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Infinite families of congruences modulo 2 for (ℓ, k)-regular partitions

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Abstract. Let $b_{\ell,k}(n)$ denote the number of (ℓ, k)-regular partition of n. Recently, some congruences modulo 2 for $(3, 8), (4, 7)$-regular partition and modulo 8, modulo 9 and modulo 12 for $(4, 9)$-regular partition have been studied. In this paper, we use theta function identities and Newman results to prove some infinite families of congruences modulo 2 for $(2, 7)$, $(5, 8)$, $(4, 11)$-regular partition and modulo 4 for $(4, 5)$-regular partition.

Keywords. partition function, regular-partition, Ramanujan congruences.

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1. Introduction

We shall use the following standard notation throughout the paper: for complex numbers a, q with $| q |< 1$, define

$$f_i := (q^i; q^i)_\infty, \quad i = 1, 2, 3, \dots, \text{where } (a; q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m)$$

Recall that a partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An ℓ-regular partition is a partition in which none of the parts is divisible by ℓ. Denote by $b_\ell(n)$ the number of ℓ-regular partitions of n with the convention $b_\ell(0) = 1$. Then the generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{f_\ell}{f_1}. \tag{1.1}$$

Recently, Kathiravan, Srinivas and Usha [KSS] studied the (ℓ, k)-regular partition denoted by $b_{\ell,k}(n)$, which count the number of partition of n if none of the parts is divisible by ℓ or k. Then the generating function for $b_{\ell,k}(n)$ is given by

$$\sum_{n=0}^{\infty} b_{\ell,k}(n) q^n = \frac{f_\ell f_k}{f_1 f_{\ell k}}. \tag{1.2}$$

They proved infinite families of congruences modulo 2 for $(3, 8)$, $(4, 7)$-regular partitions and modulo 8, 9 and 12 for $(4, 9)$-regular partition. For example, the authors have shown that for $n \geq 0$,

$$b_{4,9}(16n + 15) \equiv 0 \pmod 8 \tag{1.3}$$

$$b_{4,7}(14(7n + j) + 13) \equiv 0 \pmod 2, \quad 1 \leq j \leq 6. \tag{1.4}$$

In [MHS16], Naika, Hemanthkumar and Bharadwaj have proved infinite families of congruences modulo 2 for $b_{3,5}(n)$-regular partition. For example, they showed that [MHS16, Theorem 1.1] if p is an odd prime $\left(\frac{-15}{p}\right)_L = 1$, $1 \leq i \leq p - 1$ and $j, n \geq 0$,

$$b_{3,5}\left(2 \times p^{2j+2} n + \frac{(6i + 2p) \times p^{2j+1} + 1}{3}\right) \equiv 0 \pmod 2.$$

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The aim of this paper is to prove several infinite families of congruences modulo 2 for $(2,7)$ and $(5,8)$-regular partitions, and modulo 4 for $(4,5)$-regular partition. The following are our main results

Theorem 1.1. *Let $a(n)$ be defined by*

$$\sum_{n=0}^{\infty} a(n)q^n = f_1^2 f_7. \tag{1.5}$$

Let $p \geq 5$ be a prime and let $\left(\frac{\star}{p}\right)$ denote the Legendre symbol. Define

$$\omega(p) := a\big(\frac{3}{8}(p^2-1)\big) + \left(\frac{-14}{p}\right)\left(\frac{-\frac{3}{8}(p^2-1)}{p}\right). \tag{1.6}$$

(i) If $\omega(p) \equiv 0 \pmod 2$, then for $n, j \geq 0$, if $p \nmid n$, we have

$$b_{2,7}\big(2p^{4j+3}n + \frac{3}{4}p^{4j+4} + \frac{1}{4}\big) \equiv 0 \pmod 2. \tag{1.7}$$

(ii) If $\omega(p) \not\equiv 0 \pmod 2$, then for $n, j \geq 0$,

(a) if $p \nmid n$, we have

$$b_{2,7}\big(2p^{6j+5}n + \frac{3}{4}p^{6j+6} + \frac{1}{4}\big) \equiv 0 \pmod 2. \tag{1.8}$$

(b) if $p \nmid (8n+3)$ and $p \neq 7$, we have

$$b_{2,7}\big(2p^{6j+2}n + \frac{3}{4}p^{6j+2} + \frac{1}{4}\big) \equiv 0 \pmod 2. \tag{1.9}$$

