brown's. The new Theorem 1 will enable us to improve the results of Heath-Brown and Iwaniec [10]. Later on we shall establish one deeper results: for

$$x^{\theta} \leq y \leq x\ exp(-(log\ x)^{1/6}),$$

we have (1.19) with $\theta = 6/11 + \varepsilon$ or $\theta = 7/13 + \varepsilon$.

Moreover, we can improve (1.19) further but only at the cost of much arduous computation.

§ 2 Proof of Theorem 1 and Theorem 2.

The proof of Theorem 1 is much along the method that was used by Heath-Brown [5].

Our starting point is based on a formal identity (see [5]) :

$$log\ \zeta(s)\prod(s) = \sum_{1\leq k\leq\infty} (-1)^{k-1}k^{-1}(\zeta(s)\prod(s) - 1)^k \tag{2.1}$$

$$= \sum_{1\leq t\leq\infty}\sum_{p\geq z}\frac{1}{tp^{ts}}, \tag{2.2}$$

where

$$\prod(s) = \prod_{p<z}(1 - \frac{1}{p^s}).$$

We pick out the coefficients of n^{-s} for those terms in (2.1) and (2.2) with $n \in |^y$. Thus in (2.2), these coefficients total

$$\sum_{1\leq t<\infty} \left(\pi\left(x^{\frac{1}{t}}\right) - \pi\left((x-y)^{\frac{1}{t}}\right)\right)\frac{1}{t} = \pi(x) - \pi(x-y) + O\left(yx^{-\frac{1}{3}}\right). \tag{2.3}$$

On the other hand, the Dirichlet series for $\zeta(s)\prod(s) - 1$ is

$$\sum_{n\geq z} c_n n^{-s}, \tag{2.4}$$

where c_n is 0 or 1 according to n has a prime factor $< z$ or not. It follows from (1.7) that there are no term of n^{-s} in (2.2) with $n \in |^y$ corresponding to exponents $k \geq k_0$. Henceforth we consider only the terms with $k < k_0$.

Let

$$((\zeta(s)\prod(s)-1)^k = \sum_{1\leq n<\infty} a_n(k)n^{-s} \tag{2.5}$$

By (2.4),

$$((\zeta(s)\prod(s)-1)^k = \left(\sum_{1\leq n<\infty} c_n n^{-s}\right)^k. \tag{2.6}$$

Then

$$\sum_{1\leq n<\infty} a_n(k)n^{-s} = \left(\sum_{1\leq n<\infty} c_n n^{-s}\right)^k$$

and

$$a_n(k) = \sum_{d_1\cdots d_k=n} c_{d_1}\cdots c_{d_k}.$$

Write

$$a_n(k) =\mid \{(d_1,\cdots,d_k) : n = d_1\cdots d_k, p(d_i)\geq z, 1\leq i\leq k\}\mid, \tag{2.7}$$

where $(d_1,\cdots,d_k) = (d'_1,\cdots,d'_k)$ means $d_i = d'_i$ for $i = 1,\cdots,k$. Therefore

$$\sum_{n\in|^y} a_n(k) = \sum_{\substack{d_1\cdots d_k=n\\ p(d_i)\geq z,1\leq i\leq k\\ n\in|^y}} 1, \tag{2.8}$$

and in (2.1), the coefficients total

$$\sum_{1\leq k\leq k_0}(-1)^{k-1}k^{-1}\sum_{n\in|^y} a_n(k) = \sum_{1\leq k<\infty}(-1)^{k-1}k^{-1}\sum_{\substack{d_1\cdots d_k=n\\ p(d_i)\geq z,1\leq i\leq k\\ n\in|^y}} 1. \tag{2.9}$$

We have that

$$\pi(x)-\pi(x-y) = \sum_{1\leq k<\infty}(-1)^{k-1}k^{-1}\sum_{\substack{d_1\cdots d_k=n\\ p(d_i)\geq z,1\leq i\leq k\\ n\in|^y}} 1 + O\left(yx^{-\frac{1}{3}}\right). \tag{2.10}$$

since (2.3) and (2.9).

By conditions (A_1) and (A_2), we have that

$$\begin{aligned}\pi(x)-\pi(x-y) &= \sum_{1\le k\le k_0}(-1)^{k-1}k^{-1}\sum_{H\in \mathbf{H}_k} c_H \sum_{d\in H} 1 + O\left(yx^{-\frac{1}{3}}\right) + O\left(\tfrac{y}{\log^2 x}\right)\\ &= y\sum_{1\le k\le k_0}(-1)^{k-1}k^{-1}\sum_{H\in \mathbf{H}'_k} c_H E_k(H,z) + R(y) + O\left(\tfrac{y}{\log^2 x}\right).\end{aligned}$$

Let

$$E(x,z)=\sum_{1\le k\le k_0}(-1)^{k-1}k^{-1}\sum_{H\in \mathbf{H}'_k} c_H E_k(H,z).$$

This completes the proof.

