Hardy-Ramanujan Journal
Vol.17 (1994) 1-31

ESTIMATE OF SUMS OF DIRICHLET SERIES

Shituo Lou and Qi Yao

§ 1. INTRODUCTION.

Let x be a large positive number, ε be a small positive number and k be a finite integer. Let $y = x^{\theta}, 1/2 < \theta < 7/12, T = x^{1-\theta+\varepsilon/2}$, and let $M_1, \cdots, M_k$ and L be real numbers such that

$$M_1 \cdots M_k L = x/2. \tag{1.1}$$

$U_1, \cdots, U_k$ and W be real numbers such that

$$W \ll L^{1/2-\varepsilon} \tag{1.2}$$

and

$$U_i \ll M_i^{1/2}, 1 \le i \le k. \tag{1.3}$$

Let A be a fixed integer, and $| S |$ be the number of elements of set S. We discuss the set $S(U_1, \cdots, U_k, W)$ which satisfies following conditions, for $1 \le i \le k$ and j be a positive integer,

(1.4) $| S(U_1, \cdots, U_k, W) | \ll U_i^{-2j}(M_i^j + T)(\log x)^A$ and $| S(U_1, \cdots, U_k, W) | \ll W^{-2j}(L^j + T)(\log x)^A$;

(1.5) $| S(U_1, \cdots, U_k, W) | \ll (U_i^{-2j} M_i^j + U_i^{-6j} M_i^j T)(\log x)^A$;

(1.6) $| S(U_1, \cdots, U_k, W) | \ll (W^{-2j} L^j + W^{-6j} L^j T)(\log x)^A$;

(1.7) $W^4 \mid S(U_1,\cdots,U_k,W) \mid \ll T(\log x)^A$.

Let $a_j = \log M_j/\log x$. In § 2 we shall give some functions $h_i(a_1,\cdots,a_k)$ for $1 \le i \le 3$, such that

$$U_1 \cdots U_k W \mid S(U_1,\cdots,U_k,W) \mid \ll x^{1/2-\varepsilon} + x^{h_i(a_1,\cdots,a_k)}. \tag{1.8}$$

In § 3, we discuss those $S(U_1,\cdots,U_k,W)$'s which satisfy one more inequality : for $1 \le j \le k$,

$$\mid S(U_1,\cdots,U_k,W) \mid \ll W^{-4}U_i^{-2}(M_j^2T^{1/2} + M_j^{5/4}T^{3/4} + T)T^{\varepsilon}. \tag{1.9}$$

If $T^{1/5} \le M_j \le T^{1/3}$, we replace (1.9) by

$$\mid S(U_1,\cdots,U_k,W) \mid \ll W^{-4}U_i^{-2}M_j^{5/4}T^{3/4+\varepsilon}, \tag{1.10}$$

and, if $M_j \le T^{1/5}$, we replace (1.9) by

$$\mid S(U_1,\cdots,U_k,W) \mid \ll W^{-4}U_i^{-2}T^{1+\varepsilon}. \tag{1.11}$$

In § 3 we will give $h_i(a_1,\cdots,a_k)$ for $4 \le i \le 10$ with (1.8).

Heath-Brown and Iwaniec [1] discussed the gaps between consecutive primes using sieve method. The remainder term

$$R(x;M_1,\cdots,M_k) = \sum_{\substack{M_i<m_i\le 2M_i \\ 1\le i\le k}} a_{m_1,1},\cdots,a_{m_k,k}r_{m_1\cdots m_k}, \tag{1.12}$$

where

$$r_d = \left[\frac{x}{d}\right] - \left[\frac{x-y}{d}\right],$$

$M_i < y$ and $\mid a_{m_i},i \mid \le 1$, be considered. Applying the method which is very close to Heath-Brown and Iwaniec's, in § 5, we will show that

$$R(x;M_1,\cdots,M_k) \ll x^{\theta-\varepsilon}, \tag{1.13}$$

if (1.8) holds with

$$h_i(a_1,\cdots,a_k) < 1/2, \tag{1.14}$$

In fact, they proved that (see [1]) if $\theta = 11/20 + \varepsilon, k = 2, M_1 \ll x^{0.46-\varepsilon}$, and $M_2 \ll x^{0.46-\varepsilon}$, (1.13) holds. Consequently, they obtained that for $y = x^{\theta}, \theta \geq 11/20 + \varepsilon$,

$$\pi(x) - \pi(x-y) > \frac{1}{212}\ \frac{y}{log\ x}, \tag{1.15}$$

where $\pi(x)$ be the number of primes $\leq x$.

In [2], Heath-Brown discussed some kind of products of Dirichlet series

$$W(s) = X(s)\prod_{j=1}^{k_0} Y_j(s),$$

where

$$X(s) = \sum_{L_0 < n \leq 2L_0} n^{-s},$$

$$Y_j(s) = \sum_{M_j < n \leq 2M_j} b_m n^{-s}, \mid b_m \mid \leq 1;$$

and

$$L_0 \prod_{j=1}^{k_0} M_j = \frac{x}{2}.$$

Using Heath-Brown's method that was used in [2], in § 5, we shall show that

$$\int_T^{2T} \mid W(\frac{1}{2} + it) \mid dt \ll x^{\frac{1}{2}} exp(-(log\ x)^{\frac{1}{3}}(loglog\ x)^{\frac{2}{3}}) \tag{1.16}$$

for

$$T_1 \leq T \leq x^{1-\theta+\varepsilon},$$

where θ is a fixed positive constant with $1/2 < \theta < 1$, and

$$T_1 = exp((log\ x)^{\frac{1}{3}}(loglog\ x)^{-\frac{1}{3}}),$$

if (1.8) holds with (1.14).

In [3], and [4], we shall apply the results of this paper to investigate gaps between consecutive primes and prove that (1.11) holds for $\theta = 11/20 + \varepsilon$

in [3] and $\theta = 6/11 + \varepsilon$ in [4]. In Section 4, we also discuss the case of $\theta = 11/20 + \varepsilon$ that will be used in [3] to prove that

$$0.99\frac{y}{log\ x} < \pi(x) - \pi(x-y) < 1.01\frac{y}{log\ x}.$$

In Section 4 we discuss the case of $\theta = 6/11 + \varepsilon$ as well that will be used in [4] to prove that

$$\frac{0.969y}{log\ x} < \pi(x) - \pi(x-y) < \frac{1.031y}{log\ x},$$

for $\theta = 6/11 + \varepsilon$.

In this paper δ be another positive small number with $\delta \ll \varepsilon$ and may be difference in some paragraphs. Also, we use

$$c\delta < \delta.$$

§ 2. ESTIMATE OF $| S(U_1, \cdots, U_k, W) |$.

In this section we estimate $| S(U_1, \cdots, U_k, W) |$. First we divide $M_1, \cdots, M_k$, and L into $k_0 + 1$ parts (in this paper $2 \le k_0 \le 6$), denote the products of elements in one part by $N_1, \cdots, N_{k_0}$, and L_0, respectively. Also denote the product of those U_h's or W that correspond to N_j or L_0 by U_j or W, again. We now replace (1.1), (1.2) and (1.3) by

$$N_1 \cdots N_{k_0} L_0 = \frac{x}{2}, \tag{2.1}$$

$$U_j \le N_j^{\frac{1}{2}} A_j, \quad 1 \le j \le k_0, \tag{2.2}$$

and

$$W \le L_0^{\frac{1}{2}} A_0 \tag{2.3}$$

where A_j (or A_0) $= x^{-\delta}$ if L belongs to N_j (or L_0), otherwise $A_j = 1$. Thus we have that

$$W \prod_{i=1}^{k} U_i \ll \left(L_0 \prod_{i=1}^{k} M_i \right)^{\frac{1}{2}-\delta} \ll x^{\frac{1}{2}-\delta}. \tag{2.4}$$

It is clear that if L_0 satisfies (1.6) with $i = 2$ and $L_0 \ll T^{1/2}$, then L_0 satisfies (1.7). In this paper, assume that

$$L_0 \ll T^{1/2} \text{ if } L_0 \neq L. \tag{2.5}$$

Replacing L by L_0, we shall show that the function $\mid S(U_1, \cdots, U_{k0}, W) \mid$ with (1.4), (1.5), (1.6), (1.7) and (2.4) such that (1.8).

We now renew the notations $a_i = log\ N_i/log\ x, i = 1, \cdots, k_0, \sigma = log\ L_0/log\ x = 1 - a_1 - \cdots - a_{k_0}$ and $t_0 = log\ T/log\ x$. Let

$$F = \mid S(U_1, \cdots, U_{k0}, W) \mid x^{-\delta} \tag{2.6}$$

We replace $\mid S(U_1, \cdots, U_{k0}, W) \mid$ by F, it makes that we can drop the factor $log^A x$ in the right hand side of (1.4), (1.5), (1.6) and (1.7), and drop T^ϵ in (1.9).

If $F \geq 2U_i^{-2j} N_i^j$, then we have that

$$N_i \ll T^{1/j} \tag{2.7}$$

since $F \leq U_i^{-2j}(N_i^j + T)$.

Suppose j such that

$$\sigma \leq t_0/j \text{ and } j = 2 \text{ if } \sigma \geq t_0/2. \tag{2.8}$$

Thus, if $j > 2$, we have $\sigma \leq t_0/j$, i.e. $W^j \leq T$. Then

$$F \ll W^{-2j}T.$$

We prove following lemmas :

Lemma 2.1. *Let $k_0 = 2$, we have*

$$U_1 U_2 \mid S(U_1, U_2, W) \mid \ll x^{h_0(a_1, a_2) + \epsilon/2} + x^{1/2 - \delta}, \tag{2.9}$$

where $h_0(a_1, a_2) = h_0(a_2, a_1), a_1' = max\{t_0, a_1\}$ and

$$h_0(a_1, a_2) = \begin{cases} \frac{1}{2} - \varepsilon, & a_1 \geq a_2 \geq t_0 \quad (2.10) \\ \frac{t_0}{2} + \frac{a_1}{2} + min\left\{\frac{a_2}{4j}, \frac{\sigma}{6} + \frac{a_2}{12j}\right\}, & \text{otherwise.} \quad (2.11) \end{cases}$$

Proof. We have that, by (1.4), (1.5) and (1.6),

$$F \le min\{U_1^{-2}(N_1+T), U_2^{-2}(N_2+T), U_1^{-2}N_1 + U_1^{-6}N_1T, U_2^{-2}N_2 + U_2^{-6}N_2T,$$
$$W^{-2j}L_0^j + W^{-6j}L_0^jT, W^{-2j}T\},$$

and proceed to show that

$$U_1U_2WF \ll x^{h_0(a_1,a_2)} + x^{\frac{1}{2}-\delta}. \qquad (2.12)$$

We now consider four cases :

Case 1. $F \le 2U_1^{-2}N_1$ and $F \le 2U_2^{-2}N_2$.