(c) if $7 \nmid (8n+3)$, we have

$$b_{2,7}\big(2 \cdot 7^{6j+4}n + \frac{3}{4}7^{6j+4} + \frac{1}{4}\big) \equiv 0 \pmod 2. \tag{1.10}$$

Remark 1. *Using mathematica, we find that $a(9) \equiv 1 \pmod 2$. Using Theorem 1.1 with $p = 5$, we see that $\omega(5) \equiv 0 \pmod 2$ and hence by Theorem 1.1[(i)] we obtain that for any integer n not divisible by 5, we have $b_{2,7}(250n + 469) \equiv 0 \pmod 2$.*

Theorem 1.2. *Let $b(n)$ be defined by*

$$\sum_{n=0}^{\infty} b(n)q^n = f_1^2 f_5. \tag{1.11}$$

Let $p \geq 5$ be a prime and let $\left(\frac{\star}{p}\right)$ denote the Legendre symbol. Define

$$\omega(p) := b\big(7(p^2-1)/24\big) + \left(\frac{-10}{p}\right)\left(\frac{-7(p^2-1)/24}{p}\right). \tag{1.12}$$

(i) If $\omega(p) \equiv 0 \pmod 2$, then for n such that $p \nmid n$ and for all $j \geq 0$, we have

$$b_{5,8}\big(20p^{4j+3}n + \frac{35}{6}p^{4j+4} + \frac{7}{6}\big) \equiv 0 \pmod 2. \tag{1.13}$$

(ii) If $\omega(p) \not\equiv 0 \pmod 2$, then for all $j \geq 0$,

(a) if $p \nmid n$, we have

$$b_{5,8}\big(20p^{6j+5}n + 35p^{6j+6}/6 + 7/6\big) \equiv 0 \pmod 2. \tag{1.14}$$

(b) if $p \nmid 24n+7$ and $p \neq 5$, we have

$$b_{5,8}\big(20p^{6j+2}n + \frac{35}{6}p^{6j+2} + \frac{7}{6}\big) \equiv 0 \pmod 2, \tag{1.15}$$

(c) if $5 \nmid 24n+7$, we have

$$b_{5,8}\big(4 \cdot 5^{6j+5}n + \frac{7}{6}5^{6j+5} + \frac{7}{6}\big) \equiv 0 \pmod 2, \tag{1.16}$$

Remark 2. *Using mathematica, we find that $b(14) = 0$. Using Theorem 1.2 with $p = 7$, we see that $\omega(7) = 0$ and hence by Theorem 1.2[(i)] we obtain that for any integer n not divisible by 7, we have $b_{5,8}(6860n + 14007) \equiv 0 \pmod 2$.*

Theorem 1.3. *For $n, k \geq 0$ and $m \in \{2, 4\}$, we have*

$$b_{4,5}\left(4 \cdot 5^{4k}n + \frac{5^{4k}+1}{2}\right) \equiv b_{4,5}(4n+1) \pmod 4, \tag{1.17}$$

$$b_{4,5}\left(4 \cdot 5^{4k+1}n + \frac{5^{4k}(8m+1)+1}{2}\right) \equiv 0 \pmod 4, \tag{1.18}$$

$$b_{4,5}\left(4 \cdot 5^{4k+3}n + \frac{5^{4k+2}(8m+1)+1}{2}\right) \equiv 0 \pmod 4. \tag{1.19}$$

2. Preliminaries

In this section, we collect some lemmas to prove our main results. We recall that for $|\ ab\ | < 1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.20}$$

Now from Jacobi's triple product identity [Ber91, Entry 19, p. 35]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{2.21}$$

it follows that (see [Ber91, Entry 22, p.36])

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.22}$$

$$\psi(q) := f(q;q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{2.23}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{2.24}$$

Now we shall state few lemmas which will be required for the proof of our results.

Lemma 2.1. *(Hirschhorn and Sellers [HiSe10, Theorem 1]) The following 2-dissection holds*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{2.25}$$

Lemma 2.2. *The following 2-dissection holds (See [Ber91, p.315]):*

$$f_1 f_5^3 = f_2^3 f_{10} - q\frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3 \frac{f_4^4 f_{40}^2 f_{10}}{f_2 f_8^2}, \tag{2.26}$$

$$f_1^3 f_5 = \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3 + 2q^2 \frac{f_4^6 f_{10} f_{40}^2}{f_2 f_8^2 f_{20}^2}. \tag{2.27}$$

