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Identities among some combinatorial objects involving special values of multiple zeta functions

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Abstract. In the article, we establish some identities involving special values of multiple zeta functions among the counting functions of number of representations of an integer by a linear combination of figurate numbers such as triangular numbers, square numbers, pentagonal numbers, etc. More precisely, we provide our result for $\delta_k(n)$, $r_k(n)$ and $\mathcal{N}_k^a(n)$ (for a fixed $a \geq 3$), the number of representations of n as a sum of k-triangular numbers, as a sum of k-square numbers and as a sum of k-higher figurate numbers (for a fixed $a \geq 3$), respectively. Moreover, these identities also occur when one of $\delta_k(n)$, $r_k(n)$ and $\mathcal{N}_k^a(n)$ is replaced by the k-colored partition functions.

Keywords. q-series, k-coloured partition function, Trigonometric functions, Integer composition.

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1. Preliminaries

We define the q-Pochhammer symbol, also known as the q-shifted factorial, given as the product of the form

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), \tag{1.1}$$

with $(a;q)_0 = 1$. It is a q-analogue of the Pochhammer symbol $(x)_n = x(x+1)\cdots(x+n-1)$ for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, in the sense that

$$\lim_{q\to 1}\frac{(q^x;q)_n}{(1-q)^n} = (x)_n. \tag{1.2}$$

The q-Pochhammer symbol is a major building block in the theory of hypergeometric series and the theory of integer partitions. The q-Pochhammer symbol can be extended to an infinite product given by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.3}$$

Moreover, for $a = q$, $(a;q)_\infty$ is the Dedekind eta function $\eta(\tau)$ upto a multiple of q-power, where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$ (complex upper half-plane), i.e., $\eta(\tau) = q^{1/24}(q;q)_\infty$. A product of the form $\prod\limits_{d|N}\eta^{r_d}(d\tau)$; with $r_d \in \mathbb{Z}$, is commonly known as a eta-quotient of level N and weight $\frac{1}{2}\sum\limits_{d|N} r_d$, in the theory of modular forms. These are the building blocks for finding the formulas for $r_k(n)$, $\delta_k(n)$ and $\mathcal{N}_k^a(n)$(for a fixed $a \geq 3$) (See [RV23]).

For a fixed integer $a \geq 1$, we define the n^{th}-figurate number by $f_a(n) := \frac{an^2+(a-2)n}{2}$, following Ono-Robins-Wahl [ORW95]. Note that the function $f_a(n)$ denotes the n^{th}-triangular number (resp. square number and pentagonal number) when $a = 1$ (resp. 2 and 3). In the literature, the n^{th} triangular number is denoted by T_n, and for a fixed $a \geq 3$, the integer $f_a(n)$ is known as the n^{th}-higher figurate number. These functions are associated with counting the number of vertices of some

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geometric objects. We denote the generating functions for these figurate numbers as $\Psi(\tau)$ (for triangular numbers), $\theta(\tau)$ (for square numbers) and $\Phi_a(\tau)$ (for $a \geq 3$), and they are given respectively by

$$\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}, \qquad \Psi(\tau) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}, \qquad \text{and} \qquad \text{for } a \geq 3, \quad \Phi_a(\tau) := \sum_{n\in\mathbb{Z}} q^{f_a(n)}. \tag{1.4}$$

where $q \in \mathbb{H}$. It is known from [ORW95, Proposition 1, Theorem 10] that the generating function for the figurate numbers defined in (1.4) are explicitly obtained from Jacobi triple product identity given by

$$\theta(\tau) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \quad \Psi(\tau) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \tag{1.5}$$

and for $a \geq 3$,

$$\Phi_a(\tau) = \prod_{n=1}^{\infty} (1-q^{an})(1-q^{an-1})(1-q^{an-(a-1)}) = (q^a;q^a)_\infty (q^{-1};q^a)_\infty (q^{-(a-1)};q^a)_\infty. \tag{1.6}$$

Let $\delta_k(n)$, $r_k(n)$ and $\mathcal{N}_k^a(n)$(for a fixed $a \geq 3$) denote the number of representations of n as a sum of k-triangular numbers, as a sum of k-square numbers and as a sum of k-higher figurate numbers (for a fixed $a \geq 3$) respectively, i.e.,

$$r_k(n) = \#\left\{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^{k} x_i^2\right\},$$

$$\delta_k(n) = \#\left\{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^{k} \frac{x_i(x_i+1)}{2}\right\}$$

and for a fixed integer a ≥ 3, $\mathcal{N}_k^a(n) = \#\left\{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^{k} f_a(x_i)\right\}.$

It is easy to see that

$$\theta^k(\tau) = 1 + \sum_{n=1}^{\infty} r_k(n)\, q^n, \quad \Psi^k(\tau) = 1 + \sum_{n=1}^{\infty} \delta_k(n)\, q^n \text{ and } \quad \Phi_a^k(\tau) = 1 + \sum_{n=1}^{\infty} \mathcal{N}_k^a(n)\, q^n$$

for $a \geq 3$. For more details on these arithmetical functions, we refer to [ORW95] and [RV23].