The proof of Theorem 2

By Prime Number Theorem,

$$\pi(x)=\int_2^x \frac{dt}{\log t}+O(x\ exp(-\log x)^{1/2}).$$

We have

$$\pi(x)-\pi(x-y_0)=\int_{x-y_0}^{x}\left(\frac{1}{\log t}-\frac{1}{\log x}\right)dt+\frac{y_0}{\log x}+O(x\ exp(-(\log x)^{1/2})$$

$$=\int_{x-y_0}^{x}\frac{\log\frac{x}{t}}{\log t\ \log x}dt+\frac{y_0}{\log x}+O(x\ exp(-(\log x)^{1/2}). \qquad (2.11)$$

Clearly, for $x-y_0\le t\le x$,

$$\log\frac{x}{t}\le \log\frac{x}{x-y_0}\le\frac{y_0}{x-y_0}=O\left(\frac{y_0}{x}\right).$$

Therefore, (3.1) is

$$\pi(x)-\pi(x-y_0)=\frac{y_0}{\log x}+O(y_0 exp(-(\log x)^{1/6}). \qquad (2.12)$$

Using Theorem 1 with $y=y_0, S'_k=\mathbf{H}_k$, and

$$R(y_0)=\sum_{1\le k\le k_0}(-1)^{k-1}k^{-1}R_k(y_0)+O\left(\frac{y_0}{\log x}\right),$$

we have

$$\pi(x) - \pi(x - y_0) = y_0 E(x, z) + \sum_{1 \le k \le k_0} (-1)^{k-1} k^{-1} R_k(y_0) + O\left(\frac{y_0}{log^2 x}\right). \tag{2.13}$$

Comparing (2.12) with (2.13), we have

$$\frac{y_0}{log\ x} = y_0 E(x, z) + \sum_{1 \le k \le k_0} (-1)^{k-1} k^{-1} R_k(y_0) + O\left(\frac{y_0}{log^2 x}\right), \tag{2.14}$$

hence

$$E(x, z) = \frac{1}{log\ x} - \frac{1}{y_0} \sum_{1 \le k \le k_0} (-1)^{k-1} k^{-1} R_k(y_0) + O\left(\frac{y_0}{log^2 x}\right). \tag{2.15}$$

By (2.15) and (1.16),

$$\frac{1 - e_1}{log\ x} < E(x, z) < \frac{1 + e_1'}{log\ x}. \tag{2.16}$$

Using Theorem 1 again,

$$yE(x, z) = \pi(x) - \pi(x - y) - \sum_{1 \le k \le k_0} (-1)^{k-1} k^{-1} R_k(y) + O\left(\frac{y}{log^2 x}\right). \tag{2.17}$$

By (1.17), (2.15) and (2.17), we have

$$\frac{(1 - e_1 - e_2')y}{log\ x} < \pi(x) - \pi(x - y) < \frac{(1 + e_1' + e_2)y}{log\ x}. \tag{2.18}$$

This completes the proof.

§ 3. "Good Set"

Let c_0 be a constant that will be defined later on. Let I_0 be an interval $[a_0, b_0]$ which contains in $[1, x]$ and $I_j (1 \le j \le r)$ be a subset of interval $[a_j, b_j]$ contains in $[x^{c_0}, x]$ also. Denote $D = I_0 \cdots I_r$ be a direct product of I_j. Let $i_j = log\ a_j / log\ x$ and $i_j' = log\ b_j / log\ x$ and let $d_j = x^{\theta_j}$ with $i_j \le \theta_j \le i_j'$ and $0 \le j \le r$. For convenience, we write $d = \{\theta_0, \theta_1, \cdots, \theta_r\} \in D$, and a set

$$\mathbf{D} = \{\{\theta_0, \theta_1, \cdots, \theta_r\} : 1/2 \ge 1 - \theta_1 - \cdots - \theta_r = \theta_0 \ge \theta_1 \ge \cdots \ge \theta_r\}. \tag{3.1}$$

For short, we denote $\{\theta_j\} = \{\theta_0, \theta_1, \cdots, \theta_r\}$.

Let $\mathbf{D} \cap |^y$ be a set of integers, $d \in \mathbf{D} \cap |^y$ if and only if $d \in \mathbf{D}$ and $d \in |^y \cdot d = d'$ with $d, d' \in \mathbf{D} \cap |^y$ means $d = d_0 \cdots d_r$ and $d' = d'_0 \cdots d'_r$ with $d_j = d'_j$ for $0 \le j < r$. We shall show the sufficient conditions for $\mathbf{D} \cap |^y$ be a "good set", i.e. for a fixed z with $x^{1/5} > z = x^c$, there exists a function $E_{\mathbf{D}}(x, z)$, independent of y, which satisfies that

$$\sum_{d \in \mathbf{D} \cap \mathbf{I}^y} 1 = y E_{\mathbf{D}}(x, z) + O(y\ exp(-log^{1/7} x)), \tag{3.2}$$

where $E_{\mathbf{D}}(x, z)$ and constant in "O" are uniformly for

$$x^{\theta} \le y \le x\ exp(-4(log\ x)^{\frac{1}{3}}(loglog\ x)^{-\frac{1}{3}}).$$

Let $\theta = 11/20 + \varepsilon, t_0 = 1 - \theta + \varepsilon/2$ and $z = x^c$ with $c = 1/2 - 8t_0/9$. Define

$$\mathbf{D}(6) = \{\{\theta_0, \theta_1, \cdots, \theta_5\} : \{\theta_0, \theta_1, \cdots, \theta_5\} \in \mathbf{D}, 2t_0/5 \ge 1 - \theta_1 - \cdots - \theta_5 =$$
$$\theta_0 \ge \theta_1 \ge \cdots \ge \theta_5 \ge 1/2 - 8t_0/9\} \tag{3.3}$$

and

$$\mathbf{D}(8) = \{\{\theta_0, \theta_1, \cdots, \theta_7\} : \{\theta_0, \theta_1, \cdots, \theta_7\} \in \mathbf{D}, 2t_0/7 \ge 1 - \theta_1 - \cdots - \theta_7 =$$
$$\theta_0 \ge \theta_1 \ge \cdots \ge \theta_7 \ge 1/2 - 8t_0/9\} \tag{3.4}$$

In this section, we shall prove that :