Lemma 2.3. *(Baruah and Ahmed [AhBa15, Eqn. (2.4)])*

$$\frac{1}{f_1 f_{11}} = \frac{f_{24}^2 f_{132}^5}{f_2^2 f_{12} f_{22}^2 f_{66}^2 f_{264}^2} + q\frac{f_4^2 f_6 f_{24} f_{88} f_{132}^2}{f_2^3 f_8 f_{12} f_{22}^2 f_{44} f_{264}} + q^6 \frac{f_8 f_{12}^2 f_{44}^2 f_{66} f_{264}}{f_2^2 f_4 f_{22}^3 f_{24} f_{88} f_{132}} + q^{15} \frac{f_{12}^5 f_{264}^2}{f_2^2 f_6^2 f_{22}^2 f_{24}^2 f_{132}}. \tag{2.28}$$

Lemma 2.4. *The following 5−dissection holds:*

$$f_1 = f_{25}\left(\frac{1}{F(q^5)} - q - q^2 F(q^5)\right) \tag{2.29}$$

as well as the identities

$$\frac{f_1^6}{f_5^6} = \left(\frac{1}{R^5} - 11q - q^2 R^5\right) \tag{2.30}$$

and

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}(F^4(q^5) + qF^3(q^5) + 2q^2F^2(q^5) + 3q^3F(q^5) + 5q^4 - 3q^5F^{-1}(q^5) + 2q^6F^{-2}(q^5) - q^7F^{-3}(q^5) + q^8F^{-4}(q^5)), \tag{2.31}$$

where $F(q) := q^{-1/5}R(q)$ and $R(q)$ is the Rogers-Ramanujan continued fraction defined, for $|\, q \,| < 1$, by

$$R(q) := \frac{q^{1/5}}{1}\,\frac{}{+}\,\frac{q}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+\cdots}.$$

The identities (2.29), (2.30) and (2.31) are same as (8.4.1), (8.4.3) and (8.4.4) respectively in [Hir17].

Lemma 2.5. *[CuGu13, Theorem 2.2] Let $p \geq 5$ be a prime such that*

$$\frac{\pm p - 1}{6} := \begin{cases} \dfrac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \dfrac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

With the notations as defined in (2.20) and (2.24), we have

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}.$$

Furthermore, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}, k \neq \frac{\pm p-1}{6}$, then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

Lemma 2.6. *(Ahmed and Baruah [AhBa16, Lemma 2.3]) If $p \geq 3$ is prime, then*

$$f_1^3 = \sum_{\substack{k=0 \\ k \neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn + 2k + 1) q^{pn \cdot \frac{pn+2k+1}{2}} + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} f_{p^2}^3.$$

Furthermore, if $k \neq \frac{p-1}{2}, 0 \leq k \leq p-1$, *then* $\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$.

The following result of M. Newman [New62, Theorem 3] will play a crucial role in the proof of our theorems, therefore we shall quote is as a lemma. Following the notations of Newman's paper, we shall let p, q denote distinct primes, let r, s be integers such that $r, s \neq 0, r \not\equiv s \pmod{2}$. Set

$$\phi(\tau) = \prod (1 - x^n)^r (1 - x^{nq})^s = \sum c(n) x^n, \tag{2.32}$$

$\varepsilon = \frac{1}{2}(r+s), t = (r+sq)/24, \Delta = t(p^2-1), \theta = (-1)^{\frac{1}{2}-\varepsilon} 2q^s$, $\left(\frac{\star}{p}\right)$ is the well-known Legendre symbol. Then the result is as follows.

Lemma 2.7. *With the notations defined as above, the coefficients $c(n)$ of $\phi(\tau)$ satisfy*

$$c(np^2 + \Delta) - \gamma(n) c(n) + p^{2\varepsilon - 2} c\big((n - \Delta)/p^2\big) = 0, \tag{2.33}$$

where

$$\gamma(n) = p^{2\varepsilon-2} c - \left(\frac{\theta}{p}\right) p^{\varepsilon - 3/2} \left(\frac{n - \Delta}{p}\right). \tag{2.34}$$

3. Proof of Theorems

3.A. Proof of Theorem 1.1

By definition

$$\sum_{n=0}^{\infty} b_{2,7}(n) q^n = \frac{f_2 f_7}{f_1 f_{14}} \equiv \frac{f_2}{f_1 f_7} \pmod{2}. \tag{3.35}$$

Now from [Yao14, Lemma 2.1], we have

$$\sum_{n=0}^{\infty} p_{[1^1 7^1]}(2n+1) q^n = \frac{f_2^2 f_{14}^2}{f_1^3 f_7^3}, \tag{3.36}$$

where $p_{[1^1 7^1]}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{[1^1 7^1]}(n) q^n = \frac{1}{f_1 f_7}.$$

