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On a general divisor problem related to a certain Dedekind zeta-function over a specific sequence of positive integers

Anubhav Sharma and Ayyadurai Sankaranarayanan

Abstract. We investigate the average behavior of coefficients of the Dirichlet series of positive integral power of the Dedekind zeta-function $\zeta_{\mathbb{K}_3}(s)$ of a non-normal cubic extension $\mathbb{K}_3$ of $\mathbb{Q}$ over a certain sequence of positive integers. More precisely, we prove an asymptotic formula with an error term for the sum

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} a_{k,\mathbb{K}_3}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2),$$

where $(\zeta_{\mathbb{K}_3}(s))^k := \sum_{n=1}^{\infty}\frac{a_{k,\mathbb{K}_3}(n)}{n^s}$.

Keywords. Dedekind zeta-function, Hecke eigenvalues, non-normal cubic extension.

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1. Introduction

Let $\mathbb{K}$ be an algebraic extension of degree m of rational field $\mathbb{Q}$. Define

$$\zeta_{\mathbb{K}}(s) := \sum_{\alpha}\frac{1}{N(\alpha)^s},$$

for $\Re(s) > 1$, where the summation is running over all the integral ideals α of $\mathbb{K}$ and norm of integral ideal α is denoted by $N(\alpha)$. The function $\zeta_{\mathbb{K}}(s)$ can also be written as

$$\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty}\frac{a_{\mathbb{K}}(n)}{n^s},$$

where $a_{\mathbb{K}}(n)$ denotes the number of integral ideals of $\mathbb{K}$ with norm m. It is shown (by Chandrasekharan and Good [ChGo83]) that these coefficients are multiplicative and satisfies the upper bound

$$a_{\mathbb{K}}(n) \leq d(n)^m,$$

where m is the degree of extension, i.e., $m = [\mathbb{K} : \mathbb{Q}]$ and $d(n)$ is the number of divisors of n.

In 1949, Landau [Lan49] showed that

$$\sum_{n\leq x} a_{\mathbb{K}}(n) = cx + O\left(x^{1-\frac{2}{m+1}+\epsilon}\right),$$

where c is the residue of $\zeta_{\mathbb{K}}(s)$ at its simple pole at $s = 1$, which is further improved to

$$\sum_{n\leq x} a_{\mathbb{K}}(n) = cx + O\left(x^{\frac{23}{73}}\log^{\frac{315}{146}} x\right),$$

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for quadratic field by Huxley and Watt [HuWa00]. Some further improvement is also available for cubic fields by Müller [Mül88]. In 1993, W.G. Nowak [Now93] established that

$$\sum_{n\leq x} a_{\mathbb{K}}(n) = cx + \begin{cases} O\left(x^{1-\frac{2}{m}+\frac{8}{m(5m+2)}}\log^{\frac{10}{5m+2}} x\right) & \text{for } 3 \leq m \leq 6, \\ O\left(x^{1-\frac{2}{m}+\frac{3}{2m^2}}\log^{\frac{2}{m}} x\right) & \text{for } m \geq 7. \end{cases}$$

We also have some significant results (by Chandrasekharan and Narasimhan [ChNa63] and by Chandrasekharan and Good [ChGo83]) of $\sum_{n\leq x} a_{\mathbb{K}}(n)^k$ for some higher powers k, if $\mathbb{K}$ is the Galois extension of $\mathbb{Q}$.

If h is the class number of $\mathbb{K}$ and $[\mathbb{K} : \mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of real conjugate fields and $2r_2$ is the number of complex conjugate fields, then we can write

$$\sum_{n\leq x} a_{\mathbb{K}}(n) = h\lambda x + E(x),$$

where

$$\lambda := \frac{2^{r_1+r_2}\pi^{r_2}R}{w|\Delta|^{\frac{1}{2}}}.$$

Here, w is the number of roots of unity in $\mathbb{K}$; R is the regulator of $\mathbb{K}$ and Δ is the discriminant of $\mathbb{K}$.

When $[\mathbb{K} : \mathbb{Q}] = m \geq 10$, B. Paul and A. Sankaranarayanan proved that

$$E(x) \ll x^{1-\frac{3}{m+6}+\epsilon},$$

where implied constants depend only on $\mathbb{K}$ and ϵ (see [PaSa20]).

Also, if $\mathbb{K} = \mathbb{Q}(\zeta_l)$, where l is some positive integer and $[\mathbb{K} : \mathbb{Q}] = m \geq 8$, then,

$$E(x) \ll x^{1-\frac{3}{m+5}+\epsilon},$$

where the implied constants depend only on $\mathbb{K}$ and ϵ (see [PaSa20]).

It is of great interest to study the L-functions related to primitive holomorphic cusp forms. For many years, it has been a profound area in which many authors have contributed.

Let $L(s, f)$ be the L-function connected with the primitive holomorphic cusp form f of weight w for the full modular group $SL(2, \mathbb{Z})$ and $\lambda_f(n)$ are the normalized n^{th} Fourier coefficients of Fourier expansion of $f(z)$ at the cusp ∞, i.e.,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{w-1}{2}} e^{2\pi i n z}$$

where $\Im(z) > 0$, then the L-function attached to $\lambda_f(n)$ is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

for $\Re(s) > 1$, where $\lambda_f(n)$ are Hecke eigenvalues of Hecke operators T_n.

Also,

$$L^k(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_{k,f}(n)}{n^s},$$

where

$$\lambda_{k,f}(n) = \sum_{n=n_1n_2\ldots n_k} \lambda_f(n_1)\lambda_f(n_2)\ldots\lambda_f(n_k).$$

In 2012, Kanemitsu, Sankaranarayanan and Tanigawa [KST02] proved that for $k \geq 2$,

$$\sum_{n\leq x}\lambda_{k,f}(n) \ll x^{1-\frac{3}{2k+2}+\epsilon},$$

where implied constant depends only on f and ϵ, which is further improved by Lü in [Lü12].

For such divisor problems connected to holomorphic cusp forms, see the work of H.F. Liu [Liu18], [LiuZha19] and Lü [Lü12]. Recently, several authors considered the average behavior of $\lambda_{\mathrm{sym}^j f}(n)$ over certain sequences of positive integers and established some interesting asymptotic formulas (see, for instance [ShSa22a, ShSa22b, ShSa22c, Hua22]).

