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On certain partial sums involving squares of Hecke eigenvalues

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Abstract. Let $a(n)$ be the nth Fourier coefficient of a cuspidal Hecke eigenform of even integral weight $k \geq 2$ and trivial character that is a normalized new form for some level N. We show that the partial sums

$$H_n = \sum_{m=1}^{n} a(m)^2/m^k$$

are not integral for $n \geq n_0$.

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1. Introduction and Results

Let

$$h_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \qquad (n \in \mathbf{N})$$

be the n-th harmonic number. Harmonic numbers have been studied since antiquity and are important in various branches of number theory. They also have interesting arithmetic properties. For example, a result of Theisinger [The1915] asserts that h_n for $n \geq 2$ never is an integer. Except for the use of "Bertrand's postulate" (cf. below) the proof is completely elementary.

If one seeks for a generalization in the context of modular forms one naturally is led to the partial sums formally attached to the Rankin-Selberg L-series at $s = k$ where k is the weight of the form in question. (For properties of the Rankin-Selberg zeta function we refer, e.g. to [Bu98, sect. 1.6].) Indeed, more precisely let

$$f(z) = \sum_{n \geq 1} a(n) e^{2\pi i n z}$$

($z \in \mathcal{H}$ = complex upper half-plane) be a cuspidal Hecke eigenform of even integral weight $k \geq 2$ and trivial character that is a normalized new form of some level N. Recall that $a(1) = 1$ and $a(n)$ is the eigenvalue of the n-th Hecke operator $T(n)$. It is well-known that the $a(n)$ are algebraic integers and generate a totally real number field K. Let

$$D_f(s) := \sum_{n \geq 1} a(n)^2 n^{-s} \qquad (\Re(s) \gg 1)$$

be the Rankin-Selberg L-series attached to f. Recall that $\sigma_0 = k$ is the abscissa of convergence of $D_f(s)$. We define

$$H_n = H_n(f) := \sum_{m=1}^{n} a(m)^2 m^{-k} \qquad (n \in \mathbf{N}). \tag{1.1}$$

Then $H_n \in K$ and the H_n could be viewed as (at least) formal generalizations of the numbers h_n.

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We prove:

Theorem 1. *Assume that $n \geq 8$ is an integer such that $a(p) \neq 0$ for some prime $p \in (n/2, n]$. Then H_n is not an algebraic integer.*

The proof of Theorem 1 works in a similar way as the proof in [The1915] with an additional use of Deligne's deep result

$$|a(p)| \leq 2p^{\frac{k-1}{2}}. \tag{1.2}$$

The assumption $n \geq 8$ (which implies that $p \geq 5$) comes from the fact that seemingly we do not have good information about the exact p-power dividing $a(p)$ for $p = 2, 3$, as will be clear from the proof.

2. Proof of Theorem 1

Assume that $n \geq 8$. Let p be any prime number in the interval $(n/2, n]$ which is guaranteed to exist by the hypothesis of Theorem 1. Further, $p \geq 5$. We denote by $\mathcal{O} \subset K$ the ring of algebraic integers of K. Suppose that $H_n \in \mathcal{O}$.

Let $C(n,p)$ be the n-th factorial, with the factor p omitted; i.e.:

$$C(n,p) := (p-1)! \cdot \prod_{\nu=1}^{n-p} (p+\nu).$$

We multiply both sides of (1.1) with $C(n,p)^k$. Our assumption implies that $C(n,p)^k H_n \in \mathcal{O}$. Hence, we infer that

$$C(n,p)^k \cdot \frac{a(p)^2}{p^k} \in \mathcal{O}$$

since the denominators of all other terms of the resulting sum on the right-hand side cancel. Therefore also

$$C(n,p)^{k/2} \cdot \frac{a(p)}{p^{k/2}} \in \mathcal{O}. \tag{2.3}$$

Now note that p does not divide any of the numbers $p+1, p+2, \ldots, n$, because the next multiple of p, namely $2p$ is larger than n.

By the unique factorization of integral ideals into products of prime ideals in the Dedekind domain $\mathcal{O}$, applied to the principal ideals in question and using that p does not divide $C(n,p)$, we deduce from (2.3) that $p^{k/2} | a(p)$; i.e., $a(p) = p^{k/2}\alpha_p$ with $\alpha_p \in \mathcal{O}$.

From (1.2) we therefore find that

$$|\alpha_p| \leq \frac{2}{\sqrt{p}} < 1 \tag{2.4}$$

where for the last inequality in (2.4) we have used that $p \geq 5$.

Let σ be one of the finitely many embeddings of K over $\mathbf{Q}$ into an algebraic closure of $\mathbf{Q}$. Then as is well-known

$$f^\sigma(z) := \sum_{n \geq 1} a(n)^\sigma e^{2\pi i n z} \qquad (z \in \mathcal{H})$$

is a normalized cuspidal Hecke eigenform of weight k, trivial character and level N. We have $a(p)^\sigma = p^{k/2}\alpha_p^\sigma$ and

$$|\alpha_p^\sigma| \leq \frac{2}{\sqrt{p}} < 1.$$

From the above we therefore find that the norm of α_p satisfies

$$|N(\alpha_p)| = |\prod_{\sigma} \alpha_p^{\sigma}| < 1.$$

However, $N(\alpha_p)$ is a rational integer. Hence, we should have that $a(p) = 0$, which contradicts our assumption. This proves Theorem 1.