From (3.36) and (3.35), we obtain

$$\sum_{n=0}^{\infty} b_{2,7}(2n+1) q^n \equiv f_1^2 f_7 \pmod{2}. \tag{3.37}$$

By hypothesis of the theorem and equation (3.37), we have

$$\sum_{n=0}^{\infty} a(n) q^n = f_1^2 f_7 \equiv \sum_{n=0}^{\infty} b_{2,7}(2n+1) q^n \pmod{2}. \tag{3.38}$$

Putting $r = 2, q = 7$ and $s = 1$ in (2.32), we have by Lemma 2.7, for any $n \geq 0$

$$a\Big(p^2 n + \frac{3}{8}(p^2-1)\Big) = \gamma(n)a(n) - p\,a\Big(\frac{1}{p^2}\big(n - \frac{3}{8}(p^2-1)\big)\Big), \tag{3.39}$$

where

$$\gamma(n) = pc - \left(\frac{-14}{p}\right)\left(\frac{n - \frac{3}{8}(p^2-1)}{p}\right) \tag{3.40}$$

and c is a constant. Setting $n = 0$ in (3.39) and using the facts that $a(0) = 1$ and $a\left(\frac{-3(p^2-1)/8}{p^2}\right) = 0$, we obtain

$$a\Big(\frac{3}{8}(p^2-1)\Big) = \gamma(0). \tag{3.41}$$

Setting $n = 0$ in (3.40) and using (3.41), we obtain

$$pc = a\Big(\frac{3}{8}(p^2-1)\Big) + \left(\frac{-14}{p}\right)\left(\frac{-\frac{3}{8}(p^2-1)}{p}\right) := \omega(p). \tag{3.42}$$

Now rewriting the equation (3.39), by referring (3.40) and (3.42), we obtain

$$a\Big(p^2 n + \frac{3}{8}(p^2-1)\Big) = \left(\omega(p) - \left(\frac{-14}{p}\right)\left(\frac{n - \frac{3}{8}(p^2-1)}{p}\right)\right) a(n) - pa\Big(\frac{1}{p^2}\big(n - \frac{3}{8}(p^2-1)\big)\Big), \tag{3.43}$$

Now, replacing n by $pn + \frac{3}{8}(p^2-1)$ in (3.43), we obtain

$$a\big(p^3 n + \frac{3}{8}(p^4-1)\big) = \omega(p)a\big(pn + \frac{3}{8}(p^2-1)\big) - pa(\frac{n}{p}). \tag{3.44}$$

Case - 1: $\omega(p) \equiv 0 \pmod 2$

Since $\omega(p) \equiv 0 \pmod 2$, from equation (3.44) we obtain

$$a\big(p^3 n + 3(p^4-1)/8\big) \equiv pa(n/p) \pmod 2. \tag{3.45}$$

Now, replacing n by pn in (3.45), we obtain

$$a\big(p^4 n + \frac{3}{8}(p^4-1)\big) \equiv pa(n) \equiv a(n) \pmod 2. \tag{3.46}$$

Since $p^{4j}n + \frac{3}{8}(p^{4j}-1) = p^4\big(p^{4j-4}n + \frac{3}{8}(p^{4j-4}-1)\big) + \frac{3}{8}(p^4-1)$, using equation (3.45), we obtain that for every integer $j \geq 1$,

$$a\big(p^{4j} n + \frac{3}{8}(p^{4j}-1)\big) \equiv a\big(p^{4j-4} n + \frac{3}{8}(p^{4j-4}-1)\big) \equiv a(n) \pmod 2. \tag{3.47}$$

Now if $p \nmid n$, then (3.45) yields

$$a\big(p^3 n + \frac{3}{8}(p^4-1)\big) \equiv 0 \pmod 2. \tag{3.48}$$

Replacing n by $p^3 n + 3(p^4-1)/8$ in (3.47) and using (3.48), we obtain

$$a\big(p^{4j+3} n + \frac{3}{8}(p^{4j+4}-1)\big) \equiv 0 \pmod 2. \tag{3.49}$$

Case - 2: $\omega(p) \not\equiv 0 \pmod 2$

In order to prove part (ii), we replace n by $p^2n + 3p(p^2-1)/8$ in (3.44)

$$\begin{aligned} a\big(p^5n + \frac{3}{8}(p^6-1)\big) &= a\Big(p^3\big(p^2n + \frac{3}{8}p(p^2-1)\big) + \frac{3}{8}(p^4-1)\Big) \\ &\equiv \omega(p)a\big(p^3n + \frac{3}{8}(p^4-1)\big) - p\, a\big(pn + \frac{3}{8}(p^2-1)\big) \\ &\equiv \big[\omega^2(p) - p\big]a\big(pn + \frac{3}{8}(p^2-1)\big) - p\omega(p)\, a\big(\frac{n}{p}\big). \end{aligned} \tag{3.50}$$