For $k \geq 2$, let $\Delta_k(x)$ denotes the error term in the asymptotic formula for $\sum_{n\leq x} d_k(n)$, where $\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n)n^{-s}$. The estimation of $\Delta_k(x)$ is popularly known as the general Dirichlet divisor problem. From elementary arguments, one can get $\Delta_k(x) \ll x^{\frac{k-1}{k}} \log^{k-2} x$. For $k = 2$, we have $\Delta_2(x) \ll x^{\frac{35}{108}+\epsilon}$, see [Ivi12]. For $k = 3$, $\Delta_3(x) \ll x^{\frac{43}{96}+\epsilon}$ is the best result available, due to G. Kolesnik [Kol79]. We may define the order α_k of $\Delta_k(x)$ as the least number such that $\Delta_k(x) \ll x^{\alpha_k+\epsilon}$ for every $\epsilon > 0$. The following results are known (see [Titch86]):

$$\alpha_k \leq \frac{k-1}{k} \qquad \text{for } k = 2, 3, \ldots$$

and

$$\alpha_k \leq \frac{k-1}{k+1} \qquad \text{for } k = 2, 3, \ldots.$$

But the exact value of α_k has not been determined for any value of k. For an extensive literature and detailed discussion of general divisor problem, see [Titch86, Chapter 12].

Estimating the average behavior of some special functions over polynomial values has been of interest since the early 1950s. In 1952, Erdös [Erd52] proved that

$$c_1 x \log x < \sum_{k=1}^{x} d(f(k)) < c_2 x \log x$$

and

$$\sum_{p\leq x} d(f(p)) \ll x,$$

where $f(x) \in \mathbb{Z}[x]$ and c_1, c_2 are some positive constants. For a quadratic polynomial $f(x)$, McKee [Mck95, McK99] proved that

$$\sum_{k=1}^{x} d(f(k)) \sim \lambda(f)x \log x,$$

where $\lambda(f)$ can be written in terms of Hurwitz class numbers. No similar results have been established for polynomials that have higher degrees. Then, Titchmarsh considered the linear polynomials $f(x) = a + x$ $(a \neq 0)$ and observed the average behavior of divisor function over shifted primes. More precisely, he proved that

$$\sum_{p\leq x} d(p + a) \sim C(a)x,$$

where $C(a)$ is some constant depending upon a. Motivated by the previous result of Titchmarsh, many authors studied the problem of finding an optimal error term of $\sum_{p\le x} d(n_p)$, where the n_p's are quantities of arithmetic significance, for instance see [Pol16, AkDr12, Chi22].

One topic that has drawn a lot of attention involves figuring out the average order of $d_k(n)$ over sparse sequences of values taken by polynomials, i.e.,

$$D_k(f(x_1,x_2,\dots,x_n),x) := \sum_{|f(x_1,x_2,\dots,x_n)|\le x} d_k(|f(x_1,x_2,\dots,x_n)|).$$

For $f(x_1,x_2)$ a binary irreducible cubic form, Greaves [Gre70] proved that there exist real constants $c_1>0$ and c_2 depending only on $f(x_1,x_2)$, such that

$$D_2(f(x_1,x_2,\dots,x_n),x) = c_1 x^{\frac{2}{3}}\log x + c_2 x^{\frac{2}{3}} + O\left(x^{\frac{9}{14}+\epsilon}\right),$$

for any $\epsilon>0$ as $x\to\infty$. If $f(x_1,x_2)$ is an irreducible quartic form, Daniel [Dan99] proved that

$$D_2(f(x_1,x_2,\dots,x_n),x) = c_1 x^{\frac{1}{2}}\log x + O\left(x^{\frac{1}{2}}\log\log x\right),$$

where $c_1>0$ is a constant depending only on f. More related work can be found in [LaZh21].

In 1998, Friedlander and Iwaniec [FrIw98] established the asymptotic formula for the distribution of prime values of a^4+b^2. More precisely, they proved

$$\sum_{\substack{a=1\\ a^2+b^4\le x}}^{\infty}\sum_{b=1}^{\infty}\Lambda(a^2+b^4) = \frac{4}{\pi}\kappa x^{\frac{3}{4}}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right),$$

where κ is some constant. Motivated from the above result, they [FrIw06] replaced Λ by d_k and established the following asymptotic formula for $d_3(a^6+b^2)$:

$$\sum_{\substack{a=1\\ a^2+b^4\le x\\ (a,b)=1}}^{\infty}\sum_{b=1}^{\infty} d_3(a^6+b^2) = c\kappa x^{\frac{2}{3}}(\log x)^2 + O\left(x^{\frac{2}{3}}(\log x)^{\frac{7}{4}}(\log\log x)^{\frac{1}{2}}\right),$$

where c and κ are some constants. For irreducible binary definite quadratic forms f, Daniel [Dan97] proved an asymptotic formula for $D_k(f(x_1,x_2,\dots,x_n),x)$ for any $k\ge 2$. With the help of circle method, Sun and Zhang [SuZh16] proved that

$$\sum_{1\le a_1,a_2,a_3\le x} d_3(a_1^2+a_2^2+a_3^2) = c_1x^3(\log x)^2 + c_2x^3\log x + c_3x^3 + O\left(x^{\frac{11}{4}+\epsilon}\right),$$

where c_1,c_2,c_3 are some constants and ϵ is any positive number. Finally, Blomer [Blo18] proved an asymptotic formula for $D_k(f(x_1,x_2,\dots,x_n),x)$ for any $k\ge 2$ where $f(x_1,x_2,\dots,x_n)$ is a form of degree k in $n=k-1$ variables, coming from incomplete norm form. Very recently, Lapkova and Zhou [LaZh21] investigated the average sum of the k^{th} divisor function over values of quadratic polynomials f, not necessarily homogenous, in $n\ge 3$ variables for any $k\ge 2$.

Let $\mathbb{K}_3$ be a non-normal cubic extension of a rational field $\mathbb{Q}$ given by an irreducible polynomial $f(x)=x^3+ax^2+bx+c$ of discriminant $D(<0)$. It is natural to study the k^{th} integral power of Dedekind zeta function, i.e.,

$$(\zeta_{\mathbb{K}_3}(s))^k = \sum_{n=1}^{\infty}\frac{a_{k,\mathbb{K}_3}(n)}{n^s}, \tag{1.1}$$

for $\Re(s) > 1$, where $a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1n_2\ldots n_k} a_{\mathbb{K}_3}(n_1)a_{\mathbb{K}_3}(n_2)\ldots a_{\mathbb{K}_3}(n_k)$.