Choosing k' such that $L_0^{k'} \ge T$, we have $F \ll W^{-2k'}L_0^{k'}$, and then

$$\begin{aligned} U_1U_2WF &\ll U_1U_2W(U_2^{-2}N_2)^{\frac{1}{2}-\frac{1}{2(2k'+1)}}(U_1^{-2}N_2)^{\frac{1}{2}-\frac{1}{2(2k'+1)}}(W^{-2k'}L_0^{k'})^{\frac{1}{2k'+1}} \\ &\ll (U_1U_2W)^{\frac{1}{2k'+1}}(N_1N_2L_0)^{\frac{1}{2}-\frac{1}{2(2k'+1)}} \ll x^{\frac{1}{2}-\delta} \end{aligned}$$

by (2.4). Thus (2.12) holds. (Note : when $a_1 \ge a_2 \ge t_0$, we have that

$$F \le 2U_1^{-2}N_1 \text{ and } F \le 2U_2^{-2}N_2$$

by (1.4) with $i = 1$. This completes the proof of (2.10).)

Case 2. $F \le 2U_1^{-2}N_1, F > 2U_2^{-2}N_2$.

In this case we have that $F \ll U_2^{-2}T, F \ll U_2^{-6}N_2T$ and $N_2 \ll T$ since (1.4), (1.5) (take $M_i = N_i$) (2.7) and $F > 2U_2^{-2}N_2$. Thus

$$\begin{aligned} F &\le 2min\{U_1^{-2}N_1, U_2^{-2}T, U_2^{-6}N_2T, W^{-2j}L_0^j, W^{-2j}T\} \\ &+ 2min\{U_1^{-2}N_1, U_2^{-2}T, U_2^{-6}N_2T, W^{-2j}L_0^jT, W^{-2j}T\} \\ &\le 2(U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-3)/4j}(U_2^{-6}N_2T)^{1/4j}(W^{-2j}min\{L_0^j, T\})^{1/2j} \\ &+ 2min\{(U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-3)/4j}(U_2^{-6}N_2T)^{1/4j}(W^{-2j}T)^{1/2j}, \\ &\qquad (U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-1)/4j}(U_2^{-6}N_2T)^{1/12j}(W^{-6j}L_0^jT)^{1/6j}\} \\ &\ll (U_1U_2W)^{-1}N_1^{1/2}T^{1/2-1/2j}N_2^{1/4j}min\left\{L_0^{\frac{1}{2}}, T^{\frac{1}{2j}}\right\} \\ &\quad +(U_1U_2W)^{-1}(TN_1)^{1/2}min\left(N_2^{1/4j}, N_2^{1/12j}L_0^{1/6}\right). \end{aligned} \qquad (2.13)$$

If $L_0 \ll T^{1/j}$, we have that

$$L_0^{\frac{1}{2}} \ll T^{\frac{1}{2j}} \text{ and } N_2^{\frac{1}{6j}} L_0^{\frac{1}{3}} \ll T^{\frac{1}{2j}}.$$

Thus the first term on the right hand side of (2.13) is less than the second term. If $L_0 \gg T^{1/j}$, by (2.7), we have $j = 2$. We have that the first term on the right hand side of (2.13) is less than the second term again. Then

$$F \ll (U_1U_2W)^{-1}(TN_1)^{1/2} min\left(N_2^{1/4j}, N_2^{1/12j}L_0^{1/6}\right).$$

Thus (2.12) and (2.11) follows.

Case 3. $F \le 2U_2^{-2}N_2, F \ge 2U_1^{-2}N_1$.

The proof here is the same as in Case 2.

Case 4. $F > 2U_1^{-2}N_1, F > 2U_2^{-2}N_1$.

We have that

$$\begin{aligned} F &\le 2min\left\{U_1^{-2}T, U_2^{-2}T, U_1^{-6}N_1T, U_2^{-6}N_2T, W^{-2j}L_0^j, W^{-2j}T\right\} \\ &+ 2min\left\{U_1^{-2}T, U_2^{-2}T, U_1^{-6}N_1T, U_2^{-6}N_2T, W^{-6j}L_0^j, W^{-2j}T\right\}. \end{aligned}$$

Then

$$\begin{aligned} F &\ll (U_1^{-2}T)^{\frac{1}{2}}(W^{-2j}L_0^j)^{\frac{1}{6j}}(W^{-2j}T)^{\frac{1}{3j}}(U_2^{-2}T)^{\frac{1}{2}-\frac{3}{4j}}(U_2^{-6}N_2T)^{\frac{1}{4j}} \\ &+ (U_1^{-2}T)^{\frac{1}{2}}(W^{-6j}L_0^jT)^{\frac{1}{6j}}(U_2^{-2}T)^{\frac{2j-1}{4j}}(U_2^{-6}N_2T)^{\frac{1}{12j}} \\ &\ll (U_1U_2W)^{-1}\left(T^{1-\frac{1}{6j}}L_0^{\frac{1}{6}}N_2^{\frac{1}{4j}} + TL_0^{\frac{1}{6}}N_2^{\frac{1}{12j}}\right) \\ &\le (U_1U_2W)^{-1}TL_0^{\frac{1}{6}}N_2^{\frac{1}{12j}}, \end{aligned}$$

since $N_2 \ll T$ by (2.7) with $j = 1$. Thus we get (2.12) and (2.11) again.

Lemma 2.2. *If $N_2 \le T$, we have*

$$U_1U_2W \mid S(U_1, U_2, W) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_1^i(a_1,a_2)}, \tag{2.14}$$

where

$$h_1(a_1, a_2) = \frac{i+1}{2(i+2)}t_0 + \frac{1}{4(i+2)} + \frac{a_1' + \sigma_i}{2} - \frac{a_1 + \sigma}{4(i+2)},$$

$a_1' = max\{t_0, log\ N_1/log\ x\}$ *and* $\sigma_i = max\left\{\frac{t_0}{(i+2)}, 1 - a_1 - a_2\right\}$.

Proof. We consider two cases :

Case 1. $F \geq 2U_2^{-2}N_2$

In this case, $N_2 \ll T$, then

$$\begin{aligned} F &\ll min\left\{U_1^{-2}x^{a_1'}, U_2^{-2}T, U_2^{-6}N_2T, W^{-2(i+2)}x^{(i+2)\sigma_i}\right\} \\ &\ll (U_1^{-2}x^{a_1})^{1/2}(W^{-2(i+2)}x^{(i+2)\sigma_i})^{1/2(i+2)}(U_2^{-2}T)^{(2i+1)/4(i+2)}(U_2^{-6}N_2T)^{1/4(i+2)} \\ &\ll U_1^{-1}U_2^{-1}W^{-1}x^{(a_1'+\sigma_i)/2}T^{(i+1)/2(i+2)}N_2^{1/4(i+2)} \\ &\ll U_1^{-1}U_2^{-1}W^{-1}x^{1/4(i+2)}x^{a_1'/2-a_1/4(i+2)}T^{(i+1)/2(i+2)}x^{\sigma_i/2-\sigma/4(i+2)}, \end{aligned}$$

since $N_2 = x/N_1L_0 = x^{1-a_1-\sigma}$.

Then (2.14) follows.

Case 2 $F < 2U_2^{-2}N_2$

If $F < 2U_1^{-2}N_1$, the proof of this lemma is same as Case 1 of Lemma 2.1. If $F \geq 2U_1^{-2}N_1$, then $N_1 \leq T$. Interchanging N_1 with N_2, we may reduce this case to Case 1.

Lemma 2.3. *Let* $k_0 = 3$ *and* $\varepsilon < \sigma = 1 - a_1 - a_2 - a_3 \leq t_0/2$, *then*

$$U_1U_2U_3W \mid S(U_1, U_2, U_3, W) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_2(a_1,a_2,a_3)}, \tag{2.15}$$

where

$$h_2(a_1, a_2, a_3) = \frac{a_1'}{2} + \frac{a_2' + a_3'}{2} + \frac{t_0}{6} + \frac{\sigma}{12}, \tag{2.16}$$

$$a_1' = \frac{log\ N_1'}{log\ x} = max\{a_1, t_0\}, a_2' = \frac{log\ N_2'}{log\ x} = max\left\{a_2, \frac{t_0}{3}\right\}$$

and $a_3' = \frac{log\ N_3'}{log\ x} = max\left\{a_3, \frac{t_0}{3}\right\}$.

Proof. In this case we have that $L_0 \ll T^{1/2}$. Then

$$\begin{aligned} F \leq min\{&U_1^{-2}(N_1 + T), U_2^{-6}(N_2^3 + T), U_3^{-6}(N_3^3 + T), W^{-4}(L_0^2 + T), \\ &W^{-4}L_0^2 + W_0^{-12}L_0^2T, U_3^{-6}N_3^3 + U_3^{-18}N_3^3T, U_1^{-2}N_1'\}. \end{aligned}$$

We consider two cases :

Case 1. $F \leq 2W^{-4}L_0^2$.