Now, as $\omega(p) \not\equiv 0 \pmod 2$, we have $\omega^2(p) - p \equiv 0 \pmod 2$, and therefore (3.50) becomes

$$a\big(p^5n + \frac{3}{8}(p^6-1)\big) \equiv a(n/p) \pmod 2. \tag{3.51}$$

Replacing n by pn in (3.51), we obtain

$$a\big(p^6n + 3(p^6-1)/8\big) \equiv a(n) \pmod 2. \tag{3.52}$$

Using equation (3.52) repeatedly, we see that for every integer $j \geq 1$,

$$a\big(p^{6j}n + 3(p^{6j}-1)/8\big) \equiv a(n) \pmod 2. \tag{3.53}$$

Observe that if $p \nmid n$, then $a(n/p) = 0$. Thus (3.51) yields

$$a\big(p^5n + \frac{3}{8}(p^6-1)\big) \equiv 0 \pmod 2. \tag{3.54}$$

Replacing n by $p^5n + 3(p^6-1)/8$ in (3.53) and using (3.54), we obtain

$$a\big(p^{6j+5}n + \frac{3}{8}(p^{6j+6}-1)\big) \equiv 0 \pmod 2. \tag{3.55}$$

Assume that $p \nmid (8n+3)$, then $n \not\equiv -\frac{3}{8} \pmod p$ and hence $p \nmid n - \frac{3}{8}(p^2-1)$, which in-turn implies that $a\Big(\frac{1}{p^2}\big(n - \frac{3}{8}(p^2-1)\big)\Big) = 0$ and $\big(\frac{n-\frac{3}{8}(p^2-1)}{p}\big) \neq 0$. Thus for $p \geq 5$ and $p \neq 7$, from equation (3.43) we obtain

$$a\big(p^2n + \frac{3}{8}(p^2-1)\big) \equiv 0 \pmod 2. \tag{3.56}$$

Replacing n by $p^2n + 3(p^2-1)/8$ in (3.53) and using (3.56), we obtain

$$a\big(p^{6j+2}n + \frac{3}{8}(p^{6j+2}-1)\big) \equiv 0 \pmod 2. \tag{3.57}$$

Finally, assume that $p = 7$ and $p \nmid (8n+3)$, that is $7 \nmid (n-18)$. From (3.44) we obtain

$$\begin{aligned} a\big(7^4n + \frac{3}{8}(7^4-1)\big) &= a\Big(7^2\big(7^2n + \frac{3}{8}(7^2-1)\big) + \frac{3}{8}(7^2-1)\Big) \\ &\equiv a\big(7^2n + \frac{3}{8}(7^2-1)\big) + a(n) \pmod 2 \\ &\equiv 2a(n) + a\big(\frac{n-18}{49}\big) \equiv 0 \pmod 2. \end{aligned} \tag{3.58}$$

Replacing n by $7^4n + 7(7^4-1)/24$ in (3.53) and using (3.58), we obtain

$$a\big(7^{6j+4}n + \frac{3}{8}(7^{6j+4}-1)\big) \equiv 0 \pmod 2. \tag{3.59}$$

The proof of Theorem 1.1 follows from the fact that $b_{2,7}(2n+1) \equiv a(n) \pmod 2$.

3.B. Proof of Theorem 1.2

By definition

$$\sum_{n=0}^{\infty} b_{5,8}(n)q^n = \frac{f_5 f_8}{f_1 f_{40}} \equiv \frac{f_1^3 f_5 f_4}{f_{40}} \pmod 2. \tag{3.60}$$

Substituting (2.27) into (3.60) and extracting terms involving q^{2n+1} from both sides, we get

$$\sum_{n=0}^{\infty} b_{5,8}(2n+1)q^{2n+1} \equiv q\frac{f_2 f_4 f_{10}^3}{f_{40}} \pmod 2. \tag{3.61}$$

Cancelling q from both sides, then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} b_{5,8}(2n+1)q^n \equiv \frac{f_1 f_5^3 f_2}{f_{20}} \equiv \frac{f_1^3 f_5}{f_{10}} \pmod 2. \tag{3.62}$$