Theorem 4. *Suppose* $\theta = 11/20 + \varepsilon, t_0 = 1 - \theta + \varepsilon/2, z = x^{c_0}$ *with* $c = 1/2 - 8t_0/9, \mathbf{D}'$ *be a subset of* $\mathbf{D}$, *and*

$$\mathbf{D}' \cap (\mathbf{D}(6) \cup \mathbf{D}(8)) = \emptyset, \tag{3.5}$$

then $\mathbf{D}'$ *satisfies (3.2), i.e.* $\mathbf{D}'$ *is a good set.*

Obviously, the subset of $\mathbf{D}$ *with* $r \neq 5$ *or* 7 *are good sets.* $\mathbf{D}(6)$ *and* $\mathbf{D}(8)$ *are called exceptional sets.*

We discuss those sequences $d = \{\theta_0, \theta_1, \cdots, \theta_r\} = \{\theta_j\}$ in $\mathbf{D}$. For such $\{\theta_j\}$, we define a corresponding set Θ of all of sequences $\{\theta_0', \theta_1, \cdots, \theta_r, \theta_{r+1}, \cdots, \theta_{r+r_1}\}$ with $\theta_0' \leq \theta_0, \theta_1 \geq \cdots \geq \theta_r \geq \log z/\log x > \theta_{r+1} \geq \cdots \geq \theta_{r+r_1}$ and

$$\theta_0' + \theta_1 + \cdots \theta_{r+r_1} = 1. \tag{3.6}$$

By (3.6) and (3.1), we have that if $r_1 = 0$, then

$$\theta_0' = \theta_0 \geq \theta_1. \tag{3.7}$$

For short, write $\{\theta_j\}' = \{\theta_0', \theta_1, \cdots, \theta_r, \theta_{r+1}, \cdots, \theta_{r+r_1}\}$. $\{\theta_j\}$ and $\{\theta_j\}' \in \Theta$. Let $\theta_0' = \log X/\log x, \theta_j = \log X_d^{(j)}/\log x$ $(1 \leq j \leq r)$ and $\theta_{r+j} = \log Z_j/\log x$ $(1 \leq j \leq r_1)$. For each $\{\theta_0', \theta_1, \cdots, \theta_r, \theta_{r+1}, \cdots, \theta_{r+r_1}\}$, we define a product of Dirichlet series :

$$W(s, \{\theta_j\}'\} = W(s) = X(s)\prod_{j=1}^{r} X_d^{(j)}(s) Y(s) \prod_{j=1}^{r_1} Z_j(s) \tag{3.8}$$

where

$$\begin{aligned} X(s) &= \sum_{X<n\leq 2X} n^{-s}; \\ X_d^{(j)}(s) &= \sum_{X_d^{(j)}<m\leq 2X_d^{(j)}} f_m^{(j)} m^{-s}, \mid f_m^{(j)} \mid \leq 1; \\ Z_j(s) &= \sum_{Z_j<l\leq 2Z_j} c_l l^{-s}, \mid c_l \mid \leq 1; \\ Y(s) &= \sum_{Y<t\leq 2Y} \mu(t) v_t t^{-s}, \mid v_t \mid \leq 1. \end{aligned}$$

with $Y = O(x^\delta), \delta$ be a sufficient small number with $\delta \not\leq \epsilon$. Each $\{\theta_j\} \in \mathbf{D}$ corresponds all of $W(s, \{\theta_j\}')$'s for which $\{\theta_j\}' \in \Theta$. Define that $\mathbf{W}(\mathbf{D})$ is a set of all of such $W(s, \{\theta_j\}')$. For short, we write $W(s, \{\theta_j\}') = W(s)$. In [7], we proved that

Theorem A. *If* $\mathbf{D}$ *satisfies one of following conditions*

(1) $a_0 \geq x^{1/2}$;

(2) *all of* $W(s) \in \mathbf{W}(\mathbf{D})$ *such that*

$$\int_T^{2T} \left| W\left(\frac{1}{2} + it\right) \right| dt \ll x^{\frac{1}{2}} exp\left(-(\log x)^{\frac{1}{3}} (\log\log x)^{-\frac{2}{3}}\right) \tag{3.9}$$

for

$$T_1 \le T \le \frac{x^{1-\Delta}}{y},$$

where Δ *is any fixed positive constant, and*

$$T_1 = exp\left((log\ x)^{\frac{1}{3}}(loglog\ x)^{-\frac{1}{3}}\right).$$

Then (1.2) holds i.e. **D** *is a good set*

Let $\theta_0, \theta_1, \cdots, \theta_k$ be positive numbers. In [7], we discussed the sequence $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with positive number k such that

$$\theta_0 + \theta_1 + \cdots + \theta_k = 1, \tag{3.10}$$

defined a set $E(\theta)$ of some $\{\theta_0, \theta_1, \cdots, \theta_k\}$'s and acutely proved that [7,§ 5]).

Theorem B. *Let* $\{\theta_j\} \in$ **D**. *For each* $\{\theta_j\}' \in \Theta$ *define*

$$W'(s) = X(s)\prod_{j=1}^{r} X_d^{(j)}(s)\prod_{j=1}^{r_1} Z_j(s).$$

If $\{\theta_j\}' \in E(\theta)$, *then*

$$\int_T^{2T} \mid W'\left(\frac{1}{2} + it\right) \mid dt \ll x^{1/2-\varepsilon}. \tag{3.11}$$

Moreover, (3.9) holds.

We now describe the set $E(\theta)$.