For positive integers $k_1, k_2, \ldots, k_w \geq 2$, we define the multiple zeta function as follows:

$$\zeta(k_1, k_2, \ldots, k_w) = \sum_{m_1 < m_2 < \cdots < m_w} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_w^{k_w}}$$
$$\text{and} \qquad \zeta^{odd}(k_1, k_2, \ldots, k_w) = \sum_{\substack{m_1 < m_2 < \cdots < m_w \\ m_i - \text{odd}}} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_w^{k_w}}. \tag{1.7}$$

When $k_i = k$ for all i, we simply denote it as $\zeta_w(k)$ and $\zeta_w^{odd}(k)$, respectively. By convention, we write $\zeta(k) = \zeta_1(k)$, $\zeta_0(k) = 1$, $\zeta^{odd}(k) = \zeta_1^{odd}(k)$ and $\zeta_0^{odd}(k) = 1$.

Trigonometric identities

For each $z \in \mathbb{C}$, we have the following trigonometric function with their product expansion and Taylor series expansion given as follows

$$\begin{aligned} \sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}, \\ \cos z = \prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} \sin^2 z = z^2\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)^2 &= \sum_{n=1}^{\infty}\left(\sum_{r=0}^{n-1}\binom{2n}{2r+1}\right)\frac{(-1)^{n-1}}{(2n)!}z^{2n} \\ \text{and}\quad \cos^2 z = \prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)^2 &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{2n}{2r}\right)\frac{(-1)^n}{(2n)!}z^{2n}. \end{aligned} \tag{1.9}$$

Here, $\binom{n}{r}$ denotes the number of ways to choose r objects from a set of n objects.

2. Identities for $\delta_k(n)$, $r_k(n)$ and $\mathcal{N}_k^a(n)$

In this section, we establish certain identities for $r_k(n)$, $\delta_k(n)$ and $\mathcal{N}_k^a(n)$(for a fixed $a \geq 3$) which involves the special values of multiple zeta function. We make use of trigonometric functions to prove our results. Before stating our results, we define the set $S(n)$ as a collection of all possible compositions of non-zero positive integers $\{a_1, a_2, \ldots, a_r\}$ such that $a_1 + a_2 + \cdots + a_r = n$, i.e.,

$$S(n) = \{\{a_1, a_2, \ldots, a_r\} \mid a_i > 0 \text{ for all } i, \text{ and } a_1 + a_2 + \cdots + a_r = n\}.$$

For example, given $n = 6$, the elements $\{3, 2, 1\}$, $\{2, 3, 1\}$ and $\{1, 2, 3\}$ are different integer compositions of $S(n)$.

Now, we present our results.

Theorem 2.1. *For a given positive integer n, we have the following identities for $r_k(n)$, $\delta_k(n)$ and $\mathcal{N}_k^a(n)$(for a fixed $a \geq 3$), respectively involving the special values of multiple zeta function $\zeta_k(2)$ given by*

$$\begin{aligned} &\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}r_{2k}(n) = \sum_{k=0}^{\infty}\frac{(-1)^k\zeta_k(2)}{\pi^{2k}}r_{2k}(n) \\ &= \sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell \\ m_i-distinct}}\frac{(-1)^\ell}{\pi^{2\ell}}\frac{r_2(s_1)r_2(s_2)\cdots r_2(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{m\neq m_1,m_2,\ldots,m_\ell}\left(1-\frac{1}{m^2\pi^2}\right), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\delta_{2k}(n) = \sum_{k=0}^{\infty}\frac{(-1)^k\zeta_k(2)}{\pi^{2k}}\delta_{2k}(n) \\ &= \sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell \\ m_i-distinct}}\frac{(-1)^\ell}{\pi^{2\ell}}\frac{\delta_2(s_1)\delta_2(s_2)\cdots\delta_2(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{m\neq m_1,m_2,\ldots,m_\ell}\left(1-\frac{1}{m^2\pi^2}\right) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\mathcal{N}_{2k}^a(n) = \sum_{k=0}^{\infty}\frac{(-1)^k\zeta_k(2)}{\pi^{2k}}\mathcal{N}_{2k}^a(n) \\ &= \sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell \\ m_i-distinct}}\frac{(-1)^\ell}{\pi^{2\ell}}\frac{\mathcal{N}_2^a(s_1)\mathcal{N}_2^a(s_2)\cdots\mathcal{N}_2^a(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{m\neq m_1,m_2,\ldots,m_\ell}\left(1-\frac{1}{m^2\pi^2}\right). \end{aligned} \tag{2.12}$$