Suppose $F \leq 2U_3^{-6}N_3^3$ and $F \leq 2U_2^{-6}N_2^3$. By (2.4), we have that

$$\begin{aligned} F &\ll \; min\,\{U_1^{-2}N_1, W^{-4}L_0^2, U_2^{-6}N_2^3, U_3^{-6}N_3^3\} \\ &\ll \; (U_1^{-2}N_1)^{\frac{1}{2}-\frac{1}{26}}(U_2^{-6}N_2^3)^{\frac{1}{6}-\frac{1}{78}}(U_2^{-6}N_3^3)^{\frac{1}{6}-\frac{1}{78}}(W^{-4}L_0^2)^{\frac{1}{4}-\frac{1}{52}} \\ &\ll \; (U_1U_2U_3W)^{-1+\frac{1}{13}}(N_1N_2N_3L_0)^{\frac{1}{2}-\frac{1}{26}} \ll x^{\frac{1}{2}-\delta}. \end{aligned}$$

Suppose $F > 2U_3^{-6}N_3^3$ and $F > 2U_2^{-6}N_2^3$, then

$$N_2 \ll T^{1/3}, N_3 \ll T^{1/3} \text{ and } F \ll U_2^{-18}N_2^3T,$$

since $2U_i^{-6}N_i^3 \leq F \leq U_i^{-6}(N_i^3 + T), F \leq U_2^{-18}N_2^3T + U_2^{-6}N_2^3$ for $i =$ 2 or $3, L_0 \ll T^{1/2}$ and $N_2 \ll T^{1/3}$. Therefore

$$\begin{aligned} F &\ll \; (U_1^{-2}N_1)^{\frac{1}{2}}(W^{-4}L_0^2)^{\frac{1}{4}}(U_3^{-6}T)^{\frac{1}{6}}(U_2^{-6}T)^{\frac{1}{24}}(U_2^{-18}N_2^3T)^{\frac{1}{24}} \\ &\ll \; (U_1U_2U_3W)^{-1}T^{\frac{1}{4}}N_2^{\frac{1}{8}}N_1^{\frac{1}{2}}L_0^{\frac{1}{2}} \qquad (2.17) \\ &\ll \; (U_1U_2U_3W)^{-1}x^{h_2(a_1,a_2,a_3)} \end{aligned}$$

Suppose $F > 2U_3^{-6}N_3^3$ and $F \leq 2U_2^{-6}N_2^3$, then

$$\begin{aligned} F &\ll \; (U_1^{-2}N_1)^{\frac{1}{2}}(U_2^{-6}N_2^3)^{\frac{1}{6}}(W^{-4}L_0^2)^{\frac{1}{4}}(U_3^{-6}T)^{\frac{1}{24}}(U_3^{-18}N_3^3T)^{\frac{1}{24}} \\ &\ll \; (U_1U_2U_3W)^{-1}N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}L_0^{\frac{1}{2}}T^{\frac{1}{12}}N_3^{\frac{1}{8}} \ll (U_1U_2U_3W)^{-1}x^{h_2(a_1,a_2,a_3)} \qquad (2.18) \end{aligned}$$

since $a_3' = t_0/3$ and $N_3 \ll T^{1/3}$ in this case. Suppose $F < 2U_3^{-6}N_3^3$ and $F \geq 2U_2^{-6}N_2^3$.

On interchanging N_2 with N_3, we may reduce this case to (2.18).

Case 2. $F > 2W^{-4}L_0^2$.

We have that

$$F \ll (U_1^{-2}N_1)^{\frac{1}{2}}(U_2^{-6}N_2'^3)^{\frac{1}{6}}(U_3^{-6}N_3'^3)^{\frac{1}{6}}(W^{-4}T)^{\frac{1}{8}}(W^{-12}L_0^2T)^{\frac{1}{24}},$$

then

$$F \ll (U_1U_2U_3W)^{-1}N_1^{\frac{1}{2}}N_2'^{\frac{1}{2}}N_3'^{\frac{1}{2}}T^{\frac{1}{6}}L_0^{\frac{1}{12}} \ll (U_1U_2U_3W)^{-1}x^{h_2(a_1,a_2,a_3)}.$$

The proof of Lemma 2.3 is now completed.

Lemma 2.4. *If* $k_0 = 3, a_1 \geq t_0, a_2 \geq t_0/2$, *then*

$$U_1U_2U_3W \mid S(U_1, U_2, U_3, W) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_3(a_1,a_2,a_3)}, \tag{2.19}$$

where

$$h_3(a_1, a_2, a_3) = \frac{a_1 + a_2}{2} + \frac{a_3'}{2} + \frac{t_0}{8} + \frac{\sigma}{6},$$

$a_3' = log\ N_3'/log\ x = max\{t_0/4, a_3\}$ *and* $\sigma = 1 - a_1 - a_2 - a_3$.

Proof. We have that

$$F \ll min\{U_1^{-2}N_1, U_2^{-4}N_2^2, U_3^{-8}(N_3^4+T), W^{-6}(L_0^3+T), W^{-18}L_0^3T+W^{-6}L_0^3\}.$$

Suppose $F \leq 2W^{-6}L_0^3$, we are in Case 1 of Lemma 2.1, so that

$$U_1U_2U_3WF \ll x^{\frac{1}{2}-\delta}.$$

Suppose $F > 2W^{-6}L_0^3$, then

$$\begin{aligned} F &\ll (U_1^{-2}N_1)^{\frac{1}{2}}(U_2^{-4}N_2^2)^{\frac{1}{4}}(U_3^{-8}x^{4a_3'})^{\frac{1}{8}}(W^{-6}T)^{\frac{5}{48}}(W^{-18}L_0^3T)^{\frac{1}{48}} \\ &\ll (U_1U_2U_3W)^{-1}N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}x^{\frac{a_3'}{2}}T^{\frac{1}{8}}L_0^{\frac{1}{6}} \ll (U_1U_2U_3W)^{-1}x^{h_3(a_1,a_2,a_3)}. \end{aligned}$$

Thus the lemma follows.

§ 3. $L_0 = L$.

In this section we discuss $\mid S(U_1, \cdots, U_k, W) \mid$ with (1.4), (1.5), (1.6), (1.7) and (1.9). We use the same notations as in § 2 and assume $L_0 = L$.

Lemma 3.1. *Let* $L = L_0$. *If* $L_0 \geq T^{1/2+\epsilon}$; *then*

$$U_1 \cdots U_{k_0}W \mid S(U_1, U_{k_0}, W) \mid \ll x^{\frac{1}{2}-\delta}.$$

Proof. Take j such that

$$N_1^j \cdots N_{k_0}^j \geq T.$$

We have

$$F \ll min\{U_1^{-2j} \cdots U_{k_0}^{-2j}N_1^{2j}, W^{-4}T\}.$$

Thus

$$\begin{aligned} F &\ll (U_1^{-2j}\cdots U_{k_0}^{-2j}N_1^{2j}\cdots N_{k_0}^{2j})^{3/4}(W^{-4}T)^{1/4} \\ &\ll (U_1^{-2j}\cdots U_{k_0}^{-2j}N_1^{2j}\cdots N_{k_0}^{2j})^{1/2j}(W^{-4}T)^{1/4} \\ &\ll (U_1\cdots U_{k_0}W)^{-1}N_1^{1/2}\cdots N_{k_0}^{1/2}T^{1/4}, \end{aligned}$$

since $U_1^{-2j}\cdots U_{k_0}^{-2j}N_1^{2j}\cdots N_{k_0}^{2j}\le 1$. We obtain

$$U_1\cdots U_{k_0}WF \ll x^{1/2-\varepsilon/2}$$

by $L_0 \ge T^{1/2+\varepsilon}$.

Lemma 3.2. *$k_0 = 2, N_1 \ge T, N_2 < T^{1/5}$ and*

$$L_0 N_2^{1/2} \ge T^{1/2+\varepsilon},$$

then

$$U_1U_2W \mid S(U_1,U_2,W) \mid \ll x^{\frac{1}{2}-\delta}.$$

Proof. We have that, choosing j such that $N_2^j \ge T$,

$$F \ll min\{W^{-4}U_2^{-2}T, U_1^{-2}N_1, U_2^{-j}N^j\}.$$

Thus

$$\begin{aligned} F &\ll (W^{-4}U_2^{-2}T)^{1/4}(U_1^{-2}N_1)^{1/2}(U_2^{-2j}N_2^j)^{1/4j} \\ &\ll (U_1U_2W)^{-1}N_1^{1/2}N_2^{1/4}T^{1/4} = (U_1U_2W)^{-1}(x/L_0N_2)^{1/2}N_2^{1/4}T^{1/4} \\ &\ll x^{\frac{1}{2}-\delta}, \end{aligned}$$

since $L_0^{1/2}N_2^{1/4} \ge T^{1/4+\varepsilon/2}$.

Lemma 3.3. *$k_0 = 2, N_1 \ge T, N_2 \ge T^{1/5}$ and*

$$L_0 \ge T^{3/8+\varepsilon}N_2^{1/8},$$

then

$$U_1U_2W \mid S(U_1,U_2,W) \mid \ll x^{\frac{1}{2}-\delta}.$$

Proof. We have that, choosing j such that $N_2^j \ge T$,

$$F \ll min\{W^{-4}U_2^{-2}N_2^{5/4}T^{3/4}, U_1^{-2}N_1, U_2^{-2j}N^j\}.$$

Thus, by $U_2^{-2j}N_2^j \leq 1$,

$$\begin{aligned} F &\ll (W^{-4}U_2^{-2}N_2^{5/4}T^{3/4})^{1/4}(U_1^{-2}N_1)^{1/2}(U_2^{-2j}N_2^j)^{1/4j} \\ &\ll (U_1U_2W)^{-1}N_1^{1/2}N_2^{9/16}T^{3/16} = (U_1U_2W)^{-1}(x/L_0N_2)^{1/2}N_2^{9/16}T^{3/16} \\ &\ll x^{1/2-\delta}. \end{aligned}$$

Write

$$c_1 = \frac{1}{2j_1} + \frac{1}{2j_2} - \frac{3}{10}, c_2 = 7 + \frac{5}{j_2} + \frac{5}{j_1}, c_3 = \frac{7}{8} - \frac{3}{4j_2} - \frac{3}{4j_3} - \frac{3}{8j_4},$$

$$c_4 = \frac{1}{4}\left(\frac{1}{j_2} + \frac{1}{j_3} + \frac{1}{2j_4} - \frac{1}{2}\right), c_5 = \frac{3}{8} - \frac{1}{4j_2} - \frac{3}{4j_3} - \frac{3}{8j_4},$$

and

$$c_6 = \frac{1}{4j_2} + \frac{1}{4j_3} + \frac{1}{8j_4} - \frac{1}{8}.$$

Define

$$N_i' = max\left\{N_i, T^{\frac{1}{j_i}}\right\}, i = 3 \text{ and } 4. \tag{3.1}$$

Lemma 3.4. *Let $K_0 = 4, a_3 \geq a_4, t_0/3 \geq a_4 \geq t_0/5, \sigma = 1 - a_1 - a_2 - a_3 - a_4 \geq t_0/j_1, j_1$ and j_2, j_3, j_4 be integers with $c_i \geq 0$ $(1 \leq i \leq 6)$, then*

$$U_1U_2U_3U_4W \mid S(U_1, U_2, U_3, U_4, W) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_4(a_1,a_2,a_3,a_4)} \tag{3.2}$$

where

$$h_4(a_1, a_2, a_3, a_4) = \frac{a_3'}{2} + \frac{a_4'}{4} + \frac{5a_4}{16} + \frac{3t_0}{16}$$
$$+max\left\{\frac{a_1}{2} + c_6j_2a_2 + (c_5 + c_6)t_0, c_4a_1 + \frac{a_2}{2} + (c_3 + c_4)t_0\right\}.$$

Proof. We have that, by $t_0/3 \geq a_4 \geq t_0/5, \sigma \geq t_0/5$ and $a_3 \geq a_4$,

$$\begin{aligned} F \ll\ & min\{W^{-4}U_4^{-2}N_4^{\frac{5}{4}}T^{\frac{3}{4}}, U_1^{-2}(N_1 + T), U_1^{-6}N_1T + U_1^{-2}N_1, U_2^{-2j_2}(N_2^{j_2} + T) \\ & U_2^{-2j_2}N_2 + U_2^{-6j_2}N_2^{j_2}T, U_3^{-10}N_3^5, U_4^{-10}N_4^5, W^{-2j_1}L_0^{j_1}, U_3^{-2j_3}N_3'^{j_3}, \\ & U_4^{-2j_4}(N_4^{j_4} + T)\}. \end{aligned}$$

We discuss following cases

Case 1. $F \le 2U_1^{-2}N_1$ and $F \le 2U_2^{-2j_2}N_2^{j_2}$.