Suppose $\{a_1, a_2, \sigma\}$ or $\{a_1, a_2, a_3, \sigma\}$ be a complementary partial sum (it means that each θ_j belongs one and only one set and their sum in a set be σ or a_i) of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with $\sigma = \theta_0$ or $\sigma \le t_0/2$, then

$$a_1 + a_2 + \sigma = 1. \tag{3.12}$$

or

$$a_1 + a_2 + a_3 + \sigma = 1. \tag{3.13}$$

Later on, we only define two of a_1, a_2 and σ if (3.12) holds; or define three of a_1, a_2, a_3 and σ if (3.13) holds. Suppose $\theta = 11/20 + \varepsilon$ and $t_0 = 9/20 - \varepsilon/2$.

We define $E(\theta)$ be a set which contains all of sequence $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with (3.12) which satisfies one of following three properties :

(I) There exists at least one complementary partial sum $\{a_1, a_2, \sigma\}$ of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ which satisfies one of following conditions:

(3.14) $a_1 \leq t_0$ and $a_2 \leq t_0$ (see Lemma 4.4 of [7]);

(3.15) $\sigma > t_0/5, a_1 > 8t_0/9$ and $a_2 > 8t_0/9$ (see (4.1.3) with $i = 3$ of [7]);

(3.16) $a_1 > 6t_0/7, a_2 > 6t_0/7$ and $\sigma > t_0/4$ (see (4.1.3) with $i = 2$ of [7]);

(3.17) $a_1 \geq t_0$ and $a_2 \geq t_0$ (see (4.1.1) of [7]);

(3.18) $1/2 \geq a_1 \geq t_0$, and $\sigma < 1/2 - 8t_0/9$ (see (4.5.6) of [7]);

(3.19) $\sigma > t_0/2$ (see Lemma 4.3 of [7]).

(II) There exists at least one complementary partial sum $\{a_1, a_2, a_3, \sigma\}$ of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ which satisfies

(3.20) $a_1 \geq t_0, a_2 \geq t_0/3, a_3 \geq t_0/3$ and $\sigma > 2t_0/5$ (see (4.2.2) of [7]).

For a fixed σ, in [7] we proved that there exists a pair of numbers (m_σ, M_σ) with the properties

$$M_\sigma - m_\sigma > 1/2 - 8t_0/9 \text{ if } \sigma \geq 1/2 - 8t_0/9; \tag{3.21}$$

$$M_\sigma - m_\sigma < \sigma \text{ if } \sigma < 1/2 - 8t_0/9; \tag{3.22}$$

$$M_\sigma > t_0 > m_\sigma; \tag{3.23}$$

and

$$M_\sigma + m_\sigma + \sigma = 1 \tag{3.24}$$

(III) Suppose $\{a_1, a_2, \sigma\}$ or $\{a_1, a_2, a_3, \sigma\}$ be a complementary sum of $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with

$$m_\sigma < a_i < M_\sigma, \quad (i = 1 \text{ or } 2), \tag{3.25}$$

(See Lemma 4.5 of [7]).

Applying Theorem A and Theorem B, Theorem 4 follows from

Theorem 5. *Suppose* $\theta = 11/20 + \varepsilon$, *and* $\mathbf{D}'$ *be a subset of* $\mathbf{D}$ *such that*

$$\mathbf{D}' \cap (\mathbf{D}(6) \cup \mathbf{D}(8)) = \phi,$$

Then for every $\{\theta_j\} \in \mathbf{D}'$, *the all of corresponding* $\{\theta_j\}' \in \Theta$ *contain in* $E(\theta)$.

§ 4. LEMMAS.

Let $\theta = 11/20 + \varepsilon$ and $\{\theta_0, \theta_1, \cdots, \theta_k\}$ with (3.11), i.e.

$$\theta_0 + \theta_1 + \cdots + \theta_k = 1.$$

In this section, we shall show some sufficient conditions for $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$. By the definition of $E(\theta)$ we check that $\{\theta_j\}$ satisfies at least one of conditions (3.14) - (3.20) and (3.25).

Lemma 4.1. *Suppose there exist two elements* θ' *and* θ'' *of* $\{\theta_0, \theta_1, \cdots, \theta_k\}$ *with* $\theta' \le t_0/2$ *and* $\theta'' < 1/2 - 8t_0/9$. *If there exists a partial sum* s *of* $\{\theta_0, \theta_1, \cdots, \theta_k\} \backslash \{\theta', \theta''\}$ *such that* $s < t_0$ *and* $s + \theta' \ge t_0$, *then* $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$.

Proof. We discuss following three cases :

Case 1. $t_0 \le s + \theta' < M_{\theta''}$ and $1 - s - \theta' \ge t_0$.

Let $\sigma = \theta''$ and $a_1 = s + \theta'$, we have that $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.23) and (3.25).

Case 2. $s + \theta' \ge M_{\theta''}$.

By (3.24), we have

$$1 - s - \theta' - \theta'' \le 1 - M_{\theta''} - \theta'' = m_{\theta''}$$

and, by (3.22),

$$1 - s - \theta' \le \theta'' + m_{\theta''} < M_{\theta''}.$$

Let $a_1 = 1 - s - \theta'$ and $\sigma = \theta''$, if $a_1 > m_{\theta''}$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.25). If $a_1 \le m_{\theta''} \le t_0$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.14) (since

$a_2 \le m_{\theta''} \le t_0$.)

Case 3. $t_0 \le s + \theta' < M_{\theta''}$ and $1 - s - \theta' < t_0$.

Let $a_1 = 1 - s - \theta' < t_0, a_2 = s < t_0$ and $\sigma = \theta'$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.14).

