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A LARGE VALUE THEOREM FOR $\zeta(s)$

BY

K. RAMACHANDRA

(TO PROFESSOR MATTI JUTILA ON HIS FIFTY-SECOND BIRTHDAY)

§ 1. **INTRODUCTION.** The first object of this paper is to prove the following theorem.

THEOREM 1. *Let $\varepsilon > 0$ be a sufficiently small constant and let T exceed a large positive constant depending on ε. We write $a = a(\varepsilon) = \varepsilon^{-5}$. Let $T \le t_1 < t_2 < \cdots < t_R \le 2T, t_{j+1} - t_j \ge 1$ $(j = 1, 2, \cdots, R-1)$ and $|\log \zeta(1 + it_j)| \ge \log\log\log T + \log a$, $(j = 1, 2, \cdots, R)$. Then $R \ll_\varepsilon T^\varepsilon$.*

COROLLARY. *Let $-T \le q_1 < q_2 < \cdots < q_{R'} \le T, q_{j+1} - q_j \ge 1$ $(j = 1, 2, \cdots, R'-1)$ and*

$$| \zeta(iq_j) | \ge \varepsilon^{-6}(q_j^*)^{\frac{1}{2}} \log\log q_j^* \ (j = 1, 2, \cdots, R')$$

where $q_j^* = | q_j | + 100$. Then $R' \ll_\varepsilon T^\varepsilon$.

PROOF OF THE COROLLARY. The proof follows from the functional equation.

We next use this corollary to deduce the following large value theorem from Montgomery's work (see the fundamental Lemma 8.1 in [1]).

THEOREM 2. *Let S be a set of complex numbers with the following* *properties.* (i) *Re* $s \geq \sigma_0$ *for all* $s \in S$. (ii) $1 \leq | Im(s-s') | \leq T_0$ *for all pairs* (s,s') *with* $s \neq s', s \in S, s' \in S$, *where* $T_0 \geq 100$. *Then for any integer* $N \geq 1$ *and any set* $\{a_n\}$ $(n = N, N+1, \cdots, 2N)$ *of complex numbers we have*

$$\sum_{s \in S} | \sum a_n n^{-s} |^2 \ll_\varepsilon GN + (T_0^\varepsilon + | S | \, loglog \ T_0) G T_0^{\frac{1}{2}}$$

where $G = \sum | a_n |^2 n^{-2\sigma_0}$, ε *is a sufficiently small positive constant and* T_0 *exceeds a constant depending only on* ε.

REMARKS. It suffices to prove the theorem in the case $N > 1$. For, when $N = 1$ we have $| \sum a_n n^{-s} |^2 \leq 2 \sum | a_n |^2 n^{-2\sigma_0}$. Next it suffices to prove this theorem when $\sigma_0 \leq Re \ s \leq \sigma_0 + \frac{1}{10}$ for all $s \in S$. The general case follows by dividing the range for $Re \ s$ into intervals of length $\frac{1}{10}$ and then adding up the various RHS with $| S |$ in each case replaced by the sum of the various $| S |$.

Given an inequality of the type of Theorem 2 with $loglog \ T_0$ replaced by $(log(NT_0))^2$, M.N. Huxley made an important deduction which is well-known. We deduce from Theorem 2 the following important theorem by his method.

THEOREM 3. *Let* S^* *be a set of complex numbers with the following properties.* (i) *Re* $s \geq \sigma_0$ *for all* $s \in S^*$. (ii) $| Im \ s | \leq T$ *for all* $s \in S^*$. (iii) $| Im(s-s') | \geq 1$ *for all pairs* (s,s') *with* $s \neq s', s \in S^*, s' \in S^*$. *Let* $N(\geq 1)$ *be an integer and let* $\{a_n\}$ $(n = N, N+1, \cdots, 2N)$ *be any set of complex numbers. Then for any real* $V > 0$, *the number of elements of* S^* *with* $| \sum a_n n^{-s} | \geq V$ *is*

$$\ll_\varepsilon GNV^{-2} + TG^3NV^{-6}(loglog \ T)^2 + T^\varepsilon(1 + TG^2V^{-4}),$$

where $G = \sum | a_n |^2 n^{-2\sigma_0}$ *and* T *exceeds a large positive constant depending only on* $\varepsilon, \varepsilon(> 0)$ *being any constant.*

Since Huxley's method is very simple we give the deduction here itself. We divide the range $(-T, T)$ for $Im \ s (s \in S^*)$ into at most $2(1 + TT_0^{-1})$ intervals of length T_0. Suppose we fix any ε and make the following assumptions. (i) $T_0^\varepsilon \leq | S |$ and (ii) $GT_0^{\frac{1}{2}} loglog \ T = \eta V^2$, where $\eta > 0$ is a small constant.