By (2.4), we have that

$$\begin{aligned} F &\ll (U_1^{-2}N_1)^{\frac{1}{2}-\frac{5c_1}{c_2}} \left(U_2^{-2j_2}N_2^{j_2}\right)^{\frac{1}{2j_2}-\frac{5c_1}{j_2c_2}} (U_3^{-10}N_3^5)^{\frac{1}{10}-\frac{c_1}{c_2}} (U_4^{-10}N_4^5)^{\frac{1}{10}-\frac{c_1}{c_2}} \times \\ &\times \left(W^{-2j_1}L_0^{j_1}\right)^{\frac{1}{2j_1}-\frac{5c_1}{j_1c_2}} \\ &\ll (U_1U_2U_3U_4W)^{-1+\frac{10c_1}{c_2}}(N_1N_2N_3N_4L_0)^{\frac{1}{2}-\frac{5c_1}{c_2}} \ll (U_1U_2U_3U_4W)^{-1}x^{\frac{1}{2}-\delta}, \end{aligned}$$

since $c_1/c_2 < 1/10$.

Case 2. $F > 2U_1^{-2}N_1$ and $F \le 2U_2^{-2j_2}N_2^{j_2}$.

We use $F \ll U_i^{-2j_i}N_i'^{j_i}$ for $i = 3$ or 4. We have that, by $2c_3 + 6c_4 = 1$,

$$\begin{aligned} F &\ll \left(W^{-4}U_4^{-2}N_4^{\frac{5}{4}}T^{\frac{3}{4}}\right)^{\frac{1}{4}} \left(U_4^{-2j_4}N_4'^{j_4}\right)^{\frac{1}{4j_4}} \left(U_3^{-2j_3}N_3'^{j_3}\right)^{\frac{1}{2j_3}} \left(U_2^{-2j_2}N_2^{j_2}\right)^{\frac{1}{2j_2}} \times \\ &\times (U_1^{-2}T)^{c_3}(U_1^{-6}N_1T)^{c_4} \\ &\ll (U_1U_2U_3U_4W)^{-1}N_1^{c_4}N_2^{\frac{1}{2}}N_3'^{\frac{1}{2}}N_4'^{\frac{1}{4}}N_4^{\frac{5}{16}}T^{\frac{3}{16}+c_3+c_4} \\ &\ll (U_1U_2U_3U_4W)^{-1}x^{h_4(a_1,a_2,a_3,a_4)}, \end{aligned} \tag{3.3}$$

Case 3. $F \le 2U_1^{-2}N_1$ and $F > 2U_2^{-2j_2}N_2^{j_2}$.

Then we have that

$$\begin{aligned} F &\ll \left(W^{-4}U_4^{-2}N_4^{\frac{5}{4}}T^{\frac{3}{4}}\right)^{\frac{1}{4}} \left(U_4^{-2j_4}N_4'^{j_4}\right)^{\frac{1}{4j_4}} \left(U_3^{-2j_3}N_3'^{j_3}\right)^{\frac{1}{2j_3}} (U_1^{-2}N_1)^{\frac{1}{2}} \times \\ &\times \left(U_2^{-2j_2}T\right)^{c_5} \left(U_2^{-6j_2}N_2^{j_2}T\right)^{c_6} \\ &\ll (U_1U_2U_3U_4W)^{-1}N_1^{\frac{1}{2}}N_2^{j_2c_6}N_3'^{\frac{1}{2}}N_4'^{\frac{1}{4}}N_4^{\frac{5}{16}}T^{\frac{3}{16}+c_5+c_6} \\ &\ll (U_1U_2U_3U_4W)^{-1}x^{h_4(a_1,a_2,a_3,a_4)}, \end{aligned} \tag{3.4}$$

as required.

Case 4. $F > 2U_1^{-2}N_1$ and $F > 2U_2^{-2j_2}N_2^{j_2}$.

In this case we have that $N_1 \ll T$ and $N_2 \ll T^{1/j_2}$. Thus (3.3) and (3.4) are true again, i.e. we get (3.1) again.

If take $j_2 = 2$ and $j_3 = j_4 = 5$, then

$$h_4(a_1, a_2, a_3, a_4) = \frac{a_3'}{2} + \frac{a_4'}{4} + \frac{5a_4}{16} + \frac{3t_0}{16} + max\left\{\frac{a_1}{2} + \frac{3}{20}a_2 + \frac{t_0}{10}, \frac{3}{40}a_1 + \frac{a_2}{2} + \frac{7}{20}t_0\right\} \tag{3.5}$$

Lemma 3.5. *Suppose* $k_0 = 4, a_4 \leq t_0/5, a_i \geq t_0/j_i, (i = 2, 3 \text{ and} 4)$ *and* $\sigma \geq t_0/j$ *with*

$$\frac{1}{j_2} + \frac{1}{j_3} + \frac{1}{j_4} + \frac{1}{j} > \frac{1}{2},$$

then

$$U_1U_2U_3U_4W \mid S(U_1, U_2, U_3, U_4, W) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_5(a_1,a_2,a_3,a_4)}$$

where

$$h_5(a_1, a_2, a_3, a_4) = \frac{a_3'}{2} + \frac{a_4'}{4} + \frac{t_0}{4} +$$

$$max\left\{\frac{a_1}{2} + c_6j_2a_2 + (c_5 + c_6)t_0, a_1 + \frac{a_2}{2} + (c_3 + c_4)t_0\right\}. \tag{3.7}$$

Proof. In the proof of Lemma 3.2 we replace (1.10) by (1.11). Then we replace $5a_4/16$ by $t_0/16$ and we get $h_5(a_1, a_2, a_3, a_4)$ replace $h_4(a_1, a_2, a_3, a_4)$. The lemma follows.

Lemma 3.6. *If* $k_0 = 3, a_1 \geq t_0$ *and* $t_0/3 \geq a_3 \geq t_0/5$, *then*

$$U_1U_2U_3W \mid S(U_1, U_2, U_3, W) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_6(a_1,a_2,a_3)}$$

where

$$h_6(a_1, a_2, a_3) = \frac{13t_0}{48} + \frac{a_1 + a_2'}{2} + \frac{5a_3}{16}.$$

and $a_2' = max\{a_2, t_0/3\}$.

Proof. We have

$$\begin{aligned} F &\ll \left(W^{-4}U_3^{-2}N_3^{\frac{5}{4}}T^{\frac{3}{4}}\right)^{\frac{1}{4}}\left(U_1^{-2}N_1\right)^{\frac{1}{2}}\left(U_2^{-6}x^{3a_2'}\right)^{\frac{1}{6}}\left(U_3^{-6}T\right)^{\frac{1}{12}} \\ &\ll (U_1U_2U_3W)^{-1}N_1^{\frac{1}{2}}N_3^{\frac{5}{16}}x^{\frac{a_2'}{2}}T^{\frac{13}{48}} \ll (U_1U_2U_3W)^{-1}x^{h_6(a_1,a_2,a_3,a_4)}. \end{aligned}$$

Thus the lemma follows.

Lemma 3.7. *If* $k_0 = 4$ *and the* a_i*'s satisfy : (1)* $a_1 + a_3 + a_4 \geq t_0$; *(2)*

$a_2 + a_3 + a_4 \le t_0$; *(3)* $a_1 + a_3 \le t_0$, *(4)* $a_1 + a_4 \le t_0$; *and (5)* $a_4 \le a_3 \le t_0/5$; *then*

$$U_1U_2U_3U_4W \mid S(U_1, U_2, U_3, U_4, W) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_7(a_1,a_2,a_3,a_4)},$$

where

$$h_7(a_1, a_2, a_3, a_4) = t_0 + a_2/8.$$

Proof. W.L.O.G. we can agree that $U_3 \ge U_4$, then we have

$$F \le min\{W^{-4}U_3^{-2}T, U_1^{-2}U_3^{-2}T, U_2^{-2}U_3^{-2}U_4^{-2}T, U_2^{-2}N_2 + U_2^{-6}N_2T, U_1^{-2}U_3^{-2}U_4^{-2}N_1N_3N_4\}.$$

If $F \le 2U_2^{-2}N_2$, this is Case 1 of Lemma 2.1; if $F \ge 2U_2^{-2}N_2$, from $U_3 \ge U_4$, we have

$$\begin{aligned} F &\ll (W^{-4}U_3^{-2}T)^{\frac{1}{4}}(U_1^{-2}U_3^{-2}T)^{\frac{1}{2}}(U_2^{-2}U_3^{-2}U_4^{-2}T)^{\frac{1}{8}}(U_2^{-6}N_2T)^{\frac{1}{8}} \\ &\ll (U_1U_2U_3U_4W)^{-1}TN_2^{\frac{1}{8}}U_3^{-\frac{3}{4}}U_4^{\frac{3}{4}} \ll (U_1U_2U_3U_4W)^{-1}x^{h_7(a_1,a_2,a_3,a_4)}, \end{aligned}$$

and the lemma follows.