Lemma 4.2. *Suppose* $\{a_1, a_2, \sigma\}$ *be a complementary partial sum of* $\{\theta_0, \theta_1, \cdots, \theta_k\}$ *with* $a_1 = \theta_1 + \cdots + \theta_k, a_2 = \theta_1' + \cdots \theta_k', a_1 \ge a_2, \sigma = 1 - a_1 - a_2 > 1/2 - 8t_0/9$ *and*

$$max\{\theta_1, \cdots, \theta_k\} - min\{\theta_1', \cdots, \theta_k'\} < \frac{1}{2} - \frac{8t_0}{9}; \tag{4.1}$$

then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$.

Proof. If $a_1 \le t_0$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.14); if $a_2 \ge t_0$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (2.8); if $m_\sigma < a_1 < M_\sigma, \{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.25).

Now we suppose $a_1 \ge M_\sigma$.

By (3.15) and (3.1), we have

$$\begin{aligned} \theta_1 + \cdots + \theta_{k-1} + \theta_k' &= \theta_1 + \cdots + \theta_k + (\theta_k' - \theta_k) \\ &> M_\sigma - \left(\tfrac{1}{2} - \tfrac{8t_0}{9}\right) \ge m_\sigma. \end{aligned}$$

If

$$\theta_1 + \cdots + \theta_{k-1} + \theta_k' < M_\sigma.$$

let $a_1 = \theta_1 + \cdots + \theta_{k-1} + \theta_k'$, then $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ by (3.17); if

$$\theta_1 + \cdots + \theta_{k-1} + \theta_k' \ge M_\sigma.$$

and

$$\theta_1 + \cdots + \theta_{k-2} + \theta_{k-1}' + \theta_k' < M_\sigma.$$

repeating above process, let $a_1 = \theta_1 + \cdots + \theta_{k-1}' + \theta_k'$ we also have $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$. And repeat it again, we have that, in all cases, $\{\theta_0, \theta_1, \cdots, \theta_k\} \in E(\theta)$ since

$$\theta_1' + \cdots + \theta_k' < t_0 \le M_\sigma.$$

§ 5. Proof of Theorem 3

It is sufficient to prove Theorem 4, i.e., let $\{\theta_j\} \in \mathrm{D}$ and $\{\theta_j\}' \in \Theta$ that satisfies following conditions :

(5.1) $t_0 > 1 - \theta_1 - \cdots - \theta_r = \theta_0 \geq \theta_0' \geq \theta_1 \geq \cdots \geq \theta_r > 1/2 - 8t_0/9 \geq \theta_{r+1} \geq \cdots \geq \theta_{r+r_1}$;

(5.2) $\{\theta_0, \theta_1, \cdots, \theta_5\} \notin \mathrm{D}(6)$ if $r = 5$, and $\{\theta_0, \theta_1, \cdots, \theta_7\} \notin \mathrm{D}(6)$ if $r = 7$;

(5.3) $\theta_0' + \theta_1 + \cdots + \theta_{r+r_1} = 1$,

we shall prove that $\{\theta_j\}' \in E(\theta)$.

We record (2.2) here : if $r_1 = 0$,

$$\theta_0' = \theta_0 \geq \theta_1.$$

Let k_0 be the number such that

$$\sum_{1 \leq j \leq k_0 - 1} \theta_j \leq t_0 \tag{5.4}$$

and

$$\sum_{1 \leq j \leq k_0} \theta_j > t_0. \tag{5.5}$$

By (5.1), $\theta_1 < t_0$, then we have $k_0 \geq 2$.

If $r + r_1 > k_0 > r$, then $\theta_{k_0} < 1/2 - 8t_0/9$. In Lemma 4.1, take $\theta' = \theta_{k_0}, \theta'' = \theta_{r+r_1}$, then $\{\theta_j\}' \in E(\theta)$. If $k_0 = r + r_1$, let

$$a_1 = \sum_{1 \leq j \leq k_0 - 1} \theta_j \leq t_0,$$

and

$$a_2 = \theta_{k_0} \leq 1/2 - 8t_0/9 < t_0$$

then $\{\theta_j\}' \in E(\theta)$ by (3.14). Finally, we consider $2 \leq k_0 \leq r$.

Lemma 5.1. *Suppose $r_1 = 0$. If $r \leq 2(k_0 - 1)$, or $r \geq k_0 + 4$, then $\{\theta_j\}' \in E(\theta)$.*

Proof. If $r \le 2(k_0 - 1)$, let

$$a_1 = \sum_{1 \le j \le k_0 - 1} \theta_j \le t_0,$$

and

$$a_2 = \sum_{k_0 \le j \le r} \theta_j \le a_1 \le t_0.$$

Thus $\{\theta_j\}' \in E(\theta)$ by (3.14).

If $r \ge k_0 + 4$, let

$$\begin{aligned} a_1 &= \sum_{k_0+1 \le j \le k_0+4} \theta_j > 4\left(\tfrac{1}{2} - \tfrac{8t_0}{9}\right) > \tfrac{8t_0}{9}, \\ a_2 &= \sum_{1 \le j \le k_0} \theta_j > t_0 \end{aligned}$$

and $\sigma = 1 - a_1 - a_2$. Thus $\{\theta_j\}' \in E(\theta)$ by (3.15) since $\sigma = 1 - \theta_1 - \cdots - \theta_r \ge \theta_1 > t_0/5$. The Lemma is proved.

To prove Theorem 4, we now discuss following cases :

Case 1. $k_0 \ge 3$.

By (5.1) and (5.4), we have

$$\theta_3 \le \theta_2 \le \frac{1}{2}(\theta_1 + \theta_2) \le \frac{1}{2}t_0.$$

If $r_1 > 0$, take $\theta' = \theta_3$ and $\theta'' = \theta_{r+r_1}$ in Lemma 4.1, we have that $\{\theta_j\}' \in E(\theta)$.