In 2012, Lü [Lü13] was able to refine the previously known results (by Fomenko [Fom08]) of mean square and third power of $a_{\mathbb{K}_3}(n)$ to

$$\sum_{n\leq x} a_{\mathbb{K}_3}(n)^2 = a_1 x\log x + a_2 x + O\left(x^{\frac{23}{31}+\epsilon}\right)$$

where a_1 and a_2 are constants and

$$\sum_{n\leq x} a_{\mathbb{K}_3}(n)^3 = xP_3(\log x) + O\left(x^{\frac{235}{259}+\epsilon}\right),$$

where $P_3(t)$ is a suitable polynomial in t of degree 4.

In this paper, we will consider the average of the Dirichlet coefficients $a_{k,\mathbb{K}_3}(n)$ of the k^{th} power $(\zeta_{\mathbb{K}_3}(s))^k$ of the Dedekind zeta-function of a non-normal cubic extension $\mathbb{K}_3$ of $\mathbb{Q}$ over the sequence of values of a binary quadratic form $F(x_1, x_2, \ldots, x_6) = \sum_{k=1}^{6} x_k{}^2$. More precisely, we are interested in the asymptotic formula for the sum

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} a_{k,\mathbb{K}_3}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2),$$

for any integer $k \geq 1$, where

$$a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1n_2\ldots n_k} a_{\mathbb{K}_3}(n_1)a_{\mathbb{K}_3}(n_2)\ldots a_{\mathbb{K}_3}(n_k).$$

Note that, $a_{1,\mathbb{K}_3}(n) = a_{\mathbb{K}_3}(n)$.
First, we make the following remark.

Remark 1. Let $|t| \geq 1$ and $\epsilon > 0$ be any small constant. Then we have

$$\zeta\left(\frac{1}{2}+it\right) \ll (|t|+1)^{\mu+\epsilon},$$

where $\mu = \mu(\frac{1}{2})$. Moreover, Phragmén Lindelöf principle leads to

$$\zeta(\sigma+it) \ll (|t|+1)^{2\mu(1-\sigma)+\epsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $|t| \geq 1$.

For any integer $k \geq 1$, writing,

$$\begin{aligned}\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} a_{k,\mathbb{K}_3}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) &= \sum_{n\leq x} a_{k,\mathbb{K}_3}(n) \sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} 1\\ &= \sum_{n\leq x} a_{k,\mathbb{K}_3}(n) r_6(n) \qquad (1.2)\\ &= M_{k,\mathbb{K}_3}(x) + \widetilde{E}_{k,\mathbb{K}_3}(x),\end{aligned}$$

where $M_{k,\mathbb{K}_3}(x)$ is the main term which is of the form $x^3 P_{k-1}(\log x)$, where $P_{k-1}(t)$ is a polynomial in t of degree $k-1$. We prove the following theorem.

Theorem 1. *Let $\epsilon > 0$ (be any small constant) and define (for $k \geq 1$) $\widetilde{\lambda}_k = \max(\lambda_k, \lambda'_k)$, where λ_k, λ'_k are defined in Theorems 2, 3 resp. Then we have for any integer $k \geq 1$,*

$$\widetilde{E}_{k,\mathbb{K}_3}(x) \ll x^{3-\frac{1}{2(1+\widetilde{\lambda}_k)}+3k\epsilon}.$$

To prove Theorem 1, first, we demonstrate the following theorems:

Theorem 2. *Let $\epsilon > 0$ (be any small constant) and define $\lambda_1 = 3\epsilon$, $\lambda_2 = \min\left(2\mu, \frac{1}{4}\right)$, $\lambda_3 = \min\left(\mu + \frac{1}{2}, \frac{5}{8}\right)$, $\lambda_4 = \min\left(2\mu + \frac{3}{4}, 1\right)$, $\lambda_5 = \min\left(3\mu + 1, \frac{3}{2}\right)$ and $\lambda_k = \mu(k-6) + \frac{k}{3}$ for $k \geq 6$. Then we have for any integer $k \geq 1$,*

$$E_{k,\mathbb{K}_3}(x) \ll x^{3-\frac{1}{2(1+\lambda_k)}+3k\epsilon},$$

where

$$\sum_{n\leq x} a_{k,\mathbb{K}_3}(n) l_1(n) = M_{k,\mathbb{K}_3}(x) + E_{k,\mathbb{K}_3}(x),$$

and $l_1(n)$ is defined in Section 2.

Remark 2. From [Bou17] of Bourgain, we can very well take $\mu = \frac{13}{84}$. Thus, the theorem is unconditional with $\mu = \frac{13}{84}$.

Theorem 3. *Let $\epsilon > 0$ (be any small constant) and define $\lambda'_1 = 3\epsilon$, $\lambda'_2 = \frac{1}{4}$, and $\lambda'_k = \frac{3k-5}{6}$ for $k \geq 3$. Then we have for any integer $k \geq 1$,*

$$E'_{k,\mathbb{K}_3}(x) \ll x^{3-\frac{1}{2(1+\lambda'_k)}+3k\epsilon},$$

where

$$\sum_{n\leq x} a_{k,\mathbb{K}_3}(n) v_1(n) = E'_{k,\mathbb{K}_3}(x),$$

and $v_1(n)$ is defined in Section 2.

From (1.2), one can easily see that the proof of Theorem 1 follows from the proof of Theorems 2 and 3.

2. Preliminaries and some important lemmas

Let $r_k(n) := \#\{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k : n_1^2 + n_2^2 + \cdots + n_k^2 = n\}$ allowing zeros, distinguishing signs, and order. We will be concerned with the function $r_6(n)$.

Lemma 1. *For any positive integer n, we have*

$$r_6(n) = 16\sum_{d|n} \chi(d')d^2 - 4\sum_{d|n} \chi(d)d^2, \tag{2.3}$$

where $dd' = n$, and χ is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv -1 \pmod 4 \\ 0 & \text{if } n \equiv 0 \pmod 2 \end{cases}.$$

Proof. See, for instance, Lemma 1 of [ShSa22c].

We can reframe the equation (2.3) as

$$\begin{aligned} r_6(n) &= 16\sum_{d|n}\chi(d)\frac{n^2}{d^2} - 4\sum_{d|n}\chi(d)d^2 \\ &=: 16l(n) - 4v(n). \end{aligned}$$

We write $l_1(n) = 16l(n)$, and $v_1(n) = 4v(n)$.