Lemma 3.8. *Suppose that* $k_0 = 6, t_0/3 \ge a_1 \ge \cdots \ge a_6 \ge t_0/5$ *and* $\sigma + a_1 + a_2 \ge t_0$, *then*

$$U_1U_2 \cdots U_6W \mid S(U_1, \cdots, U_6) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_8(a_1,\cdots,a_6)}$$

where

$$h_8(a_1, \cdots, a_6) = \frac{19}{48}a_6 + \frac{a_5}{12} + \frac{a_4}{12} + \frac{15}{16}t_0.$$

Proof. If $F \le U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4$, then

$$\begin{aligned} F &\ll (U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4)^{\frac{1}{2}-\frac{1}{22}}(W^{-2}U_1^{-2}U_2^{-2}L_0N_1N_2)^{\frac{1}{2}-\frac{1}{22}}(U_3^{-10}N_3^5)^{\frac{1}{10}-\frac{1}{110}} \\ &\ll (WU_6 \cdots U_1)^{-1+\frac{1}{11}}(L_0N_6 \cdots N_1)^{\frac{1}{2}-\frac{1}{22}} \ll x^{\frac{1}{2}-\delta}. \end{aligned}$$

If $F > U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4$, then

$$F \le U_6^{-6}U_5^{-6}U_4^{-6}N_6N_5N_4T,$$

and

$$\begin{aligned} F &\ll \left(W^{-4}U_6^{-2}N_6^{\frac{5}{4}}T^{\frac{3}{4}}\right)^{\frac{1}{4}} \left(U_1^{-2}U_2^{-2}U_3^{-2}T\right)^{\frac{1}{2}} \left(U_4^{-6}U_5^{-6}U_6^{-6}N_4N_5N_6T\right)^{\frac{1}{12}} \times \\ &\times \left(U_4^{-4}U_5^{-2}T\right)^{\frac{1}{8}} \left(U_5^{-6}T\right)^{\frac{1}{24}} \\ &\ll (WU_1\cdots U_6)^{-1}T^{\frac{15}{16}}N_4^{\frac{1}{12}}N_5^{\frac{1}{12}}N_6^{\frac{19}{48}} \ll (WU_1\cdots U_6)^{-1}x^{h_8(a_1,\cdots,a_6)}. \end{aligned}$$

Lemma 3.9. *Suppose that* $k_0 = 4, a_1 \geq a_2 \geq a_3 \geq a_4, a_1 + a_3 \leq t_0 < a_1 + a_3 + a_4, \sigma \geq t_0/3$ *and* $2a_4 \leq t_0/3$, *then*

$$U_1\cdots U_4W \mid S(U_1,\cdots,U_4) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_9(a_1,\cdots,a_4)}$$

where

$$h_9(a_1,\cdots,a_4) = \frac{15}{16}t_0 + \frac{a_2}{8} + \frac{5a_4}{8}.$$

Proof. If $F \leq 2U_2^{-2}N_2$, then

$$F \ll \left(U_1^{-2}U_3^{-2}U_4^2N_1N_3N_4\right)^{\frac{1}{2}-\frac{1}{30}} \left(U_2^{-2}N_2\right)^{\frac{1}{2}-\frac{1}{30}} \left(W^{-6}L_0^3\right)^{\frac{1}{6}-\frac{1}{10}} \ll x^{\frac{1}{2}-\delta}$$

by (2.4).

If $F > 2U_2^{-2}N_2$, then $F \ll min\{U_2^{-2}T, U_2^{-6}N_2T\}$. Thus

$$\begin{aligned} F &\ll \left(W^{-4}U_4^{-4}N_4^{\frac{5}{2}}T^{\frac{3}{4}}\right)^{\frac{1}{4}} \left(U_1^{-2}U_3^{-2}T\right)^{\frac{1}{2}} \left(U_2^{-2}T\right)^{\frac{1}{8}} \left(U_2^{-6}N_2T\right)^{\frac{1}{8}} \\ &\ll T^{\frac{15}{16}}N_2^{\frac{1}{8}}N_4^{\frac{5}{8}} \ll x^{h_9(a_1,\cdots,a_4)}, \end{aligned}$$

as required.

Lemma 3.10. *Suppose that* $k_0 = 3, a_2 + a_3 \leq t_0, a_1 + \sigma \geq t_0$ *and* $a_3 \leq t_0/5$, *then*

$$U_1\cdots U_4W \mid S(U_1,\cdots,U_4) \mid \ll x^{\frac{1}{2}-\delta} + x^{h_{10}(a_1,\cdots,a_4)}$$

where

$$h_{10}(a_1,\cdots,a_4) = t_0 + \frac{a_2}{8} + \frac{a_3}{24}.$$

Proof. We have that

$$F \ll min\{W^{-4}U_3^{-2}T, U_1^{-2}T, U_2^{-2}U_3^{-2}(T+N_2N_3), U_2^{-6}U_3^{-6}N_2N_3T+U_2^{-2}U_3^{-2}N_2N_3,$$

$$W^{-2}U_1^{-2}L_0N_1\}.$$

If $F \le 2U_2^{-2}U_3^{-2}N_2N_3$, then we go back to the Case 1 on Lemma 3.1, since $F \ll W^{-2}U_1^{-2}L_0N_1$, and $N_4 \ge x^{\varepsilon}$; if $F > 2U_2^{-2}U_3^{-2}N_2N_3$, then

$$\begin{aligned} F &\ll (W^{-4}U_3^{-2}T)^{\frac{1}{4}} (U_1^{\ 2}T)^{\frac{1}{2}} (U_2^{-2}U_3^{-2}T)^{\frac{1}{8}} (U_2^{-6}U_3^{-6}N_2N_3T)^{\frac{1}{24}} (U_2^{-6}N_2T)^{\frac{1}{12}} \\ &\ll (U_1U_2U_3W)^{-1}x^{h_{10}(a_1,a_2,a_3)}, \end{aligned}$$

as required.

§ 4. SET $E(\theta)$.

Let $\theta_j (1 \le j \le r)$ be positive numbers with

$$\sum_{1\le i\le r} \theta_i < 1.$$

Denote $\{\theta_j\} = \{\theta_0, \theta_1, \cdots, \theta_r\}$ and

$$\theta_0 = 1 - \sum_{1\le i\le r} \theta_i.$$

Divide the set of numbers

$$\{\theta_0, \theta_1, \cdots, \theta_r\}$$

into three subsets and call the sums of terms in each subset a_1, a_2 and σ, where σ is distinguished by being θ_0 if $\sigma > t_0/2$, and otherwise $\sigma \le t_0/2$. Since $a_1 + a_2 + \sigma = 1$, any two of a_1, a_2, σ determine the third. We attach an exactly similar meaning to $\{a_1, a_2, a_3, \sigma\}$ and $\{a_1, a_2, a_3, a_4, \sigma\}$. We refer to $\{a_1, a_2, \sigma\}$, or $\{a_1, a_2, a_3, \sigma\}$, or $\{a_1, a_2, a_3, a_4, \sigma\}$ as set of complementary partial sums. For $\{a_1, \cdots, a_k, \sigma\}$ if there exists some $h_i(a_1, \cdots, a_k)$ which satisfies (1.8) with

$$h_i(a_1, \cdots, a_k) < 1/2,$$

then we call it $\{a_1, \cdots, a_k, \sigma\} \in E(\theta)$. For short, we write that $\{\theta_j\} \in E(\theta)$ instead of $\{a_1, \cdots, a_k, \sigma\} \in E(\theta)$.

Lemma 4.1. *If there exists at least one set of complementary partial sums*

$\{a_1, a_2, \sigma\}$ *(or* $\{a_1, a_2, a_3, \sigma\}$*) of* $\{\theta_j\}$ *such that at least one of conditions* (4.1.1) - (4.1.4) *(or* (4.1.5)*) holds, then* $\{\theta_j\} \in E(\theta)$.

(4.1.1) $a_1 \geq a_2 \geq t_0$ *and* $\sigma > \varepsilon$;

This situation is covered by (2.9) *and* (2.10) *of Lemma* 2.1

(4.1.2) $a_2 \leq a_1 \leq t_0$ *and* $a_2 < 4 - 8t_0$;

Using (2.11) *of Lemma* 2.1 *with* $a_1' = t_0$ *and* $j = 2$, *we have that*

$$h_0(a_1, a_2) = t_0 + min\{a_2/8, \sigma/6 + a_2/24\} \leq t_0 + a_2/8 < 1/2.$$

(4.1.3) $a_1 \geq t_0, a_2 > \frac{2i+2}{2i+3} t_0$ and $\sigma \geq \frac{t_0}{i+2}$;

In this case, we use Lemma 2.2, $\sigma_i = max\{t_0/(i+2), 1 - a_1 - a_2\} = max\{t_0/(i+2), \sigma\} = \sigma$, *and* $a_1' = a_1$, *then*

$$\begin{aligned} h_1^i(a_1, a_2) &= \tfrac{i+1}{2(i+2)} t_0 + \tfrac{1}{4(i+2)} + \tfrac{2i+3}{4(i+2)}(a_1 + \sigma) \\ &= \tfrac{i+1}{2(i+2)} t_0 + \tfrac{1}{4(i+2)} + \tfrac{2i+3}{4(i+2)}(1 - a_2) < \tfrac{1}{2}. \end{aligned}$$

(4.1.4) $a_3 \leq a_2 \leq t_0/3 \leq a_1 \leq t_0, \sigma = 1 - a_1 - a_2 - a_3$ *and* $a_1 + \sigma/6 < 1 - t_0$.

By Lemma 2.3 *with* $a_2' = max\{t_0/3, a_2\} = t_0/3, a_3' = max\{t_0/3, a_3\} = t_0/3$; *then*

$$\begin{aligned} h_2(a_1, a_2, a_3) &= a_1/2 + (a_2' + a_3')/2 + t_0/6 + \sigma/12 \\ &= a_1/2 + t_0/2 + \sigma/12 < 1/2. \end{aligned}$$

Lemma 4.2. *If there exists at least one set of complementary partial sums* $\{a_1, a_2, a_3, \sigma\}$ *of* $\{\theta_j\}$ *such that at least one of conditions* (4.2.1) - (4.2.3) *holds, then* $\{\theta_j\} \in E(\theta)$.

(4.2.1). $a_1 \geq t_0, a_2 \geq t_0/2, a_3 \geq t_0/4$ and $\sigma > 2t_0/7$.

In this case, Lemma 2.4 is applied, we have $a_3' = max\{a_3, t_0/4\} = a_3$, *then*

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2 + a_3)/2 + t_0/8 + \sigma/16 \\ &= 1/2 + t_0/8 - 7\sigma/16 < 1/2, \end{aligned}$$

since $\sigma > 2t_0/7$.

(4.2.2). $a_1 \geq t_0, a_2 \geq a_3 \geq t_0/3$ and $\sigma > 2t_0/5$.

Using Lemma 2.3, we have $a_2' = max\{t_0/3, a_2\} = a_2$ *and* $a_3' = max\{t_0/3, a_3\} = a_3$, *then*

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2 + a_3)/2 + t_0/6 + \sigma/12 \\ &= (1 - \sigma)/2 + t_0/6 + \sigma/12 \\ &= 1/2 + t_0/6 - 5\sigma/12 < 1/2 \end{aligned}$$

since $\sigma > 2t_0/5$.

(4.2.3). $a_1 \geq t_0, a_2 \geq t_0/3 \geq a_3$ *and* $t_0/2 \geq \sigma > max\{2t_0/5, 4t_0/5 - 6a_3/5\}$.