Then from Theorem 2 it follows that $|S| \ll_{\varepsilon,\eta} GNV^{-2}$. If (i) is not satisfied then we have $|S| \leq T_0^{\varepsilon}$ and so in any case $|S| \ll_{\varepsilon,\eta} GNV^{-2} + T_0^{\varepsilon}$, provided only that we fix T_0 by (ii). Thus (since $T_0 \leq T$)

$$|S^*| \ll_{\varepsilon,\eta} (1 + TT_0^{-1})(GNV^{-2} + T^{\varepsilon})$$

$$\ll_{\varepsilon,\eta} (GNV^{-2} + T^{\varepsilon})(1 + TG^2V^{-4}(\log\log T)^2).$$

(We have assumed in this reasoning that $T_0 \gg_{\varepsilon} 1$. But if $T_0 \ll_{\varepsilon} 1$ the term $TT_0^{-1}T^{\varepsilon}$ is a trivial upper bound for $|S^*|$). Thus Theorem 3 is proved since η depends only on ε and we can replace ε by $\frac{1}{2}\varepsilon$.

REMARK. From Theorem 3 we can (proceeding as in the appendix, § A.3 of [3]) prove that $\left(\text{if } \frac{3}{4} \leq \sigma \leq \frac{3}{4} + \frac{D \log\log T}{\log T}\right)$ then

$$N(\sigma, T) \ll_D T^{A_2(\sigma)(1-\sigma)}(\log T)^{8-\frac{1}{5}}(\log\log T)^{2-\frac{2}{5}}, \ (D > 0 \text{ is any constant}),$$

with $A_2(\sigma) = (5\sigma - 3)(\sigma^2 + \sigma - 1)^{-1}$, which is a minor improvement over the result

$$N(\sigma, T) \ll T^{A_2(\sigma)(1-\sigma)}(\log T)^9$$

due to M.N. Huxley. A slightly cruder form of this improvement (with $8 - \frac{1}{5} + \varepsilon$ in place of $8 - \frac{1}{5}$ and $\ll_{\varepsilon,D}$ in place of $\ll_D$) was announced in the post-script to [3]. These refinements have also application to prime number theory similar to what has been done in [3].

NOTATION. The notation is mostly standard and whenever there is a departure it will be explained in the respective sections.

§ 2. PROOF OF THEOREM 1. Throughout this section $T \geq 100$ (since we are interested in the bound $\ll_{\varepsilon} T^{\varepsilon}$, we can assume that T exceeds a positive constant depending only on ε), $T \leq t \leq 2T$, $e(u) = Exp(-u)$, $X = Exp((\log T)^{200})$, $r = (\log X)^{-2}$, $z = x + iy$ and finally with real A, B (with $B > A > 0$) and $Re\ z \geq 1 - r$ we write

$$f(A, B, z) = \sum_{A \leq p < B} p^{-z} e\left(\frac{p}{X}\right).$$

LEMMA 1. *We have*

$$log\ \zeta(1+it)=\sum p^{-1-it}e(\frac{p}{X})+O(1),$$

where p runs over all primes and the O-constant is absolute.

PROOF. Follows (since higher powers of p contribute $O(1)$) from

$$\sum p^{-1-it}e(\frac{p}{X})+O(1)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}(log\ \zeta(1+it+w))X^w\Gamma(w)dw$$

(where $w=u+iv$ is a complex variable) on moving the line of integration suitably, using the zero-free region $\sigma\geq 1-(log\ T)^{-10}$ and the estimates for $log\ \zeta(1+it+w)$ in the zero-free region.

LEMMA 2. *The inequality* $\mid log\ \zeta(1+it)\mid\geq logloglog\ T+log\ a$ *implies at least one of the following inequalities.*

(i) $\mid f(T,\infty,1+it)\mid\geq 1$

(ii) $\mid f(Y,T,1+it)\mid\geq\frac{1}{3}log\ a$, *where* $loglog\ Y=logloglog\ T+\frac{1}{4}\ log\ a$.

PROOF. Trivial since $\sum_{p\leq Y}p^{-1}=loglog\ Y+\gamma_0+O((log\ Y)^{-1})$, where γ_0 is an absolute constant.

LEMMA 3. *If* $f(z)$ *is analytic in* $\mid z-z_0\mid\leq r$, *we have*

$$\mid f(z_0)\mid\leq\frac{1}{\pi r^2}\int\int\mid f(z)\mid dxdy$$

where the integration is over the disc $\mid z-z_0\mid\leq r$.