The functions $\chi(d)$ and $\frac{n^2}{d^2}$ are completely multiplicative functions. This implies that $\chi(d)\frac{n^2}{d^2}$ is multiplicative. If $g(d)$ is any multiplicative function, then $\sum_{d|n} g(d)$ is also multiplicative. Therefore, $l(n)$ is a multiplicative function. Similarly, $v(n)$ is also multiplicative.

Note that

$$\begin{aligned} l(p) &= p^2 + \chi(p), \\ l(p^2) &= p^4 + p^2\chi(p) + \chi(p^2), \end{aligned}$$

and

$$\begin{aligned} v(p) &= 1 + p^2\chi(p), \\ v(p^2) &= 1 + p^2\chi(p) + p^4\chi(p^2). \end{aligned}$$

Lemma 2. (**[Lü13]**) *For* $\Re(s) > 1$, *we have*

$$\zeta_{\mathbb{K}_3}(s) = \zeta(s)L(s, f),$$

where f *is a holomorphic cusp form of weight 1 with respect to the congruence group* $\Gamma_0(|D|)$ *and* $D(<0)$ *be the discriminant of* $f(x) = x^3 + ax^2 + bx + c$.

From Lemma 2, we can write

$$a_{\mathbb{K}_3}(n) = \sum_{d|n}\lambda_f(d).$$

Also, note that

$$a_{\mathbb{K}_3}(p) = 1 + \lambda_f(p).$$

Lemma 3. *For any* $\epsilon > 0$, *we have*

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^4 dt \sim \frac{T(\log T)^4}{2\pi} \tag{2.4}$$

and

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^{12} dt \ll T^{2+\epsilon} \tag{2.5}$$

uniformly for $T \geq 1$.

Proof. For the proof of (2.4) see (Theorem 5.1 of [Ivi12]), and (2.5) result is due to Heath-Brown [Hea78].

Lemma 4. ([Go82]) *For any* $\epsilon > 0$*, we have*

$$\int_1^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt \ll T \log T,$$

uniformly for $T \geq 1$ *and*

$$L(\sigma + it, \chi) \ll_\epsilon (1 + |t|)^{\frac{1}{3}(1-\sigma)+\epsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$*, and* $|t| \geq t_0$ *(where* t_0 *is sufficiently large).*

Lemma 5. ([Rama74]) *For any* $\epsilon > 0$*, we have*

$$\int_1^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll T^{1+\epsilon},$$

uniformly for $T \geq 1$*.*

Lemma 6. *For any* $\epsilon > 0$ *and for any* $T \geq 1$ *uniformly, we have*

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^2 dt \sim cT \log T \tag{2.6}$$

and

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^6 dt \ll T^{2+\epsilon}. \tag{2.7}$$

Proof. Proofs of (2.6) and (2.7) follow by A. Good [Go82] and Jutila [Jut87], respectively.

Lemma 7. *For any* $\epsilon > 0$*, we have*

$$L(\sigma + it, f) \ll (1 + |t|)^{\max(\frac{2(1-\sigma)}{3}, 0)+\epsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ *,* $|t| \geq 1$*.*

Proof. Proof follows from a result of A. Good [Go82] on using maximum-modulus principle to a suitable function.

Lemma 8. *Let* f *be defined as in Lemma 2 and* $a_{k,\mathbb{K}_3}(n)$ *be defined as in equation (1.1). If*

$$F_k(s) = \sum_{n=1}^{\infty} \frac{a_{k,\mathbb{K}_3}(n) l(n)}{n^s},$$

for $\Re(s) > 3$*, then*

$$F_k(s) = G_k(s) H_k(s),$$

where

$$G_k(s) = \zeta(s-2)^k L(s, \chi)^k L(s-2, f)^k L(s, f \otimes \chi)^k,$$

and χ *is the non-principal character modulo 4. Here,* $H_k(s)$ *is a Dirichlet series which converges uniformly and absolutely in the half plane* $\Re(s) > \frac{5}{2}$*, and* $H_k(s) \neq 0$ *on* $\Re(s) = 3$*.*

Proof. We observe that $a_{k,\mathbb{K}_3}(n)l(n)$ is multiplicative, and hence

$$F_k(s) = \prod_p \left(1 + \frac{a_{k,\mathbb{K}_3}(n)l(p)}{p^s} + \cdots + \frac{a_{k,\mathbb{K}_3}(p^m)l(p^m)}{p^{ms}} + \cdots\right).$$

Note that

$$\begin{aligned} a_{k,\mathbb{K}_3}(n)l(p) &= k a_{\mathbb{K}_3}(p)l(p) \\ &= k\left(1+\lambda_f(p)\right)\left(p^2+\chi(p)\right) \\ &= kp^2 + k\chi(p) + kp^2\lambda_f(p) + k\lambda_f(p)\chi(p) \\ &=: b(p). \end{aligned}$$

From the structure of $b(p)$, we define the coefficients $b(n)$ as

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s-2)^k L(s,\chi)^k L(s-2,f)^k L(s, f\otimes\chi)^k,$$

which is absolutely convergent in $\Re(s) > 3$. We also note that

$$\begin{aligned} &\prod_p \left(1 + \frac{b(p)}{p^s} + \cdots + \frac{b(p^m)}{p^{ms}} + \cdots\right) \\ &= \zeta(s-2)^k L(s,\chi)^k L(s-2,f)^k L(s, f\otimes\chi)^k \\ &=: G_k(s), \end{aligned}$$

for $\Re(s) > 3$. Observe that $b(n) \ll_\epsilon n^{2+\epsilon}$ for any small positive constant ϵ.

Now, we note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\begin{aligned} \left|\frac{b(p)}{p^s} + \frac{b(p^2)}{p^{2s}} + \cdots + \frac{b(p^m)}{p^{ms}} + \cdots\right| &\ll \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{m\sigma}} \\ &\leq \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{(3+2\epsilon)m}} \\ &= \sum_{m=1}^{\infty} \frac{1}{p^{(1+\epsilon)m}} \\ &= \frac{\frac{1}{p^{1+\epsilon}}}{1 - \frac{1}{p^{1+\epsilon}}} \\ &= \frac{1}{p^{1+\epsilon} - 1} \\ &< 1. \end{aligned}$$

Let us write

$$A = \frac{a_{k,\mathbb{K}_3}(p)l(p)}{p^s} + \cdots + \frac{a_{k,\mathbb{K}_3}(p^m))l(p^m)}{p^{ms}} + \cdots, \quad \text{and} \quad B = \frac{b(p)}{p^s} + \cdots + \frac{b(p^m)}{p^{ms}} + \cdots.$$