We use Lemma 2.3 with $a_2' = max\{t_0/3, a_2\} = a_2$, *and* $a_3' = max\{t_0/3, a_3\} = t_0/3$, *then*

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2)/2 + t_0/6 + t_0/6 + \sigma/12 \\ &= (1 - \sigma - a_3)/2 + t_0/3 + \sigma/12 \\ &= 1/2 + t_0/3 - 5\sigma/12 - a_3/3 < 1/2 \end{aligned}$$

since $\sigma > 4t_0/5 - 6a_3/5$.

Lemma 4.3. *Suppose* $L = L_0$ *(i.e. (1.9) holds). If* $\sigma > t_0/2$; *then* $\{\theta_j\} \in E(\theta)$.

It is covered by Lemma 3.1.

For a fixed σ, denote

$$M_\sigma = \sup_{\substack{\{a_1, a_2, \sigma\} \in \{\theta\} \\ a_2 \leq t_0 \leq a_1}} \{a_1\} \text{ and } m_\sigma = \inf_{\substack{\{a_1, a_2, \sigma\} \in \{\theta\} \\ a_2 \leq t_0 \leq a_1}} \{a_2\}.$$

Lemma 4.4. *Suppose that* $\{\theta_j\}$ *satisfies the condition* (4.4.1) : (4.4.1) *all of complementary partial sums* $\{a_1, a_2, \sigma\}$ *with* $a_2 \leq a_1 \leq t_0$ *satisfies* $\{\theta_j\} \in E(\theta)$, ($a_2 \leq a_1$ *is not necessary). Then for a fixed* σ, $\{\theta_j\} \in E(\theta)$ *if*

$$m_\sigma < a_2 < M_\sigma. \tag{4.4.2}$$

Moreover, we have that

$$m_\sigma + a_2 + M_\sigma = 1, \tag{4.4.3}$$

$$m_\sigma < t_0 < M_\sigma; \tag{4.4.4}$$

for $t_0/j \geq \sigma \geq 2t_0/(2j+1)$,

$$M_\sigma \geq 1 - \frac{6j}{6j-1}t_0 - \frac{2j-1}{6j-1}\sigma, \tag{4.4.5}$$

$$m_\sigma \leq \frac{6j}{6j-1}t_0 - \frac{4j}{6j-1}\sigma; \tag{4.4.6}$$

and for $2t_0/(2j+1) \geq \sigma \geq t_0/(j+1)$,

$$M_\sigma \geq 1 - \frac{2j}{2j+1}t_0 - \sigma, \tag{4.4.7}$$

$$m_\sigma \leq \frac{2j}{2j+1}t_0. \tag{4.4.8}$$

Proof. We use (4.13) with $j = i+1$ to prove (4.4.8). Then we have (4.4.7) by (4.4.3).

For $\theta > 11/20$, we can get much simple conditions for $\{a_1, a_2, \sigma\} \in E(\theta)$. We will use it to discuss the gap between consecutive primes in [3].

Lemma 4.5. *Suppose that* $t_0 < 9/20$. *If there exists at least one set of complementary partial sums* $\{a_1, a_2, \sigma\}$ *with* $a_1 \leq t_0$, *and* $a_2 \leq t_0$, *then* $\{\theta_j\} \in E(\theta)$. *Moreover, we have that*

$$M_\sigma - m_\sigma > \sigma, \quad \text{if} \quad \sigma < 0.1; \tag{4.5.1}$$

and

$$M_\sigma - m_\sigma > 1 - 2t_0, \quad \text{if} \quad \sigma \geq 0.1. \tag{4.5.2}$$

Proof. If $a_2 \leq a_1$, and $a_2 < 4-8t_0$, it is same as (5.1.8). If $a_1 \geq a_2 \geq 4-8t_0$, using (2.11) of Lemma 2.1 with $j = 2$, we have

$$\begin{aligned} h_0(a_1, a_2) &= t_0 + min\{a_2/8, \sigma/6 + a_2/24\} \\ &\leq t_0 + (4-8t_0)/24 + (1 - 2(4-8t_0))/6 < 1/2. \end{aligned}$$

By (4.4.4), (4.4.5) and (4.4.6), we have that

$$M_\sigma - m_\sigma \geq 1 - \frac{28}{29}t_0 - \frac{9}{29}\sigma - \left(\frac{28}{29}t_0 - \frac{20}{29}t_0\right) \geq \sigma, \tag{4.5.3}$$

if $\sigma < t_0/5$. When $t_0/5 \le \sigma < 0.1$, we have that

$$M_\sigma - m_\sigma \ge 1 - \frac{8}{9}t_0 - \sigma - \frac{8}{9}t_0 > 1 - 2t_0.$$

Then (4.5.1) holds. When $t_0/(j+1) \le \sigma \le 2t_0/(2j+1)$, by (4.4.4) and (4.4.6),

$$M_\sigma - m_\sigma \ge 1 - \frac{2j}{2j+1}t_0 - \sigma - \frac{2j}{2j+1}t_0 \ge 1 - 2t_0. \tag{4.5.4}$$

When $2t_0/(2j+1) \le \sigma \le t_0/(j+1)$, by (4.4.5) and (4.4.7)

$$M_\sigma - m_\sigma \ge 1 - \frac{6j}{6j-1}t_0 - \frac{2j-1}{6j-1}\sigma - \left(\frac{6j}{6j-1}t_0' - \frac{4j}{6j-1}\sigma\right) \ge 1 - 2t_0 \tag{4.5.5}$$

Thus (4.5.2) holds.

For $\theta > 6/11$, we can get some simple conditions for $\{a_1, a_2, \sigma\} \in E(\theta)$. We will use it to discuss the gap between consecutive primes in [3]. Moreover, we have that

Corollary 4.5.1. *Suppose that* $\theta > 6/11$. (4.5.1) *and* (4.5.2) *hold. If* $a_2 \le a_1 \le 1/2$ *and* $\sigma = 1 - a_1 - a_2 < 1/2 - 8t_0/9$, *then* $\{\theta_j\} \in E(\theta)$.

Proof. If $a_2 \ge t_0$, it is covered by (4.1.1); if $a_1 < t_0$, we have that $\{\theta_j\} \in E(\theta)$ by (4.5.1); if $a_2 \le t_0 < a_1 < 1/2$ and $\sigma \ge t_0/5$, then

$$a_2 = 1 - a_1 - \sigma > 1 - 1/2 - (1/2 - 8t_0/9) = 8t_0/9,$$

thus $\{\theta_j\} \in E(\theta)$ by (4.1.3). If $a_2 \le t_0 < a_1 < 1/2$ and $\sigma < t_0/5$, using (2.11) in Lemma 2.1 with $j = 5$, then

$$\begin{aligned} h_0(a_1, a_2) &= t_0/2 + a_1/2 + min\{a_2/8, \sigma/6 + a_2/60\} \\ &\le t_0/2 + a_1/2 + \sigma/6 + a_2/60 \\ &= t_0/2 + a_1/2 + \sigma/6 + (1 - a_1 - \sigma)/60 \\ &= t_0/2 + 1/60 + 29a_1/60 + 3\sigma/20 \\ &\le t_0/2 + 1/60 + 29/120 + 3t_0/100 < 1/2, \end{aligned}$$

since $a_1 \le 1/2$ and $\sigma \le t_0/5$.

Lemma 4.6. *Suppose $t_0 < 5/11$. If there exists at least one set of complementary partial sums $\{a_1, a_2, \sigma\}$ (or $\{a_1, a_2, a_3, \sigma\}$) of $\{\theta_j\}$ such that at least one of conditions (4.6.1) - (4.6.5) holds, then $\{\theta_j\} \in E(\theta)$.*
(4.6.1) *$a_2 \le a_1 \le t_0$ and $\sigma < 1 - 20t_0/11$.*

Proof. When $\sigma \ge t_0/3$, we apply Lemma 2.2 with $i = 1$, $\sigma_1 = max\{t_0/3, 1 - a_1 - a_2\} = max\{t_0/3, \sigma\} = \sigma$ and $a_1' = t_0$, then

$$\begin{aligned} h_1^i(a_1, a_2) &= t_0/3 + 1/12 + t_0/2 + \sigma/2 - (a_1 + \sigma)/12 \\ &\le 5t_0/6 + 1/12 + 5\sigma/12 - a_1/12 \\ &\le 5t_0/6 + 1/24 + 11\sigma/24 \\ &< 5t_0/6 + 1/24 + 11(1 - 20t_0/11)/24 = 1/2, \end{aligned}$$

since $a_1 \ge (1 - \sigma)/2$. ($a_1 \ge a_2$ and $a_1 + a_2 + \sigma = 1$).

When $\sigma < t_0/3$, we turn to (2.10) of Lemma 2.1 with $j = 2$

$$\begin{aligned} h_0(a_1, a_2) &\le t_0 + \sigma/6 + a_2/24 < t_0 + \sigma + (1 - \sigma)/48 \\ &\le t_0 + \tfrac{1}{48} + 7\sigma/48 \le 1/2. \end{aligned}$$

(4.6.2). $a_1 \ge a_2 > (2i + 2)t_0/(2i + 3)$ and $1 - 20t_0/11 > \sigma \ge t_0/i$.

Proof. If $a_2 \ge t_0$, it is already covered by (4.1.1). If $a_1 \ge t_0 > a_2 > (2i + 2)t_0/(2i + 3)$, using (4.1.3). If $t_0 > a_1 \ge a_2 > 8t_0/9$, using (4.2.1).

We can write (4.2.2) to be

(4.6.3). $t_0 \ge a_1 > (2i + 2)t_0/(2i + 3)$ and $1 - 20t_0/11 > \sigma \ge t_0/i$. (It is not necessary $a_2 \le a_1$).

Proof. If $a_2 \ge t_0$, we are back to (4.1.3). If $a_1 < t_0$, it is covered by (4.2.2).

(4.6.4). $t_0 \le a_1 \le 1/2$, and $\varepsilon < \sigma < 1/2 - 8t_0/9$;

Proof. If $\sigma \ge t_0/5$, then $a_2 = 1 - \sigma - a_1 > 8t_0/9$, $\{\theta_j\} \in E(\theta)$ by (4.1.3) with $i = 3$. If $\sigma < t_0/5$, we apply (2.10) with $j = 5$ in Lemma 2.1. We have

$$\begin{aligned} h_0(a_1, a_2) &= t_0/2 + a_1/2 + \sigma/6 + a_2/60 \\ &= t_0/2 + a_1/2 + \sigma/6 + (1 - \sigma - a_1)/60 \\ &< t_0/2 + 29/120 + (9/60)(1/2 - 8t_0/9) + 1/60 < 1/2. \end{aligned}$$

(4.6.5). $a_2 \le a_1 \le 1/2$ and $\sigma < 1/2 - 8t_0/9$.