Now may suppose that

$$r_1 = 0.$$

By Lemma 5.1, we also suppose

$$2k_0 - 1 \le r \le k_0 + 3 \tag{5.6}$$

i.e. $3 \le k_0 \le 4$.

When $k_0 = 4$, by (5.6) we have $r = 7$. Since $\{\theta_j\} \notin \mathrm{D}(8)$, we have

$$\theta_1 + \theta_2 + \theta_3 > \frac{6}{7}t_0.$$

Let $a_1 = \theta_1 + \theta_2 + \theta_3 > 6t_0/7, a_2 = \theta_4 + \theta_5 + \theta_6 + \theta_7 > 8t_0/9 > 6t_0/7$, and $\sigma = \theta_0 > 1/8 > t_0/4$, thus $\{\theta_j\}' \in E(\theta)$ by (3.16). When $k_0 = 3$, by (5.6), $r = 5$ or 6. If $r = 5$, by $\{\theta_j\} \notin \mathbf{D}(6)$, we have

$$\theta_1 > \frac{2}{5}t_0.$$

Let $a_1 = \theta_3 + \theta_4 + \theta_5$, when $\theta_3 + \theta_4 + \theta_5 \geq t_0$, let $a_2 = \theta_2 > t_0/3$ and $a_2 = \theta_1 > 2t_0/5$, then $\{\theta_j\}' \in E(\theta)$ by (3.20). When $\theta_3 + \theta_4 + \theta_5 < t_0$, let $a_2 = \theta_1 + \theta_2$, by (5.1) we have

$$\theta_1 + \theta_2 \leq \frac{2}{3}(\theta_0 + \theta_1 + \theta_2) \leq \frac{2}{3}(1 - \theta_3 - \theta_4 - \theta_5) < t_0.$$

Thus $\{\theta_j\}' \in E(\theta)$ by (3.14).

When $k_0 = 3, r = 6$ and $r_1 = 0$, we discuss following cases :

Case 1.1. $\theta_1 + \theta_3 + \theta_5 \leq t_0$ or $\theta_2 + \theta_4 + \theta_6 \geq t_0$.

Let $a_1 = \theta_1 + \theta_3 + \theta_5$ and $a_2 = \theta_2 + \theta_4 + \theta_6$, then $\{\theta_j\}' \in E(\theta)$ by (3.14) or (3.17).

Case 1.2. $\theta_1 + \theta_3 + \theta_5 > t_0 > \theta_2 + \theta_4 + \theta_6$.

If $\theta_1 - \theta_6 < 1/2 - 8t_0/9$, take $\sigma = 1 - a_1 - \cdots - a_6 \geq a_1 > 1/2 - 8t_0/9$ in Lemma 4.2, then $\{\theta_j\}' \in E(\theta)$. We consider that

$$\theta_1 - \theta_6 \geq 1/2 - 8t_0/9.$$

By (4.1), $\theta_6 \geq 1/2 - 8t_0/9$, therefore

$$\theta_1 \geq 1 - 16t_0/9$$

and $\theta_0 + \theta_1 \geq 2 - 32t_0/9 > 8t_0/9$. Let $a_1 = \theta_0 + \theta_1 > 8t_0/9, a_2 = \theta_2 + \theta_3 + \theta_4 + \theta_5 > 8t_0/9$ and $\sigma = \theta_6$, thus $\{\theta_j\}' \in E(\theta)$ by (3.15).

Case 2. $k_0 = 2$.

By (5.4), we have $\theta_1 > t_0/2$. If $r_1 = 0$, then $\theta_0' = \theta_0 > t_0/2$ and $\{\theta_j\} \in E(\theta)$ by (3.19). Now may suppose that $r_1 > 0$ and $\theta_0 \leq t_0/2$.

We discuss the following two cases :

Case 2.1. $\theta_3 > t_0/2$.

Let $a_1 = \theta_2 + \theta_3 \geq t_0$. By (5.1) and (5.3), we have that

$$\theta_2 + \theta_3 \leq 1 - \theta_0' - \theta_1 - \theta_4 - \cdots - \theta_{r+r_1} \leq 1 - \theta_2 - \theta_3;$$

then

$$\theta_2 + \theta_3 \leq 1/2. \tag{5.7}$$

Let $\sigma = \theta_{r+r_1}$, then $\{\theta_j\}' \in E(\theta)$ by (3.18) (since $r_1 > 0$ implies $\theta_{r+r_1} < 1/2 - 8t_0/9$).

Case 2.2. $\theta_3 \leq t_2/2$.

We have

$$\theta_1 + \theta_0 + \theta_3 + \cdots + \theta_{r+r_1} = 1 - \theta_2 > 1 - t_0$$

and $\theta_1 + \theta_0 + \theta_3 + \cdots + \theta_{r+r_1-1} > t_0$ (since $\theta_{r+r_1} < 1/2 - 8t_0/9 < 1 - 2t_0$). We can find a number $j = 0$ or 3 or $j \leq r + r_1 - 1$ with

$$\theta_1 + \theta_0 + \theta_3 + \cdots + \theta_{k-1} < t_0$$

and

$$\theta_1 + \theta_0 + \theta_3 + \cdots + \theta_k \geq t_0.$$

In Lemma 3.1, take $\theta' = \theta_0$ and $\theta'' = \theta_{r+r_1}$, then we have that $\{\theta_j\} \in E(\theta)$.

The proof of Theorem 1 is complete.

§ 6. Proof of Theorem 3.