Again substituting (2.27) into (3.62) and extracting terms involving q^{2n+1} and cancelling q from both sides, then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} b_{5,8}(4n+3)q^n \equiv f_1 f_5^2 \pmod 2. \tag{3.63}$$

Now substituting (2.29) into (3.63) and extracting terms involving q^{5n+1} and cancelling q from both sides, then replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} b_{5,8}(20n+7)q^n \equiv f_1^2 f_5 \pmod 2. \tag{3.64}$$

By the notation in the theorem, we have

$$\sum_{n=0}^{\infty} b(n)q^n = f_1^2 f_5 \equiv \sum_{n=0}^{\infty} b_{5,8}(20n+7)q^n \pmod 2. \tag{3.65}$$

Putting $r = 2, q = 5$ and $s = 1$ in (2.32), we have by Lemma 2.7, for any $n \geq 0$

$$b\big(p^2 n + \frac{7}{24}(p^2-1)\big) = \gamma(n)b(n) - p\, b\big(\frac{1}{p^2}\Big(n - \frac{7}{24}(p^2-1)\big)\Big), \tag{3.66}$$

where

$$\gamma(n) = pc - \left(\frac{-10}{p}\right)\left(\frac{n - \frac{7}{24}(p^2-1)}{p}\right) \tag{3.67}$$

and c is a constant. Setting $n = 0$ in (3.66) and using the facts that $b(0) = 1$ and $b\left(\frac{-7(p^2-1)/24}{p^2}\right) = 0$, we obtain

$$b\big(\frac{7}{24}(p^2-1)\big) = \gamma(0) \tag{3.68}$$

Setting $n = 0$ in (3.67) and using (3.68), we obtain

$$pc = b\big(\frac{7}{24}(p^2-1)\big) + \left(\frac{-10}{p}\right)\left(\frac{-\frac{7}{24}(p^2-1)}{p}\right) := \omega(p). \tag{3.69}$$

Now rewriting the equation (3.66), by referring (3.67) and (3.69), we obtain

$$b\big(p^2 n + \frac{7}{24}(p^2-1)\big) = \left(\omega(p) - \left(\frac{-10}{p}\right)\left(\frac{n - \frac{7}{24}(p^2-1)}{p}\right)\right) b(n) - p\, b\Big(\frac{1}{p^2}\big(n - \frac{7}{24}(p^2-1)\big)\Big). \tag{3.70}$$

Now, replacing n by $pn + 7(p^2-1)/24$ in (3.70), we obtain

$$b\big(p^3n + \frac{7}{24}(p^4-1)\big) = \omega(p)\, b\big(pn + \frac{7}{24}(p^2-1)\big) - p\, b\big(\frac{n}{p}\big). \tag{3.71}$$

Case - 1: $\omega(p) \equiv 0 \pmod 2$

Since $\omega(p) \equiv 0 \pmod 2$, we have

$$b\big(p^3n + \frac{7}{24}(p^4-1)\big) \equiv p\, b\big(\frac{n}{p}\big) \pmod 2. \tag{3.72}$$

Replacing n by pn in (3.72) we obtain

$$b\big(p^4n + \frac{7}{24}(p^4-1)\big) \equiv pb(n) \equiv b(n) \pmod 2. \tag{3.73}$$

Using equation (3.73) repeatedly, we see that for every integer $j \geq 1$,

$$b\big(p^{4j}n + \frac{7}{24}(p^{4j}-1)\big) \equiv b(n) \pmod 2. \tag{3.74}$$

Now as $p \nmid n$, then (3.72) yields

$$b\big(p^3n + \frac{7}{24}(p^4-1)\big) \equiv 0 \pmod 2. \tag{3.75}$$

Replacing n by $p^3n + 7(p^4-1)/24$ in (3.74) and utilizing (3.75), we obtain

$$b\big(p^{4j+3}n + \frac{7}{24}(p^{4j+4}-1)\big) \equiv 0 \pmod 2. \tag{3.76}$$

Case - 2: $\omega(p) \not\equiv 0 \pmod 2$

First note that $b(14) = 0$, which implies $\omega(7) = 0$ and hence we can assume that $p \neq 7$.
Now, in order to prove part (ii), we first replace n by $p^2n + \frac{7}{24}p(p^2-1)$ in (3.71) to obtain

$$\begin{aligned} b\big(p^5n + \frac{7}{24}(p^6-1)\big) &\equiv \omega(p)b\big(p^3n + \frac{7}{24}(p^4-1)\big) - p\; b\big(pn + \frac{7}{24}(p^2-1)\big) \\ &\equiv \big[\omega^2(p) - p\big]b\big(pn + \frac{7}{24}(p^2-1)\big) - p\omega(p)\; b\big(\frac{n}{p}\big). \end{aligned} \tag{3.77}$$