Proof. If $a_2 \ge t_0$, it is covfered by (4.1.1). If $a_1 < t_0, \{\theta_j\} \in E(\theta)$ by (4.1.5) since $\sigma < 1/2 - 8t_0/9 < 1 - 20t_0/11$. If $a_2 \le t_0 < a_1 < 1/2$, the proof is same as (4.5.6).

From Lemma 4.3 and (4.6.1), we have, for $\sigma < 1 - 20t_0/11$, (4.4.3) - (4.4.8) hold. Moreover, (4.5.3), (4.5.4) and (4.5.5) with $t_0 < 5/11$ can imply that

$$M_\sigma - m_\sigma > \sigma, \text{ if } \sigma < t_0/5; \tag{4.6.6}$$

and

$$M_\sigma - m_\sigma > t_0/5, \text{ if } \sigma \ge t_0/5 \tag{4.6.7}$$

since $1 - 2t_0 > t_0/5$.

(4.6.6) $t_0 \le a_1 \le 1/2$ and $a_2 < 2 - 4t_0$;

In this case, we use Lemma 2.1. By (2.11) with $j = 2$ and $a_1' = a_1$,

$$h_0(a_1, a_2) \le a_1/2 + t_0/2 + a_2/8 \le 1/4 + t_0/2 + t_0/20 < 1/2.$$

Lemma 4.7. *Suppose that $t_0 < 5/11$ and $L = L_0$. If there exists at least one set of complementary partial sums $\{a_1, a_2, \sigma\}$ (or $\{a_1, a_2, a_3, \sigma\}$) of $\{\theta_j\}$ such that at least one of conditions (4.7.1) - (4.7.6) holds, then $\{\theta_j\} \in E(\theta)$.*

(4.7.1). $k_0 = 3, 1/2 \ge a_1 \ge t_0, 2t_0/5 \ge a_2 \ge t_0/3$, and $t_0/4 \ge a_3 \ge t_0/5$.

Proof. We use Lemma 3.6 with $a_2' = max\{a_2, t_0/3\} = a_2$; then

$$\begin{aligned} h_6(a_1, a_2, a_3) &= 13t_0/48 + (a_1 + a_2)/2 + 5a_3/16 \\ &\le 13t_0/48 + 1/4 + t_0/5 + 5t_0/64 < 1/2. \end{aligned}$$

(4.7.2). $k_0 = 4, a_1 \leq 8t_0/9, a_2 \leq 4t_0/9,$ *and* $a_4 \leq a_3 \leq t_0/4$.

Proof. Using (3.5), we have that, $a_3' = a_4' = t_0/4$ and

$$h_4(a_1, a_2, a_3, a_4) = \frac{29}{64}t_0 + max\left\{\frac{3}{40}a_1 + \frac{a_2}{2} + \frac{7t_0}{20}, \frac{a_1}{2} + \frac{3a_2}{20} + \frac{t_0}{10}\right\} < \frac{1}{2}.$$

(4.7.3). $k_0 = 4, a_1 + a_3 + a_4 \geq t_0, a_2 + a_3 + a_4 \leq t_0, a_1 + a_3 \leq t_0, a_1 + a_4 \leq t_0, a_4 \leq a_3 \leq t_0/5$ *and* $a_2 \leq 4 - 8t_0$.

Proof. Using Lemma 3.7, since $a_2 \leq 4 - 8t_0$, we have

$$h_7(a_1, a_2, a_3, a_4) = t_0 + a_2/8 < 1/2.$$

(4.7.4). $k_0 = 6, 2t_0/7 \geq a_1 \geq \cdots \geq a_6 \geq t_0/5$ and $\sigma + a_1 + a_2 \geq t_0$.

Proof. Using Lemma 3.8, we have that

$$h_8(a_1, \cdots, a_6) = \frac{15}{16}t_0 + \frac{a_4}{12} + \frac{a_5}{12} + \frac{19}{48}a_6 \leq \frac{15}{16}t_0 + \frac{t_0}{21} + \frac{19}{168}t_0 < \frac{1}{2}.$$

(4.7.5). $2t_0/5 \geq \theta_1 \geq \cdots \geq \theta_4 \geq 1 - 20t_0/11, 2t_0/5 \geq 1 - \theta_1 - \cdots - \theta_6 \geq \theta_1, \geq 1 - 20t_0/11$, and $\theta_5 \geq \theta_6, 2\theta_6 \leq t_0/3, \theta_1 + \theta_2 + \theta_5 < t_0 < \theta_1 + \theta_2 + \theta_5 + \theta_6$.

Proof. In Lemma 3.9, take $a_1 = \theta_1 + \theta_2, a_2 = \theta_3 + \theta_4, a_3 = \theta_5$, and $a_4 = \theta_6$, then

$$\begin{aligned} a_4 &= \theta_6 \leq \tfrac{1}{2}(\theta_5 + \theta_6) \leq \tfrac{1}{2}\left(1 - \theta_1 - \cdots - \theta_4 - \left(1 - \sum_{1 \leq i \leq 6} \theta_i\right)\right) \\ &\leq (1 - 5a_2)/2 \end{aligned}$$

and

$$\begin{aligned} h_9(a_1, \cdots, a_4) &\leq 15t_0/16 + a_2/8 + 5(1 - 5a_2)/16 \\ &< 15t_0/16 + 5/16 - 23a_2/16 < 1/2. \end{aligned}$$

(4.7.6). $k_0 = 4, a_1 + a_4 \leq t_0, a_2 = a_3 \leq t_0, \sigma + a_1 \geq t_0, a_3 < t_0/5, a_2 < 1/3$ *and* $a_3 < t_0/5$.

Proof. In Lemma 3.10,

$$h_{10}(a_1, \cdots, a_4) = t_0 + a_2/8 + a_3/24 < t_0 + 1/24 + t_0/120 < 1/2.$$

(4.7.7). $k_0 = 5, a_3 \le a_2 \le a_1 \le t_0/2$, and $a_5 \le a_4 \le t_0/4$.

Proof. If $a_2 + a_1 > 8t_0/9$, we replace a_1 by $a_2 + a_1$ and s by a_5 with $k_0 = 2$, then $\{\theta_j\} \in E$ by (4.6.3). If $a_2 + a_1 \le 8t_0/9$, then $a_3 > 4t_0/9$ it is covered by (4.7.2).

(4.7.8) $k_0 = 2, a_1 \ge t_0/2, a_2 \le t_0/5$, and $\sigma + a_2/2 > t_0/2$.
See Lemma 3.2.

(4.7.9). $k_0 = 2, a_1 \ge t_0, t_0/3 \ge a_2 \ge t_0/5$, and $\sigma > a_2/8 + 3t_0/8$.

See Lemma 3.3.

§ 5. ANALYTIC FORM OF $R(x; M_1 \cdots M_k)$.

We shall examine the remainder term $R(x; M_1, \cdots, M_k)$ (see (1.12)) which was used in [1]. Let x be a large number, $y = x^\theta$ with $1/2 < \theta < 7/12$, and $\mathbf{A} = \{n : x - y < n < x\}$. For convenience we define $a_{m_i,i} = 0$, unless $M_i < m \le 2M_i$. We rewrite (1.12) :

$$R(x; M_1, \cdots, M_k) = \sum_{m_1, \cdots, m_k} a_{m_1,1} \cdots a_{m_k,k} \left(\left[\frac{x}{m_1 \cdots m_k} \right] - \left[\frac{x-y}{m_1 \cdots m_k} \right] \right)$$

and we write

$$\begin{aligned} L &= \frac{x}{2M_1 \cdots M_k}, \\ L(s) &= \sum_{L/2^k < I \le 3L} I^{-s}, \\ M_i(s) &= \sum_{M_i < m_i \le 2M_i} a_{m_i,i} m_i^{-s}, \text{ for } 1 \le i \le k, \end{aligned}$$

and

$$g(s) = L(s) \prod_{1 \le i \le k} M_i(s),$$

where k is a positive integer.

Theorem 2. *Suppose* $\{\theta_j\} \in E(\theta)$, *then*

$$R(x; M_1, \cdots, M_k) \ll x^{\theta - \varepsilon}.$$

Proof. Lemma 3.12 of Titchmarsh [7] is applied to the function $g(s)$, to yield

$$\sum_{\substack{m_1,\cdots,m_k \\ Im_1\cdots m_k\in \mathbf{A}}} a_{m_1,1}\cdots a_{m_k,k} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} g(s)\frac{x^s-(x-y)^s}{s}ds + O\left(\frac{x^{1-\eta}}{T}\right), \tag{5.1}$$

where $c = 1+(\log x)^{-1}$ and $T > 0$. The conditions $Im_1\cdots m_k \in \mathbf{A}$ and $M_i < m_i \le 2M_i$ imply that $L/2^k < I \le 3L$, and so that the sum on the left of (5.1) becomes

$$\sum_{m_1,\cdots,m_k} a_{m_1,1}\cdots a_{m_k,k}\left(\left[\frac{x}{m_1\cdots m_k}\right]-\left[\frac{x-y}{m_1\cdots m_k}\right]\right)$$
$$= R(x;M_1,\cdots,M_k) - y\sum_{m_1,\cdots,m_k}\frac{a_{m_1,1}\cdots a_{m_k,k}}{m_1\cdots m_k}$$

For $T_0 \le L$, in the range $\{s = c+it : | t |\le T_0 \le L\}$, we have

$$L(s) = \frac{(3L)^{1-s}-(L/2^k)^{1-s}}{s-1} + O(L^{-c})$$

by Theorem 4.11 of Titchmarsh [7]. Moreover if $T_0 y \le x$,

$$\frac{x^s-(x-y)^s}{s} = yx^{s-1} - O(| s | y^2x^{c-2}) \ll yx^{c-1}.$$

Hence

$$\frac{1}{2\pi i}\int_{c-iT_0}^{c+iT_0} g(s)\frac{x^s-(x-y)^s}{s}ds$$
$$= \frac{1}{2\pi i}\int_{c-iT_0}^{c+iT_0}\frac{x^s-(x-y)^s}{s}M_1(s)\cdots M_k(s)x^{s-1}ds + O(E_1) + O(E_2),$$

where

$$E_1 = \int_{-T_0}^{T_0} L^{-c} \,|\prod_{1\le j\le k} M_j(c-it)\,|\, yx^{c-1}dt \ll T_0M_1\cdots M_k yx^{-1},$$

and

$$\begin{aligned} E_2 &= \int_{-T_0}^{T_0} L^{-c}\,|\prod_{1\le j\le k} M_j(c-it)\,|\, y^2x^{c-2}dt \ll T_0L^{1-c}(M_1\cdots M_k)^{1-c}y^2x^{c-2} \\ &\le T_0y^2x^{-1}. \end{aligned}$$