In this section we discuss that $\theta = 11/20 + \varepsilon$.

Let

$$S_k' = \{d = d_0 d_1 \cdots d_{k-1} : d \in S_k, d_0 \geq \cdots \geq d_{k-1}\}. \tag{6.1}$$

Take $\mathbf{H}_k = S_k'$ and $c_H = k!$, then (A_1) holds.

Let

$$\mathrm{D}(6) = \left\{d : d \in S_6', d = d_0 \cdots d_5, x^{2t_0/5} \geq d_0 \geq \cdots \geq d_5 \geq x^{\frac{1}{2} - \frac{8t_0}{9}}, d_3 d_4 d_5 \geq x^{t_0}\right\} \tag{6.2}$$

and

$$\mathbf{D}(8) = \left\{ d : d \in S'_k, d = d_0 \cdots d_7, x^{\frac{2t_0}{7}} \geq d_0 \geq \cdots \geq d_7 \geq x^{\frac{1}{2} - \frac{8t_0}{9}} \right\} \quad (6.3)$$

In [8], we proved the following lemma :

Lemma 6.1. *Let* $\mathbf{H}''_6 = \mathbf{D}(6), \mathbf{H}''_8 = \mathbf{D}(8)$ *and* $\mathbf{H}''_k = \emptyset$ *for* $k \neq 6$ and 8, *then condition* (A_2) *holds, i.e. (1.10) holds for* $\mathbf{H}'_k = \mathbf{H}_k \setminus \mathbf{H}''_k$.

Proof of Theorem 3. By (1.12), we have that

$$R(y) = (5!) \sum_{d \in \mathbf{D}(6)} 1 + (7!) \sum_{d \in \mathbf{D}(8)} 1. \quad (6.4)$$

We now estimate $R(y)$. By (4.2), $d \in \mathbf{D}(6)$ implies $d = d_0 \cdots d_5$ with

$$p(d_j) \geq x^{\frac{1}{2} - \frac{8t_0}{9}} > (d_j)^{\frac{1}{2}}, 0 \leq j \leq 5. \quad (6.5)$$

Then all of d_j's be primes. Let

$$\mathbf{D}_1(6) = \{(p_0, \cdots, p_4) : p_j \text{ primes for } 0 \leq j \leq 4, x^{\frac{2t_0}{5}} \geq p_0 \geq \cdots$$
$$\geq p_4 \geq x^{\frac{1}{2}\left(1 - \frac{8t_0}{5}\right)}, p_3 \geq x^{\frac{t_0}{3}}\}.$$
$$|^{(1)} = |^{(1)}(p_0, \cdots, p_4) = [\frac{x-y}{p_0 \cdots p_4}, \frac{x}{p_0 \cdots p_4}),$$

and

$$\Delta_5 = \Delta_5^1 \cup \Delta_5^2,$$

where

$$\Delta_5^1 = \left\{ (t^0, \cdots, t^4) : \frac{2t_0}{5} \geq t^0 \geq \cdots \geq t^4 \geq \frac{t_0}{3} \right\} \quad (6.6)$$

and

$$\Delta_5^2 = \left\{ (t^0, \cdots, t^4) : \frac{2t_0}{5} \geq t^0 \geq \cdots \geq t^3 \geq \frac{t_0}{3} \geq t^4 \geq \frac{1}{2}\left(1 - \frac{8t_0}{5}\right) \right\}. \quad (6.7)$$

Then we have that

$$\sum_{d \in \mathbf{D}(6)} 1 = \sum_{(p_0, \cdots, p_4) \in \mathbf{D}_1(6)} \sum_{p_5 \in |^{(1)}} 1 \leq \sum_{(p_0, \cdots, p_4) \in \mathbf{D}_1(6)} \frac{2y}{p_0 \cdots p_4 log \frac{y}{p_0 \cdots p_4}}$$

$$\leq \frac{(2+\varepsilon)y}{\log x}\int\cdots\int_{\Delta_5}\frac{dt^0\cdots dt^4}{t^0\cdots t^4(1-t^0-\cdots-t^4)}. \tag{6.8}$$

By (6.6) and (6.7) we have that

$$\int\cdots\int_{\Delta_5^1}\frac{dt^0\cdots dt^4}{t^0\cdots t^4(1-t^0-\cdots-t^4)} \leq \frac{1}{5!}\int_{\frac{t_0}{3}}^{\frac{2t_0}{5}} dt^0\int_{\frac{t_0}{3}}^{\frac{2t_0}{5}} dt^1\cdots\int_{\frac{t_0}{3}}^{\frac{2t_0}{5}}\frac{dt^0\cdots dt^4}{t^0\cdots t^4(1-t^0-\cdots-t^4)}$$

$$\leq \frac{10}{5!}\left(\log\frac{1.8}{1.5}\right)^5 < \frac{0.002015}{5!},$$

and

$$\int\cdots\int_{\Delta_5^2}\frac{dt^0\cdots dt^4}{t^0\cdots t^4(1-t^0-\cdots-t^4)} \leq$$

$$\leq \frac{1}{4!}\int_{\frac{t_0}{3}}^{\frac{2t_0}{5}} dt^0\int_{\frac{t_0}{3}}^{\frac{2t_0}{5}} dt^1\cdots\int_{\frac{1}{2}\left(1-\frac{8t_0}{5}\right)}^{\frac{t_0}{3}}\frac{dt^0\cdots dt^4}{t^0\cdots t^4(1-t^0-\cdots-t^4)}$$

$$\leq \frac{1}{(4!)(1-0.72-0.15)}\left(\log\frac{0.18}{0.15}\right)^4\left(\log\frac{0.15}{0.14}\right) < \frac{0.002933}{5!}$$