Remarks. i) For the Ramanujan τ-function, which are the Fourier coefficients of the discriminant function Δ of weight $k = 12$ (see [Leh47]), Lehmer conjectured that $a(n) \neq 0$ for all n. If this were true, then Theorem 1 together with a well-known theorem of Tchebyshev (also known under the name of Bertrand's postulate, see [HaWr59, p. 373, notes on chap. XXII]) would apply immediately to show that H_n is not algebraic integer for all $n \geq 8$.

ii) Theorems about the vanishing of $a(p)$ for primes p were proved by Serre in his seminal paper [Ser81]. Theorem 15 on Page 174 and Proposition 18 on page 180 there, give the following:

Serre's Theorem *Assume that $k \geq 2$ and $x \geq 2$.*

i) In case f is non-CM (there is no complex quadratic field **L** *such that $a(p) = 0$ for all primes p which are inert in* **L***), we have that*

$$\#\{p \leq x : a(p) = 0\} = O\left(\frac{x}{(\log x)^{4/3}}\right).$$

ii) In case f is CM, we have that

$$\#\{p \leq x : a(p) = 0\} = \frac{x}{2\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

In either case, with n large and $x = n$, we conclude that the interval $(x/2, x]$ contains

$$\pi(x) - \pi(x/2) - \#\{p \in (x/2, x] : a(p) = 0\} \geq \frac{x}{\log x} - \frac{x/2}{\log(x/2)} - \left(\frac{x/2}{\log(x/2)} - \frac{x/4}{\log(x/4)}\right) + O\left(\frac{x}{(\log x)^{4/3}}\right) \geq \frac{x}{4\log x} + O\left(\frac{x}{(\log x)^{4/3}}\right) \quad (2.5)$$

primes p such that $a(p) \neq 0$. In the above, we used the Prime Number Theorem with the error term

$$\pi(x) = \frac{x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right) \quad \text{as } x \to \infty. \quad (2.6)$$

Since the last quantity indicated in the right–hand side of (2.5) is positive for large x, we conclude that for large n, the interval $(n/2, n]$ contains a prime p with $a(p) \neq 0$ and thanks to Theorem 1, we have that H_n is not an algebraic integer. Thus, we may write our conclusion as:

Theorem 2. *There exists $n_0 = n_0(f)$ such that H_n is not an algebraic integer for $n \geq n_0$.*

3. Comments and a challenge problem

We would like to suggest the following challenge question. Let ℓ, j be integers and put

$$H_{n,\ell,j} = \sum_{m=1}^{n} a(m)^{\ell} m^{j}.$$

Our paper discussed the case $\ell = 2$ and $j = -k$. We ask under what conditions on ℓ, j one may guarantee that $H_{n,\ell,j}$ is an algebraic integer for only finitely many n. When j is negative and large in absolute value (say $-j > \ell(k-1)/2$) the above argument applies. Indeed, the argument used in Theorem 1 creates an algebraic integer all whose conjugates have absolute values at most $2/p^{-j-\ell(p-1)/2}$ and since the exponent in the denominator is positive, this bounds p; hence, n. If $\ell \geq 1$ and $j \geq 0$, then $H_{n,\ell,j}$ is an algebraic integer for all n, and the case when $\ell < 0$ is difficult since $a(m)$ might be zero (although we may decide to sum only over the values of $a(m)$ which are nonzero). The case $\ell \geq 1$ and $j \in [-\ell(k-1)/2, -1]$ is different since the argument of Theorem 1 creates an algebraic integer with large conjugates so it does not lead to a contradiction. As a toy example, we took $a(m) = \tau(m)$, the Ramanujan function of m and studied the borderline case $\ell = 2$, $j = -1$. Let n be large and assume $s \geq 1$ is such that 13^s is the largest power of 13 smaller than or equal to n. Then the only numbers $m \leq n$ divisible by 13^s are of the form $\ell \times 13^s$ where $\ell \in \{1, 2, \ldots, K\}$ and $K \leq 12$ is some integer. Separating those ones out and multiplying everything by $A = 13^{s-1} \cdot \mathrm{lcm}[p^t \leq n, p \neq 13]$, and using the multiplicativity of $\tau(m)^2$, we get that

$$AH(n, 2, -1) = \frac{(A/13^{s-1})\tau(13^s)^2}{13} \sum_{\ell=1}^{K} \frac{\tau(\ell)^2}{\ell} \pmod{\mathbf{Z}}.$$

But $13 \| \tau(13^s)$, $13 \| A/13^{s-1}$ and a small calculation with Mathematica reveals that

$$\sum_{\ell=1}^{K} \frac{\tau(\ell)^2}{\ell}$$

is not a rational number whose numerator is divisible by 13 for any $K \in \{1, \ldots, 12\}$. Thus, $H(n, 2, -1) \notin \mathbf{Z}$ for any $n \geq 13$. Presumably there is an integer n_0 such that $H_{n,\ell,j}$ is not an integer for any $n \geq n_0$, $\ell \geq 1$ and $j < 0$. Note that since $p \mid \tau(p)$ for $p \in \{2, 3, 5, 7\}$, we have that $H_{n,\ell,j}$ is an integer for all $n \leq 10$, $\ell \geq 1$ and j negative and small in absolute value. Knowledge of non-ordinary primes p, namely primes p such that $p \mid \tau(p)$ would be useful. It has been checked that in addition to the above four, $2411, 7758337633$ are the only ones below 10^{10}. See [MoSm13] for some results on this problem.

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