From the above calculations, we observe that $|B| < 1$ in $\Re(s) \geq 3 + 2\epsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\begin{aligned}\frac{1+A}{1+B} &= (1+A)(1-B+B^2-B^3+\cdots)\\ &= 1+A-B-AB+\text{higher terms}\\ &= 1+\frac{a_{k,\mathbb{K}_3}(p^2)l(p^2)-b(p^2)}{p^{2s}}+\cdots+\frac{c_m(p^m)}{p^{ms}}+\cdots,\end{aligned}$$

with $c_m(n) \ll_\epsilon n^{2+\epsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}\prod_p\left(\frac{1+A}{1+B}\right) &= \prod_p\left(1+\frac{a_{k,\mathbb{K}_3}(p^2)l(p^2)-b(p^2)}{p^{2s}}+\cdots+\frac{c_m(p^m)}{p^{ms}}+\cdots\right)\\ &\ll_\epsilon 1.\end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}H_k(s) := \frac{F_k(s)}{G_k(s)} &= \prod_p\left(\frac{1+A}{1+B}\right)\\ &\ll_\epsilon 1,\end{aligned}$$

and also $H_k(s) \neq 0$ on $\Re(s) = 3$.

Lemma 9. *Let f be defined as in Lemma 2 and $a_{k,\mathbb{K}_3}(n)$ be defined as in equation (1.1). If*

$$\widetilde{F}_k(s) = \sum_{n=1}^{\infty}\frac{a_{k,\mathbb{K}_3}(n)v(n)}{n^s},$$

for $\Re(s) > 3$, then

$$\widetilde{F}_k(s) = \widetilde{G}_k(s)\widetilde{H}_k(s),$$

where

$$\widetilde{G}_k(s) = \zeta(s)^k L(s-2,\chi)^k L(s,f)^k L(s-2,f\otimes\chi)^k,$$

and χ is the non-principal character modulo 4. Here, $\widetilde{H}_k(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\widetilde{H}_k(s) \neq 0$ on $\Re(s) = 3$.

Proof. The proof of Lemma 9 follows along similar lines as the proof of Lemma 8.

Lemma 10. ([JiLü14]) *Let χ be a primitive character modulo q and $\mathfrak{L}^d_{m,n}(s,\chi)$ be a general L-function of degree $2A$. For any $\epsilon > 0$, we have*

$$\int_T^{2T}\left|\mathfrak{L}^d_{m,n}(\sigma+it,\chi)\right|^2 dt \ll (qT)^{2A(1-\sigma)+\epsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\epsilon$, and $T \geq 1$. Also,

$$\mathfrak{L}^d_{m,n}(\sigma+it,\chi) \ll (q(1+|t|))^{\max\{A(1-\sigma),0\}+\epsilon},$$

uniformly for $-\epsilon \leq \sigma \leq 1+\epsilon$.

3. Proof of Theorem 2

Let $k \geq 1$ be an integer. Firstly, we consider the sum $\sum_{n \leq x} a_{k,\mathbb{K}_3}(n) l_1(n)$. We begin by applying Perron's formula (see [GrSo14, Chapter 2.4]) to $F_k(s)$ with $\eta = 3 + \epsilon$ and $10 \leq T \leq x$. Thus, we have

$$\begin{aligned}\sum_{n \leq x} a_{k,\mathbb{K}_3}(n) l_1(n) &= 16 \sum_{n \leq x} a_{k,\mathbb{K}_3}(n) l(n) \\ &= \frac{16}{2\pi i} \int_{\eta - iT}^{\eta + iT} F_k(s) \frac{x^s}{s} ds + O\left(\frac{x^{3+3\epsilon}}{T}\right).\end{aligned}$$

We move the line of integration to $\Re(s) = \frac{5}{2} + \epsilon$. By Cauchy's residue theorem there is only one pole at $s = 3$ of order k, coming from the factor $\zeta(s-2)^k$.

So, we obtain

$$\begin{aligned}\sum_{n \leq x} a_{k,\mathbb{K}_3}(n) l_1(n) = \operatorname*{Res}_{s=3} &\left\{F_k(s)\frac{x^s}{s}\right\} + \frac{16}{2\pi i}\left\{\int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT}\right\} F_k(s)\frac{x^s}{s} ds \\ &+ O\left(\frac{x^{3+3\epsilon}}{T}\right) \\ &=: x^3 P_{k-1}(\log x) + \frac{16}{2\pi i}(J_1(k) + J_2(k) + J_3(k)) + O\left(\frac{x^{3+3\epsilon}}{T}\right),\end{aligned}$$

where $P_{k-1}(t)$ is a polynomial in t of degree $k-1$.

Note that the horizontal lines ($J_2(k)$ and $J_3(k)$) contribute (for any fixed integer $k \geq 1$), using Lemma 6, Lemma 7 and Remark 1

$$\begin{aligned}J_2(k) + J_3(k) &\ll (x^2) \max_{\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon} x^\sigma T^{(2k\mu + \frac{2k}{3})(1-\sigma)+\epsilon} T^{-1} \\ &\ll (x^{2+\epsilon}) \max_{\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon} \left(\frac{x}{T^{2k\mu + \frac{2k}{3}}}\right)^\sigma T^{2k\mu + \frac{2k}{3} - 1 + \epsilon}.\end{aligned}$$

For any fixed k, $\mu (> 0)$, $\left(\frac{x}{T^{2k\mu+\frac{2k}{3}}}\right)^\sigma$ is monotonic as a function of σ for $\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon$ and hence the maximum is attained at the extremities of the interval $\left[\frac{1}{2} + \epsilon, 1 + \epsilon\right]$. Thus,

$$J_2(k) + J_3(k) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3\epsilon} T^{\frac{1}{2}(2k\mu + \frac{2k}{3}) - 1}.$$

Vertical line contributions:

1. **For k=1:**

$$J_1(1) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_1(s) \frac{x^s}{s} ds.$$

Using Lemma 5, Lemma 6, Lemma 7 and Cauchy-Schwarz inequality,

$$\begin{aligned}
J_1(1) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\int_{\frac{U}{2}}^{U}\left|\zeta(\frac{1}{2}+it)\right|\left|L(\frac{1}{2}+it,f)\right|dt\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\left(\int_{\frac{U}{2}}^{U}\left|\zeta(\frac{1}{2}+it)\right|^2dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f)\right|^2dt\right)^{\frac{1}{2}}\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}\log T\max_{1\le U\le T}\left\{\frac{1}{U}U^{\frac{1}{2}+\epsilon}U^{\frac{1}{2}+\epsilon}\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}T^{3\epsilon},
\end{aligned}$$

which dominates over $J_2(1)+J_3(1)$.