Since $\omega(p) \not\equiv 0 \pmod 2$, we have $\omega^2(p) - p \equiv 0 \pmod 2$, and therefore (3.77) becomes

$$b\big(p^5n + \frac{7}{24}(p^6-1)\big) \equiv b\big(\frac{n}{p}\big) \pmod 2. \tag{3.78}$$

Replacing n by pn in (3.78), we obtain

$$b\big(p^6n + \frac{7}{24}(p^6-1)\big) \equiv b(n) \pmod 2. \tag{3.79}$$

Using equation (3.79) repeatedly, we see that for every integer $j \geq 1$,

$$b\big(p^{6j}n + \frac{7}{24}(p^{6j}-1)\big) \equiv b(n) \pmod 2. \tag{3.80}$$

Observe that if $p \nmid n$, then $b(n/p) = 0$. Thus (3.78) yields

$$b\big(p^5n + \frac{7}{24}(p^6-1)\big) \equiv 0 \pmod 2. \tag{3.81}$$

Replacing n by $p^5n + 7(p^6-1)/24$ in (3.80) and using (3.81), we obtain

$$b\big(p^{6j+5}n + \frac{7}{24}(p^{6j+6}-1)\big) \equiv 0 \pmod 2. \tag{3.82}$$

Now assume that $p \nmid (24n+7)$, then $n \not\equiv -\frac{7}{24} \pmod p$ and hence $p \nmid n - \frac{7}{24}(p^2-1)$, which in-turn implies that $b\Big(\frac{1}{p^2}(n - \frac{7}{24}(p^2-1))\Big) = 0$ and $\big(\frac{n-\frac{7}{24}(p^2-1)}{p}\big) \neq 0$. Thus for $p > 5$, from equation (3.70) we obtain

$$b\big(p^2n + \frac{7}{24}(p^2-1)\big) \equiv 0 \pmod 2. \tag{3.83}$$

Replacing n by $p^2n + 7(p^2-1)/24$ in (3.80) and using (3.83), we obtain

$$b\big(p^{6j+2}n + \frac{7}{24}(p^{6j+2}-1)\big) \equiv 0 \pmod 2. \tag{3.84}$$

Finally, assume that $p = 5$ and $p \nmid (24n+7)$, that is $5 \nmid (n-7)$. From (3.70) we obtain

$$\begin{aligned} b\big(5^4n + \frac{7}{24}(5^4-1)\big) &= b\Big(5^2\big(5^2n + \frac{7}{24}(5^2-1)\big) + \frac{7}{24}(5^2-1)\Big) \\ &\equiv b\big(5^2n + \frac{7}{24}(5^2-1)\big) + b(n) \\ &\equiv 2b(n) + b\big(\frac{n-7}{25}\big) \equiv 0 \pmod 2. \end{aligned} \tag{3.85}$$

Replacing n by $5^4n + 7(5^4-1)/24$ in (3.80) and using (3.85), we obtain

$$b\big(5^{6j+4}n + \frac{7}{24}(p^{6j+4}-1)\big) \equiv 0 \pmod 2. \tag{3.86}$$

The proof of Theorem 1.2 follows from the fact that $b_{5,8}(20n+7) \equiv b(n) \pmod 2$.

3.C. Proof of Theorem 1.3

We shall need the following result, which we present as a proposition, to prove the theorem.

Proposition 3.1. For any non-negative integer k, we have

$$\sum_{n=0}^{\infty} b_{4,5}\left(4 \cdot 5^{4k}n + \frac{5^{4k}+1}{2}\right) q^n \equiv 2f_1^3 + 2qf_2f_5^5 \pmod 4. \tag{3.87}$$

Proof. By definition

$$\sum_{n=0}^{\infty} b_{4,5}(n)q^n \equiv \frac{f_1^3 f_5}{f_{20}} \pmod 2. \tag{3.88}$$

Substituting (2.27) in (3.88) and extracting terms involving q^{2n+1} and cancelling q from both sides, then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} b_{4,5}(2n+1)q^n \equiv 2f_2^3 + 3\frac{f_1f_5^3}{f_{10}} \pmod 4. \tag{3.89}$$