PROOF. Follows from

$$\mid f(z_0)\mid\leq\frac{1}{2\pi}\int_0^{2\pi}\mid f(z_0+\lambda e^{i\theta})\mid d\theta$$

valid for all λ with $0<\lambda\leq r$. (We have only to multiply by $\lambda d\lambda$ and integrate from $\lambda=0$ to $\lambda=r$).

LEMMA 4. *The number* R_1 *of points* t_j *with* $\mid f(T,\infty,1+it_j)\mid\geq 1$ *satisfies*

$$R_1\leq\frac{1}{\pi r^2}\int\int\mid f(T,\infty,z)\mid^2 dxdy$$

where the integration is over the rectangle $|x-1| \leq r, T-1 \leq y \leq 2T+1$.

PROOF. Follows from Lemma 3 (since the discs with the centres $1+it_j$ and radii r are all disjoint).

LEMMA 5. *Uniformly in* $|x-1| \leq r$, *we have*

$$\int_{T-1}^{2T+1} |f(T,\infty,z)|^2 \, dy \ll \text{loglog } X.$$

PROOF. By the well-known Montgomery Vaughan theorem we have

$$\int_{T}^{2T} |\sum_n d_n n^{-it}|^2 \, dt \ll \sum_n (T+n) |d_n|^2$$

where $\{d_n\}(n=1,2,\cdots)$ is any sequence of complex numbers such that $\sum n |d_n|^2$ is convergent. From this the LHS of the lemma is

$$\ll \sum p^{-1+2r} e(\frac{2p}{X}) \ll \sum_{p \leq X^2} p^{-1} \ll \text{loglog } X.$$

This proves the lemma.

LEMMA 6. *Divide the interval* $[Y,T]$ *into* $\leq 3 \log T$ *intervals* I *by interposing the points* U *which are powers of* 2. *The extreme two intervals are of the type* $[Y, Y+V_1)$ *and* $[U,T)$ *with* $0 \leq V_1 \leq Y$. *Then for at least one interval* I *we have*

$$|\sum_{p \in I} p^{-1-it_j} e(\frac{p}{X})| \geq (\log T)^{-1}$$

PROOF. Trivial.

LEMMA 7. *For any interval* I *define an integer* $k = k(I) \geq 1$ *to be the least integer such that* $(\min_{p \in I} p)^k \geq T$. *Define* $a_k(n)$ *by*

$$\left(\sum_{p \in I} p^{-z} e(\frac{p}{X})\right)^k = \sum_{T \leq n \leq 2^{2k} T} a_k(n) n^{-z}.$$

Then $1 \le k \le (\log T)(\log Y)^{-1}$ *and*

$$0 \le a_k(m) \le k^k.$$

PROOF. The lemma is obvious since whenever $a_k(n) \ne 0, n$ is the power product of primes with k as the sum of the exponents.

LEMMA 8. *Let $R(I)$ denote the number of t_j satisfying the final inequality of Lemma 6. Then*

$$R(I) \le (\log T)^{2k} \frac{1}{\pi r^2} \int\int \mid \sum_{p \in I} p^{-z} e(\frac{p}{X}) \mid^{2k} dx dy$$

where the integration is over the rectangle $\mid x - 1 \mid \le r, T - 1 \le t \le 2T + 1$.

PROOF. Follows from Lemma 3 just as Lemma 4 followed from Lemma 3.

LEMMA 9. *We have*

$$\begin{aligned} R(I) &\ll (\log T)^{2k+500} \sum_{T \le n \le 2^{2k} T} \mid a_k(n) \mid^2 n \\ &\ll (\log T)^{2k+500} k^{2k+2}. \end{aligned}$$

PROOF. Follows from the well-known theorem of Montgomery and Vaughan (in view of Lemma 7).

LEMMA 10. *We have*

$$R(I) \ll T^{\epsilon}.$$

PROOF. Follows from

$$k \le (\log T)(\log Y)^{-1} \le (\log T)(\log\log T)^{-1} a^{-\frac{1}{4}}$$

and hence

$$(\log T)^k \cdot k^k \ll Exp(3k \log\log T) \ll Exp(3a^{-\frac{1}{4}} \log T)$$

where the implied constants are absolute.

LEMMA 11. *The number R_2 of numbers $t = t_j$ which satisfy the second alternative of Lemma 2 satisfies*

$$R_2 \leq \sum_I R(I) \leq T^{2\varepsilon}$$

LEMMA 12. *We have*

$$R \ll_\varepsilon T^\varepsilon.$$

PROOF. Since $R \leq R_1 + R_2$, the lemma follows by Lemmas 4,5 and 11.