Thus

$$\sum_{d\in D(6)} 1 < \frac{0.009895}{5!}\left(\frac{y}{\log x}\right). \tag{6.9}$$

Let

$$D_1(8) = \left\{(p_0,\cdots,p_6) : p_j \text{ primes for } 0\leq j\leq 6, x^{\frac{2t_0}{7}} \geq p_0 \geq\cdots\geq p_6 \geq x^{\frac{1}{2}\left(1-\frac{12t_0}{7}\right)}\right\},$$

$$|^{(2)} = |^{(2)}(p_0,\cdots,p_6) = [\frac{x-y}{p_0\cdots p_6}, \frac{x}{p_0\cdots p_6}),$$

and

$$\Delta_7 = \Delta_7^1 \cup \Delta_7^2,$$

where

$$\Delta_7^1 = \left\{(t^0,\cdots,t^6) : \frac{2t_0}{7} \geq t^0 \geq \cdots t^6 \geq \frac{t_0}{4}\right\} \tag{6.10}$$

and

$$\Delta_7^2 = \left\{(t^0,\cdots,t^6) : \frac{2t_0}{7} \geq t^0 \geq\cdots\geq t^5 \geq \frac{t_0}{3} \geq t^6 \geq \frac{1}{2}\left(1-\frac{12t_0}{7}\right)\right\}. \tag{6.11}$$

Then we have that

$$\sum_{d\in D(8)} 1 = \sum_{(p_0,\cdots,p_6)\in D_1(8)} \sum_{p_7\in I^{(2)}} 1 \le \sum_{(p_0,\cdots,p_6)\in D_1(8)} \frac{2y}{p_0\cdots p_6 \log \frac{y}{p_0\cdots p_6}}$$

$$\le \frac{(2+\varepsilon)y}{\log x}\int\cdots\int_{\Delta_7}\frac{dt^0\cdots dt^6}{t^0\cdots t^6(1-t^0-\cdots-t^6)}.$$

By (6.6) and (6.7) we have that

$$\int\cdots\int_{\Delta_7^1}\frac{dt^0\cdots dt^6}{t^0\cdots t^6(1-t^0-\cdots-t^6)} \le \frac{1}{7!}\int_{\frac{t_0}{4}}^{\frac{2t_0}{7}} dt^0 \int_{\frac{t_0}{4}}^{\frac{2t_0}{7}} dt^1 \cdots \int_{\frac{t_0}{4}}^{\frac{2t_0}{7}} \frac{dt^6}{t^0\cdots t^6(1-t^0-\cdots-t^6)}$$

$$\le \frac{10}{7!}\left(\log\frac{\frac{0.9}{7}}{\frac{0.45}{4}}\right)^7 < \frac{7.6(10^{-6})}{7!},$$

and

$$\int\cdots\int_{\Delta_7^2}\frac{dt^0\cdots dt^6}{t^0\cdots t^6(1-t^0-\cdots-t^6)} \le$$

$$\le \frac{1}{6!}\int_{\frac{t_0}{4}}^{\frac{2t_0}{7}} dt^0 \int_{\frac{t_0}{4}}^{\frac{2t_0}{7}} dt^1 \cdots \int_{\frac{1}{2}\left(1-\frac{12t_0}{7}\right)}^{\frac{t_0}{4}} \frac{dt^0\cdots dt^6}{t^0\cdots t^6(1-t^0-\cdots-t^6)}$$

$$\le \frac{1}{(6!)\left(1-\frac{12(0.45)}{7}-\frac{0.45}{4}\right)}\left(\log\frac{\frac{0.9}{7}}{\frac{0.45}{4}}\right)^6\left(\log\frac{\frac{0.45}{4}}{0.1}\right) < \frac{4.027(10^{-5})}{7!}.$$

Thus

$$\sum_{d\in D(8)} 1 < \frac{9.58(10)^{-5}}{7!}\left(\frac{y}{\log x}\right). \tag{6.12}$$

By (6.4), (6.9) and (6.12) we have that

$$R(y) < \frac{0.01\, y}{\log x}.$$

In Theorem 2 take $e_1 = e_2 = 0$ and $e_1' = e_2' = 0.01$, Theorem 3 follows.

REFERENCES

[1] Hoheisel, G., *Primahlprobleme in der analysis,* sitzungsber der Preuss. Akad. D. Wissensch. Phys-Math. Kl. Berlin, 1930, 580-588.

[2] Ingham, A.E., *On the difference between consecutive primes,* Quart. J. Math., Oxford, 8 (1937) 255-266.

[3] Montgomery. H.L., Topics in multiplicative number theory, Lecture Notes in Math., 227, Springer, Berlin, (1971).

[4] Huxley, M.N., *On the difference between consecutive primes,* Invent. Math., 15 (1972), 164-170.

[5] Heath-Brown, D.R. *The Number of Primes in a Short Interval,* J. Reine Angew, Math. 389 (1988) 22-63.

[6] Lou, S. and Qi Yao, *A Chebychev's Type of Prime Number Theorem in a Short Interval II,* Hardy-Ramanujan J., 15 (1992), 1-33.

[7] Lou, S. and Yao, Q. *Estimate of sums of Dirichlet's series,* Hardy-Ramanujan J., 17 (1994), to appear.

[8] Heath-Brown, D.R., and Iwaniec, H., *On the difference between consecutive primes,* Invent. Math. 55 (1979), 49-69.

Lou,S. and Yao, Q.
Department of Mathematics
Dalhousie University
Halifax, N.S. B3H 3J5
Canada