2. **For k=2:**

$$J_1(2) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_2(s)\frac{x^s}{s}ds.$$

Using Lemma 7, Remark 1 and Lemma 5,

$$\begin{aligned}
J_1(2) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\int_{\frac{U}{2}}^{U}\left|\zeta(\frac{1}{2}+it)\right|^2\left|L(\frac{1}{2}+it,f)\right|^2dt\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}\log T\max_{1\le U\le T}\left\{\frac{1}{U}U^{2\mu+2\epsilon}U\log U\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}T^{2\mu+4\epsilon}.
\end{aligned}$$

Note that, by Lemma 5

$$\begin{aligned}
\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f)\right|^4dt &\ll \left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f)\right|^2dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f)\right|^6dt\right)^{\frac{1}{2}}\\
&\ll U^{\frac{3}{2}+\epsilon}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
J_1(2) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\int_{\frac{U}{2}}^{U}\left|\zeta(\frac{1}{2}+it)\right|^2\left|L(\frac{1}{2}+it,f)\right|^2dt\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\left(\int_{\frac{U}{2}}^{U}\left|\zeta(\frac{1}{2}+it)\right|^4dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f)\right|^4dt\right)^{\frac{1}{2}}\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}\log T\max_{1\le U\le T}\left\{\frac{1}{U}U^{\frac{1}{2}+\epsilon}U^{\frac{1}{2}(\frac{3}{2}+\epsilon)}\right\}\text{ (using Lemma 2 and above observation)}\\
&\ll x^{\frac{5}{2}+\epsilon}T^{\frac{1}{4}+2\epsilon}.
\end{aligned}$$

Thus, we have

$$J_1(2) \ll x^{\frac{5}{2}+4\epsilon}T^{\min\left(2\mu,\frac{1}{4}\right)},$$

which dominates over $J_2(2)+J_3(2)$.

3. **For k=3:**

$$J_1(3) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_3(s)\frac{x^s}{s}ds.$$

Using Lemma 2, Lemma 5, Cauchy-Schwarz Inequality and Remark 1,

$$\begin{aligned}
J_1(3) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^3\left|L\left(\frac{1}{2}+it,f\right)\right|^3 dt\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}U^{\mu+\epsilon}\left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt\right)^{\frac{1}{2}}\right.\\
&\qquad\left.\times\left(\int_{\frac{U}{2}}^{U}\left|L\left(\frac{1}{2}+it,f\right)\right|^6 dt\right)^{\frac{1}{2}}\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}\log T\max_{1\le U\le T}\left\{\frac{1}{U}U^{\mu+\epsilon}U^{\frac{1}{2}(1+\epsilon)}U^{\frac{1}{2}(2+\epsilon)}\right\}\\
&\ll x^{\frac{5}{2}+4\epsilon}T^{\mu+\frac{1}{2}}.
\end{aligned}$$

Also, we have (using Lemma 2, Lemma 5 and above observation)

$$\begin{aligned}
J_1(3) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt\right)^{\frac{1}{4}}\left(\int_{\frac{U}{2}}^{U}\left|L\left(\frac{1}{2}+it,f\right)\right|^4 dt\right)^{\frac{3}{4}}\right\}\\
&\ll x^{\frac{5}{2}+\epsilon}\log T\max_{1\le U\le T}\left\{\frac{1}{U}U^{\frac{1}{2}+\epsilon}U^{\frac{3}{4}\left(\frac{3}{2}+\epsilon\right)}\right\}\\
&\ll x^{\frac{5}{2}+5\epsilon}T^{\frac{5}{8}}.
\end{aligned}$$

Thus, we have

$$J_1(3)\ll x^{\frac{5}{2}+5\epsilon}T^{\min\left(\mu+\frac{1}{2},\frac{5}{8}\right)},$$

which dominates over $J_2(3)+J_3(3)$.

4. **For *k*=4:** First we observe, (using Lemma 2 and Cauchy-Schwarz inequality)

$$\begin{aligned}
\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^8 dt &\ll \left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt\right)^{\frac{1}{2}}\\
&\ll U^{\frac{1}{2}(1+\epsilon)}U^{\frac{1}{2}(2+\epsilon)}\\
&\ll U^{\frac{3}{2}+\epsilon}.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^6 dt &\ll \left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{8} dt\right)^{\frac{1}{2}}\\
&\ll U^{\frac{1}{2}(1+\epsilon)}T^{\frac{1}{2}\left(\frac{3}{2}+\epsilon\right)}\\
&\ll U^{\frac{5}{4}+\epsilon}.
\end{aligned}$$

Now,

$$J_1(4):=\int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT}F_4(s)\frac{x^s}{s}ds.$$

Using Lemma 2, Lemma 5, Hölder's inequality and Remark 1,

$$J_1(4) \ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} \int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right)\right|^4 \left| L\left(\frac{1}{2}+it, f\right)\right|^4 dt \right\}$$

$$\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} U^{2\mu+2\epsilon} \left(\int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right)\right|^6 dt \right)^{\frac{1}{3}} \right.$$

$$\left. \times \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2}+it, f\right)\right|^6 dt \right)^{\frac{2}{3}} \right\}$$

$$\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \le U \le T} \left\{ \frac{1}{U} U^{2\mu+2\epsilon} U^{\frac{1}{3}(\frac{5}{4}+\epsilon)} U^{\frac{2}{3}(2+\epsilon)} \right\} \text{(using above observation)}$$

$$\ll x^{\frac{5}{2}+5\epsilon} T^{2\mu+\frac{3}{4}}.$$

Also, we have (using Lemma 7, Lemma 2, Lemma 5, and Hölder's inequality)

$$J_1(4) \ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} \left(\int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \right)^{\frac{1}{3}} \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2}+it, f\right)\right|^6 dt \right)^{\frac{2}{3}} \right\}$$

$$\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \le U \le T} \left\{ \frac{1}{U} U^{\frac{1}{3}(2+\epsilon)} U^{\frac{2}{3}(2+\epsilon)} \right\}$$

$$\ll x^{\frac{5}{2}+5\epsilon} T.$$

Thus, we have

$$J_1(4) \ll x^{\frac{5}{2}+5\epsilon} T^{\min\left(2\mu+\frac{3}{4}, 1\right)},$$

which dominates over $J_2(4) + J_3(4)$.