Substituting (2.26) in (3.89) and extracting terms involving q^{2n} and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} b_{4,5}(4n+1)q^n \equiv 2f_1^3 + 2qf_2f_5^5 \pmod 4. \tag{3.90}$$

which is the $k=0$ case of (3.87). Now suppose that (3.87) holds for some $k \geq 0$. Substituting (2.29) in (3.90) and extracting terms involving q^{5n+3} and cancelling q^3 from both sides, then replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} b_{4,5}\left(4\cdot 5^{4k+1}n+\frac{5^{4k+2}+1}{2}\right)q^n \equiv 2f_5^3+2f_1^5f_{10} \pmod 4. \tag{3.91}$$

Again substituting (2.29) in (3.91) and extracting terms involving q^{5n} and then replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} b_{4,5}\left(4\cdot 5^{4k+2}n+\frac{5^{4k+2}+1}{2}\right)q^n \equiv 2f_1^3+2\frac{f_8}{f_5} \pmod 4. \tag{3.92}$$

Again substituting (2.29) in (3.92) and extracting terms involving q^{5n+3} and cancelling q^3 from both sides, then replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} b_{4,5}\left(4\cdot 5^{4k+3}n+\frac{5^{4k+4}+1}{2}\right)q^n \equiv 2f_5^3+2q\frac{f_{40}}{f_1} \pmod 4. \tag{3.93}$$

Substituting (2.31) in (3.93) and extracting terms involving q^{5n} and then replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} b_{4,5}\left(4\cdot 5^{4k+4}n+\frac{5^{4k+4}+1}{2}\right)q^n \equiv 2f_1^3+2qf_2f_5^5 \pmod 4. \tag{3.94}$$

Thus, by induction hypothesis (3.87) is established.

Now, the proof of theorem is a direct application of the proposition which can be seen as follows. Proof of (1.17) follows by comparing equation (3.90) and (3.94). Proof of (1.18) and (1.19) follows by substituting (2.29) in (3.91) and (3.93) respectively, and then equating the coefficients of q^{5n+m} for $m=2$ and $m=4$.

Theorem 3.2. *For all $n \geq 0$,*

$$b_{4,11}(22n+2m) \equiv 0 \pmod 2, \quad \textit{where} \quad m=\{1,4,8,9,10\}. \tag{3.95}$$

Proof. By definition

$$\sum_{n=0}^{\infty} b_{4,11}(n)q^n \equiv \frac{f_2^2}{f_1f_{11}f_{22}} \pmod 2. \tag{3.96}$$

Substituting (2.28) in (3.96) and extracting the terms involving q^{2n} from both sides of the congruence and then replacing q^2 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_{4,11}(2n)q^n &\equiv \frac{1}{f_{11}^3}\left(\frac{f_{12}^2}{f_6}+q^3\frac{f_{33}^3f_2}{f_{11}}\right) \pmod 2,\\ &\equiv \frac{f_6^3}{f_{11}^3}+q^3\frac{f_{33}^3f_2}{f_{11}^4} \pmod 2.\end{aligned} \tag{3.97}$$

Taking $p=11$ in (2.5), and q replacing by q^2, we get

$$f_2=\left(\sum_{\substack{k=-5\\k\neq -2}}^{5}(-1)^k q^{3k^2+k} f\left(-q^{22(17+3k)},-q^{22(16-3k)}\right)+q^{10}f_{242}\right) \tag{3.98}$$

Note that for $-5 \le k \le 5$ and $k \neq -2$,

$$3k^2 + k \not\equiv 5 \pmod{11}.$$

Also taking $p = 11$ in (2.5), and q replacing by q^6, we get

$$f_6^3 = \sum_{\substack{k=0\\k\neq 5}}^{10} (-1)^k q^{3k(k+1)} \sum_{n=0}^{\infty} (-1)^n (22n+2k+1) q^{33n\cdot(11n+2k+1)} - 11q^{90} f_{726}^3. \tag{3.99}$$

Furthermore, if $k \neq 5, 0 \le k \le 10$, then $\frac{k^2+k}{2} \not\equiv 15 \pmod{11}$.
Employing (3.99) and (3.98) in (3.97), observe that the right-hand side doesn't contain terms of the form q^{11n+m} for $m \in S = \{1, 4, 8, 9, 10\}$. Thus

$$\sum_{n=0}^{\infty} b_{4,11}(22n+2m) q^n \equiv 0 \pmod{2}. \tag{3.100}$$

Completes the proof of theorem follows from (3.100).

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