§ 3. PROOF THEOREM 2. We begin by stating the following theorem of H.L. Montgomery (which is a special case of his fundamental Lemma 8.1 of [1]).

THEOREM 4 (H.L. MONTGOMERY). *In the notation of Theorem 2 above, we have*

$$\sum_{s\in S} \mid \sum a_n n^{-s} \mid^2 \ll (\sum \mid a_n \mid^2 b_n^{-1}) \max_{s\in S} \sum_{s'\in S} \mid B(\bar{s} + s') \mid,$$

where $b_n = \left(Exp\left(-\frac{n}{2N}\right) - Exp\left(-\frac{n}{N}\right)\right) n^{2\sigma_0}$ *and for any complex z we define $B(z)$ by*

$$B(z) = \sum_{n=1}^{\infty} b_n n^{-z}.$$

PROOF. This is a special case of Lemma 8.1 of [1].

We continue with the proof of Theorem 2. Note that we can assume $N > 1$ and also $\sigma_0 \leq Re\ s \leq \sigma_0 + \frac{1}{10}$ for all $s \in S$. Also note that $\sum \mid a_n \mid^2 b_n^{-1} \ll G$. We write $s = \sigma + it, s' = \sigma' + it'$ and we have

$$B(\bar{s} + s') = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(w + \bar{s} + s' - 2\sigma_0) N^w (2^w - 1)\Gamma(w) dw$$

where $w = u + iv$ is a complex variable. Moving the line of integration to u given by $u + \sigma + \sigma' - 2\sigma_0 = 0$ we obtain

$$\mid B(\bar{s} + s') \mid \ll N\ Exp(- \mid t - t' \mid) + \int_{-\infty}^{\infty} \mid \zeta(iv - it + it') \mid Exp(- \mid v \mid) dv.$$

For any fixed t, we have uniformly

$$\sum_{s'} | B(\bar{s}+s') | \ll N + \sum_{t'} \int_{-\infty}^{\infty} | \zeta(iv - it + it') | \, Exp(- | v |) dv.$$

When $| t - t' |$ is bounded above the last integral is bounded. Also for the remaining t' the contribution from $| v | \geq (log | t - t' |)^2$ is negligible. Thus

$$\sum_{s'} | B(\bar{s}+s') | \ll N + \sum_{t', |t-t'| \geq 1000} \int_{|v| \leq (log|t-t'|)^2} | \zeta(i(v-t+t')) | \, Exp(- | v |) dv + | S |$$

We look at the maximum $M(t - t')$ of $| \zeta(i(v - t + t')) |$ in the range of integration. Suppose this is attained at $v = \tau(t')$ say. For fixed t, the points $\tau(t') - t + t' = q_j$ with $M(t - t') \geq \varepsilon^{-7}(q_j^*)^{\frac{1}{2}} loglog\ q_j^*$ (for varying t' form a union of $\leq (log\ T_0)^3$ sets of points (at mutual distances ≥ 1) and by corollary to Theorem 1 the total number of such points t' (note that $T = 2T_0$) is $O((log\ T_0)^3 T_0^{\varepsilon}) = O(T_0^{2\varepsilon})$. Hence with the exception of $O(T_0^{2\varepsilon})$ points t' we have $M(t - t') \leq \varepsilon^{-7}(| t - t' | + 100)^{\frac{1}{2}} loglog(| t - t' | + 100)$. Thus the contribution to $\sum_{s'} | B(\bar{s} + s') |$ from the exceptional points t' is $O(T_0^{\frac{1}{2}+3\varepsilon})$, (we employ the trivial bound $M(t - t') = O(T_0^{\frac{1}{2}} log\ T_0)$ for exceptional points). The other points contribute $O(| S | T_0^{\frac{1}{2}} loglog\ T_0)$ and this proves Theorem 2 completely (on noting that we can replace ε by $\frac{1}{4}\varepsilon$ if necessary).

FINAL REMARK. The analogue of Theorem 1 to "short intervals" is somewhat delicate and was investigated in [2]. The upper bound therein is in fact $O_\varepsilon(1)$, provided $| log\ \zeta(1 + it) | \geq \varepsilon\ loglog\ T$. The present work although self contained may in some sense be considered as a continuation of [2].

REFERENCES

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ADDRESS OF THE AUTHOR:

SChOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
COLABA
BOMBAY 400 005, INDIA

e-mail : KRAM@TIFRVAX.TIFR.RES.IN

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P.S: After writing up the paper I noticed that Theorem 1 can be deduced easily from any zero density estimate which ensures that

$$\lim_{\sigma}\lim_{T}\{(logT)^{-1}log(N(\sigma,T)+1)\}=0$$

as $T\to\infty$ and then $\sigma\to 1-0$.