5. **For k=5:**

$$J_1(5) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_5(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Lemma 5, Hölder's inequality and Remark 1,

$$J_1(5) \ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} \int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right)\right|^5 \left| L\left(\frac{1}{2}+it, f\right)\right|^5 dt \right\}$$

$$\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} U^{3\mu+3\epsilon} \left(\int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \right)^{\frac{1}{6}} \right.$$

$$\left. \times \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2}+it, f\right)\right|^6 dt \right)^{\frac{5}{6}} \right\}$$

$$\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \le U \le T} \left\{ \frac{1}{U} U^{3\mu+3\epsilon} U^{\frac{1}{6}(2+\epsilon)} U^{\frac{5}{6}(2+\epsilon)} \right\}$$

$$\ll x^{\frac{5}{2}+6\epsilon} T^{3\mu+1}.$$

Also, we have

$$\begin{aligned}J_1(5) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{12}dt\right)^{\frac{5}{12}}\left(\int_{\frac{U}{2}}^{U}\left|L\left(\frac{1}{2}+it,f\right)\right|^{5.\frac{12}{7}}dt\right)^{\frac{7}{12}}\right\}\\ &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}U^{\frac{5}{12}(2+\epsilon)}U^{\frac{7}{12}(2+\epsilon)}U^{\frac{18}{7}\cdot\frac{1}{3}\cdot\frac{7}{12}+2\epsilon}\right\}\\ &\ll x^{\frac{5}{2}+5\epsilon}T^{\frac{3}{2}}.\end{aligned}$$

Thus, we have

$$J_1(5)\ll x^{\frac{5}{2}+5\epsilon}T^{\min\left(3\mu+1,\frac{3}{2}\right)},$$

which dominates over $J_2(5)+J_3(5)$.

6. **For $k\ge 6$:**

$$J_1(k):=\int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT}F_k(s)\frac{x^s}{s}ds.$$

Using Lemma 2, Cauchy-Schwarz Inequality, Lemma 5, Lemma 6, and Remark 1 we get

$$\begin{aligned}J_1(k) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{k}\left|L\left(\frac{1}{2}+it,f\right)\right|^{k}dt\right\}\\ &\ll x^{\frac{5}{2}+\epsilon}\left\{\max_{1\le U\le T}\ \frac{1}{U}U^{(k-6)(\mu+\epsilon)}U^{(k-3)(\frac{1}{3}+\epsilon)}\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{6}\left|L\left(\frac{1}{2}+it,f\right)\right|^{3}dt\right\}\\ &\ll x^{\frac{5}{2}+2k\epsilon}\left\{\max_{1\le U\le T}\ U^{\mu(k-6)+\frac{1}{3}(k-3)-1}\left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{12}dt\right)^{\frac{1}{2}}\right.\\ &\qquad\left.\times\left(\int_{\frac{U}{2}}^{U}\left|L\left(\frac{1}{2}+it,f\right)\right|^{6}\right)^{\frac{1}{2}}dt\right\}\\ &\ll x^{\frac{5}{2}+2k\epsilon}\left\{\max_{1\le U\le T}\ U^{\mu(k-6)+\frac{1}{3}(k-3)-1}U^{\frac{1}{2}(2+\epsilon)}U^{\frac{1}{2}(2+\epsilon)}\right\}\\ &\ll x^{\frac{5}{2}+3k\epsilon}T^{\mu(k-6)+\frac{k}{3}}.\end{aligned}$$

Define

$$\lambda_k:=\mu(k-6)+\frac{k}{3},$$

for $k\ge 6$, then

$$J_1(k)\ll x^{\frac{5}{2}+3k\epsilon}T^{\lambda_k}.$$

Therefore, we have (for $k\ge 1$)

$$\sum_{n\le x}a_{k,\mathbb{K}_3}(n)l_1(n)=x^3P_{k-1}(\log x)+E_{k,\mathbb{K}_3}(x),$$

where $E_{k,\mathbb{K}_3}(x)\ll\dfrac{x^{3+3\epsilon}}{T}+x^{\frac{5}{2}+3k\epsilon}T^{\lambda_k}$. We choose T such that $\dfrac{x^3}{T}\sim x^{\frac{5}{2}}T^{\lambda_k}$, i.e., $T^{1+\lambda_k}\sim x^{\frac{1}{2}}$, i.e., $T\sim x^{\frac{1}{2(1+\lambda_k)}}$.

So finally, we have

$$E_{k,\mathbb{K}_3}(x)\ll x^{3-\frac{1}{2(1+\lambda_k)}+3k\epsilon}.$$

This proves the theorem.

4. Proof of Theorem 3

Let $k \geq 1$ be an integer. Now, we consider the sum $\sum_{n\leq x} a_{k,\mathbb{K}_3}(n)v_1(n)$. We begin by applying Perron's formula (see [GrSo14, Chapter 2.4]) to $\widetilde{F}_k(s)$ with $\eta = 3+\epsilon$ and $10 \leq T \leq x$. Thus, we have

$$\sum_{n\leq x} a_{k,\mathbb{K}_3}(n)v_1(n) = 4\sum_{n\leq x} a_{k,\mathbb{K}_3}(n)v(n)$$
$$= \frac{4}{2\pi i}\int_{\eta-iT}^{\eta+iT} \widetilde{F}_k(s)\frac{x^s}{s}ds + O\left(\frac{x^{3+3\epsilon}}{T}\right).$$

We move the line of integration to $\Re(s) = \frac{5}{2}+\epsilon$. There is no singularity in the rectangle obtained and the function $\widetilde{F}_k(s)\frac{x^s}{s}$ is analytic in this region. Thus, using Cauchy's theorem for rectangles pertaining to analytic functions, we get

$$\sum_{n\leq x} a_{k,\mathbb{K}_3}(n)v_1(n) = \operatorname*{Res}_{s=3}\left\{\widetilde{F}_k(s)\frac{x^s}{s}\right\} + \frac{4}{2\pi i}\left\{\int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT}\right\}\widetilde{F}_k(s)\frac{x^s}{s}ds$$
$$+ O\left(\frac{x^{3+3\epsilon}}{T}\right)$$
$$=: \frac{4}{2\pi i}(J_1'(k) + J_2'(k) + J_3'(k)) + O\left(\frac{x^{3+3\epsilon}}{T}\right).$$