Moreover, on integrating termwise as in the proof of Lemma 3.1 in Titchmarsh [7], we have

$$\frac{1}{2\pi i}\int_{c-iT_0}^{c+iT_0}\frac{x^{-s}-(x-y)^s}{s}M_1(s)\cdots M_k(s)x^{s-1}ds$$

$$=\sum_{\substack{m_1,\cdots,m_k\\ 1m_1\cdots m_k\in A}}\frac{a_{m_1,1}\cdots a_{m_k,k}}{m_1\cdots m_k}+O\left(\frac{1}{T_0}\right).$$

It follows from the above estimate, choosing $T_0=L^{1/2}$, that

$$R(x;M_1,\cdots,M_k)=\frac{1}{2\pi i}\left(\int_{c-iT}^{c+iT_0}+\int_{c+iT_0}^{c+iT}\right)g(s)\frac{x^s-(x-y)^s}{s}ds+O(x^{\theta-\epsilon}). \tag{5.2}$$

We divide the latter range into at most 2 $\log x$ subintervals of type $[T_1,T_2]$, where $T_2\le 2T_1$ and bound the interval over such a range by a sum over well spaced points $T_1\le t_1<\cdots<t_k\le T_2(t_{r+1}-t_r\ge 1)$. Thus, for some T_1 with $T_0\le T_1\le T$, and some set of t_r, we have

$$R(x;M_1\cdots M_k)\ll x^{\theta-\epsilon}+(\log x)yx^{c-1}\sum_{1\le r\le k}g(c+it_r). \tag{5.3}$$

In the following sections we shall estimate (5.3).

We denote the right hand-side of (5.3) by $R(x;N_1,\cdots,N_{k_0})$. We have

$$g(s)=N_1(s)\cdots N_{k_0}(s)L(s), \tag{5.4}$$

where the definition of $N_i(s)$ or $L_0(s)$ is the same as (5.1) and one of $N_i(s)$ or $L_0(s)$ is equal $L(s)M_{i_1}(s)\cdots M_{i_n}(s)$.

We trivially have

$$|L_0(c+it_r)|\le 4,\text{ and }|N_j(c+it_r)|\le 4\quad(j\le k).$$

Hence those t_r for which

$$|L_0(c+it_r)|\le x^{-2},\text{ or }|N_j(c+it_r)|\le x^{-2}\quad(j\le k_0)$$

contribute a total

$$\ll(\log x)yx^{c-1}T_2x^{-2}\ll x^{\theta-\epsilon}$$

to $R(x;N_1,\cdots,lN_{k_0})$. Such points may therefore be neglected. We now divide the remaining t_r into at most $(4\ log\ x)^{k_0+1}$ sets $S(U_1,\cdots,U_{k_0},W)$ for which

$$\begin{aligned} U_i &< N_i^{c-1/2} \mid N_i(c+it_r) \mid < 2U_i, i=1,\cdots,k_0, \\ W &\le L_0^{c-1/2} \mid L_0(c+it_r) \mid < 2W, \end{aligned}$$

where $x^{-2} \le L^{1/2-c}W \le 2^{-u} \le 1$, for some integer u and similarly for $U_1,\cdots,U_{k_0}$. It follows that

$$R(x;N_1,\cdots,N_{k_0}) \ll x^{\theta-\varepsilon}+(log\ x)^{k_0+2}yx^{-1/2}U_1\cdots U_{k_0}W \mid S(U_1,\cdots,U_{k_0},W) \mid \tag{5.5}$$

for some T, some $U_1,\cdots,U_{k_0},W$ and some well-spaced set $S(U_1,\cdots,U_{k_0},W)$ having the property (1.4) - (1.7): the mean-value technique of Montgomery [8]

$$\mid S(U_1,\cdots,U_{k_0},W) \mid \ll U_i^{-2j}(N_i^j+T)log^A x, 1 \le i \le k_0, \tag{5.6}$$

where A is constant; Halasz's method in the form due to Huxley

$$\mid S(U_1,\cdots,U_{k_0},W) \mid \ll (U_i^{-2j}N_i + U_i^{-6j}N_iT)log^A x; \tag{5.7}$$

while from the inequality

$$\mid L_0(c+it_r) \mid^2 \ge U^2 L_0^{1-2c},$$

we have

$$\mid S(U_1,\cdots,U_{k_0},W) \mid \ll (W^{-4}L_0^2 + W^{-12}L_0^2T)(log\ x)^A. \tag{5.8}$$

Our last estimate for $\mid S(U_1,\cdots,U_{k_0},W) \mid$ depend on the formula

$$L(s) = \frac{1}{2\pi i}\int_{\frac{1}{2}-c-iT}^{\frac{1}{2}-c+iT} \zeta(s+z)\left((3L)^z - \left(\frac{L}{2^k}\right)^z\right)\frac{dz}{z} + O\left(L^{\frac{1}{2}-c}log\ x\right)$$

for $s = c+it, T_1 \le t \le 2T_1$. This yield, if $L_0 = L$,

$$W^4 \mid S(U_1,\cdots,U_{k_0},W) \mid \le L_0^{4c-2}S \mid L_0(c+it_r) \mid^4 \ll T(log\ x)^A. \tag{5.9}$$

Deshouillers and Iwaniec [6] proved: if $N \geq 1, T \geq 1$, the coefficients d_n are complex numbers and $\varepsilon > 0$, then

$$\int_{-2T}^{2T} |\zeta(\frac{1}{2}+it)|^4 |\sum_{n<n\leq 2N} d_n n^{it}|^2 \, dt \ll T^{\varepsilon}(N_2 T^{1/2}+N^{5/4}T^{3/4}+T) \sum_{n\leq n\leq 2N} |d_n|^2 \tag{5.10}$$

From (4.1), we have

$$W^4 U_j^2 | S(U_1,\cdots,U_{k_0},W) | \leq L^{4c-2} N^{2c-1} \sum_{r} | L(c+it_r) |^4 | N_j(c+it_r) |^2$$

$$\ll L^{4c-2} N_j^{2c-1} \sum_{r} \int_{-2T}^{2T} |\zeta(\frac{1}{2}+it)|^4 L^{2-4c} N_j^{1-2c} |N_j(\frac{1}{2}+it)|^2 \frac{dt}{1+|\tau - t_r|}$$

$$+ | S(U_1,\cdots,U_{k_0},W) || (\log x) |^4 \ll T^{\varepsilon}(N_k^2 T^{1/2} + N_k^{5/4} T^{3/4} + T) \tag{5.11}$$

whence, for $1 \leq k \leq j$,

$$| S(U_1,\cdots,U_{k_0},W) | \ll W^{-4} U_k^{-2} (N_k^2 T^{1/2} + N_k^{5/4} T^{3/4} + T) T^{\varepsilon}. \tag{5.12}$$

We have that

$$U_1 \cdots U_k W \,| S(U_1,\cdots,U_{k_0},W) | \ll x^{1/2-\varepsilon} + x^{h(a_1,\cdots,a_{k_0})} \tag{5.13}$$

with $h(a_1,\cdots,a_{k_0}) < 1/2$ for a certain constant $\varepsilon > 0$ since $\{\theta_j\} \in E(\theta)$. This will complete the proof of

$$R(x; M_1,\cdots,M_k) \ll x^{\theta-\varepsilon}.$$

Theorem 3. *Suppose* $\{\theta_j\} \in E(\theta)$. *then*

$$\int_{T}^{2T} | W(\frac{1}{2}+it) | \, dt \ll x^{\frac{1}{2}-\varepsilon}, \tag{5.14}$$

for

$$T_1 \leq T \leq x^{1-\theta+\varepsilon}$$

where θ *is a fixed positive constant, and*

$$T_1 = exp\left((log\ x)^{\frac{1}{3}}(loglog\ x)^{-\frac{1}{3}}\right).$$

Proof. Defined $S(U_1, \cdots, U_k, W)$ be a set of t_r into for which

$$U_i < M_i^{c-1/2} \mid M_i(c+it_r) \mid < 2U_i, \; i = 1, \cdots, k_0,$$
$$W \leq L_0^{c-1/2} \mid L_0(c+it_r) \mid < 2W,$$

where $x^{-2} \leq L^{1/2-c}W \leq 2^{-u} \leq 1$, for some integer u and similarly for $U_1, \cdots, U_{k_0}$. In Lemma 16 of [8], Heath-Brown proved

$$\int_T^{2T} \mid W(\frac{1}{2}+it) \mid dt \ll x^{\frac{1}{2}-\epsilon} + x^{-\delta} \sum \mid S(U_1, \cdots, U_k, W) \mid$$

where $\sum$ runs over $(U_1, \cdots, U_k, W)$ such that (1.1), (1.2) and (1.3), and $\mid S(U_1, \cdots, U_k, W) \mid$ satisfies (1.4), (1.5), (1.6), (1.7) and (1.9). Since $\{\theta_j\} \in E(\theta)$, we have that

$$\mid S(U_1, \cdots, U_k, W) \mid < x^{\frac{1}{2}-\epsilon},$$

then

$$\sum \mid S(U_1, \cdots, U_k, W)) \mid < x^{\frac{1}{2}-\epsilon},$$

since at most $(4 \log x)^{k+1}$ terms in above "$\sum$". Thus (5.14) follows.

References

[1] Heath-Brown, D.R. and Iwaniec, H., *On the Difference Between Consecutive Primes,* Invent. Math. 55, 49-69 (1979).

[2] Heath-Brown, D.R., *The Number of Primes in a Short Interval,* J. Reine Angew, Math. 389, 22-63 (1988).

[3] Lou, S. and Yao, Q., *The Number of Primes in a Short Interval,* Hardy-Ramanujan J., 16 (1993), 21-43.

[4] Lou, S. and Yao, Q., *A Chebychev's Type of Prime Number Theorem in a Short Interval-*II, Hardy-Ramanujan J., 15 (1992), 1-33.

[5] Lou, S. and Yao, Q., *On the Difference Between Consecutive Primes,* (to be submitted).

[6] Deshouillers J.-M. and Iwaniec H., *Power mean-value of the Riemann zeta-function,* Mathematika, 29 (1982), 202-212.

[7] Titchmarsh E.C., The Theory of the Riemann zeta-function, Oxford (1951).

[8] Montgomery H.L., Topics in Multiplicative Number Theory, Lecture Notes in Mathematics, Springer-Verlag, 227 (1971).

Lou, Shituo and Yao, Qi
Department of Math.
Dalhousie University
Halifax, Nova Scotia B3H 3J5
Canada