Note that the horizontal lines ($J_2'(k)$ and $J_3'(k)$) contribute (for any fixed integer $k \geq 1$), using Lemma 9, Lemma 10 and Lemma 6

$$J_2'(k) + J_3'(k) \ll \left(x^2\right)\max_{\frac{1}{2}+\epsilon\leq\sigma\leq 1+\epsilon} x^{\sigma}T^{(\frac{k}{3}+\frac{2k}{3})(1-\sigma)+\epsilon}T^{-1}$$
$$\ll \left(x^{2+\epsilon}\right)\max_{\frac{1}{2}+\epsilon\leq\sigma\leq 1+\epsilon}\left(\frac{x}{T^k}\right)^{\sigma}T^{k-1+\epsilon}.$$

For any fixed k, $\mu(>0)$, $\left(\frac{x}{T^k}\right)^{\sigma}$ is monotonic as a function of σ for $\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon$ and hence the maximum is attained at the extremities of the interval $\left[\frac{1}{2}+\epsilon, 1+\epsilon\right]$. Thus,

$$J_2'(k) + J_3'(k) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3\epsilon}T^{\frac{k}{2}-1}.$$

Vertical line contributions:

1. **For k=1:**

$$J_1'(1) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT}\widetilde{F}_1(s)\frac{x^s}{s}ds.$$

Using Lemma 9, Lemma 3, Lemma 10 and Cauchy-Schwarz inequality,

$$J_1'(1) \ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\leq U\leq T}\ \frac{1}{U}\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,\chi)\right|\left|L(\frac{1}{2}+it,f\otimes\chi)\right|dt\right\}$$
$$\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\leq U\leq T}\ \frac{1}{U}\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,\chi)\right|^2dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f\otimes\chi)\right|^2dt\right)^{\frac{1}{2}}\right\}$$
$$\ll x^{\frac{5}{2}+\epsilon}\log T\max_{1\leq U\leq T}\left\{\frac{1}{U}U^{\frac{1}{2}+\epsilon}U^{\frac{1}{2}+\epsilon}\right\}$$
$$\ll x^{\frac{5}{2}+\epsilon}T^{3\epsilon},$$

which dominates over $J_2'(1)+J_3'(1)$.

2. **For** $\boldsymbol{k=2}$

$$J_1'(2):=\int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT}\widetilde{F}_2(s)\frac{x^s}{s}ds.$$

Note that, by Lemma 4

$$\begin{aligned}\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f\otimes\chi)\right|^4dt &\ll\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f\otimes\chi)\right|^2dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f\otimes\chi)\right|^6dt\right)^{\frac{1}{2}}\\ &\ll U^{\frac{3}{2}+\epsilon}.\end{aligned}$$

Using Lemma 4 and Lemma 10,

$$\begin{aligned}J_1'(2) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,\chi)\right|^2\left|L(\frac{1}{2}+it,f\otimes\chi)\right|^2dt\right\}\\ &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,\chi)\right|^4dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|L(\frac{1}{2}+it,f\otimes\chi)\right|^4dt\right)^{\frac{1}{2}}\right\}\\ &\ll x^{\frac{5}{2}+\epsilon}\log T\max_{1\le U\le T}\left\{\frac{1}{U}U^{\frac{1}{2}+\epsilon}U^{\frac{1}{2}(\frac{3}{2}+\epsilon)}\right\}\text{ (using above observation)}\\ &\ll x^{\frac{5}{2}+\epsilon}T^{\frac{1}{4}+2\epsilon},\end{aligned}$$

which dominates over $J_2'(2)+J_3'(2)$.

3. **For** $\boldsymbol{k\ge 3}$**:**

$$J_1'(k):=\int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT}\widetilde{F}_k(s)\frac{x^s}{s}ds.$$

Using Lemma 8, Cauchy-Schwarz Inequality, Lemma 4, Lemma 5, and Lemma 6 we get

$$\begin{aligned}J_1'(k) &\ll x^{\frac{5}{2}+\epsilon}\log T\left\{\max_{1\le U\le T}\ \frac{1}{U}\int_{\frac{U}{2}}^{U}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^k\left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^kdt\right\}\\ &\ll x^{\frac{5}{2}+\epsilon}\left\{\max_{1\le U\le T}\ \frac{1}{U}U^{(k-2)(1-\sigma)}U^{\frac{2}{3}(k-3)(1-\sigma)}\int_{\frac{U}{2}}^{U}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2\left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^3dt\right\}\\ &\ll x^{\frac{5}{2}+2k\epsilon}\left\{\max_{1\le U\le T}\ U^{\frac{k-2}{6}+\frac{k-3}{3}-1}\left(\int_{\frac{U}{2}}^{U}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^4dt\right)^{\frac{1}{2}}\right.\\ &\qquad\left.\times\left(\int_{\frac{U}{2}}^{U}\left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^6\right)^{\frac{1}{2}}dt\right\}\\ &\ll x^{\frac{5}{2}+2k\epsilon}\left\{\max_{1\le U\le T}\ U^{\frac{k-2}{6}+\frac{k-3}{3}-1}U^{\frac{1}{2}(1+\epsilon)}U^{\frac{1}{2}(2+\epsilon)}\right\}\\ &\ll x^{\frac{5}{2}+3k\epsilon}T^{\frac{k}{2}-\frac{5}{6}},\end{aligned}$$

which dominates over $J_2'(k)+J_3'(k)$.

Therefore, we have (for $k \geq 1$)

$$\sum_{n \leq x} a_{k,\mathbb{K}_3}(n) v_1(n) = E'_{k,\mathbb{K}_3}(x),$$

where $E'_{k,\mathbb{K}_3}(x) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3k\epsilon} T^{\lambda'_k}$. We choose T such that $\frac{x^3}{T} \sim x^{\frac{5}{2}} T^{\lambda'_k}$, i.e., $T^{1+\lambda'_k} \sim x^{\frac{1}{2}}$, i.e., $T \sim x^{\frac{1}{2(1+\lambda'_k)}}$.

So finally, we have

$$E'_{k,\mathbb{K}_3}(x) \ll x^{3-\frac{1}{2(1+\lambda'_k)}+3k\epsilon}.$$

This proves the theorem.

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Anubhav Sharma
Department of Mathematics
Indian Institute of Technology Delhi, Hauz Khas
New Delhi 110016. India
e-mail: anubhav6595@gmail.com

Ayyadurai Sankaranarayanan
School of Mathematics and Statistics
University of Hyderabad Central University PO, Prof. C.R. Rao Road
Gachibowli, Hyderabad 500046. India
e-mail: sank@uohyd.ac.in