Proof. We prove the above result for $r_k(.)$ and other identities can be obtained following similar arguments. Let us consider the product formula of $\sin z$ given by $\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$. To get the required result, we consider the product $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$ and expand it in two different ways by substituting the q-product of generating function for square numbers, i.e., $z = \prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}$ (given in (1.5)) and observe that $z^k = \sum_{n=0}^{\infty} r_k(n)q^n$. More precisely, the expansion of $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$ gives that

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right) = 1+\sum_{k=1}^{\infty}\frac{(-1)^k}{\pi^{2k}}\left(\sum_{\substack{n_1,n_2,\ldots,n_k\\ n_i-\text{distinct}}}\frac{1}{n_1^2n_2^2\cdots n_k^2}\right)z^{2k} = 1+\sum_{k=1}^{\infty}\frac{(-1)^k\zeta_k(2)}{\pi^{2k}}z^{2k}.$$

Now, we substitute $z^{2k} = \sum_{n=0}^{\infty} r_{2k}(n)q^n$ and see that the constant term (q^0-th coefficient) and the coefficient of q^n $(n>0)$ is given by;

$$1+\sum_{k=1}^{\infty}\frac{(-1)^k\zeta_k(2)}{\pi^{2k}} \qquad \text{and} \qquad \sum_{k=1}^{\infty}\frac{(-1)^k\zeta_k(2)}{\pi^{2k}}r_{2k}(n), \tag{2.13}$$

respectively. On the other hand, we first substitute $z^2 = \sum_{n=0}^{\infty} r_2(n)q^n$ in $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$ to get

$$\prod_{n=1}^{\infty}\left(\left(1-\frac{1}{n^2\pi^2}\right)q^0-\sum_{k=1}^{\infty}\frac{r_2(k)}{n^2\pi^2}\,q^k\right). \tag{2.14}$$

We expand the product in (2.14) using the Cauchy product of infinite series to obtain the coefficients of q^n. The coefficients of q^n in this process is given by

$$\sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell\\ m_i-distinct}}\frac{(-1)^\ell}{\pi^{2\ell}}\frac{r_2(s_1)r_2(s_2)\cdots r_2(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{m\neq m_1,m_2,\ldots,m_\ell}\left(1-\frac{1}{m^2\pi^2}\right). \tag{2.15}$$

Since the q^n-coefficients given in (2.13) and (2.15) are the q^n-coefficients of the expansions of the product $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$ by putting the value of $z = \prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}$. Hence, we have the required identity. The first equality is obvious by the Taylor series expansion of $\sin z$. Thus, we have the required result.

Theorem 2.2. *For a given positive integer n, we have the following identities for $r_k(n)$, $\delta_k(n)$ and $\mathcal{N}_k^a(n)$ (for a fixed $a\geq 3$), respectively involving the special values of multiple zeta function $\zeta_k^{odd}(2)$ given by*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}r_{2k}(n) &= \sum_{k=0}^{\infty}\frac{(-1)^k4^k\zeta_k^{odd}(2)}{\pi^{2k}}r_{2k}(n)\\ &= \sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell\\ m_i-\text{odd \& distinct}}}\frac{(-4)^\ell}{\pi^{2\ell}}\frac{r_2(s_1)r_2(s_2)\cdots r_2(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{\substack{m\neq m_1,m_2,\ldots,m_\ell\\ m-\text{odd}}}\left(1-\frac{1}{m^2\pi^2}\right),\end{aligned} \tag{2.16}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\delta_{2k}(n)=\sum_{k=0}^{\infty}\frac{(-1)^k4^k\zeta_k^{odd}(2)}{\pi^{2k}}\delta_{2k}(n)$$
$$=\sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell\\ m_i-\text{odd \& distinct}}}\frac{(-4)^\ell}{\pi^{2\ell}}\frac{\delta_2(s_1)\delta_2(s_2)\cdots\delta_2(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{\substack{m\neq m_1,m_2,\ldots,m_\ell\\ m-\text{odd}}}\left(1-\frac{1}{m^2\pi^2}\right) \tag{2.17}$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\mathcal{N}_{2k}^a(n)=\sum_{k=0}^{\infty}\frac{(-1)^k4^k\zeta_k^{odd}(2)}{\pi^{2k}}\mathcal{N}_{2k}^a(n)$$
$$=\sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell\\ m_i-\text{odd \& distinct}}}\frac{(-4)^\ell}{\pi^{2\ell}}\frac{\mathcal{N}_2^a(s_1)\mathcal{N}_2^a(s_2)\cdots\mathcal{N}_2^a(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{\substack{m\neq m_1,m_2,\ldots,m_\ell\\ m-\text{odd}}}\left(1-\frac{1}{m^2\pi^2}\right). \tag{2.18}$$

Proof. We use the same arguments to prove the above result for $\delta_k(.)$ and the other identities. Let us consider the product formula of $\cos z$ given by $\cos z=\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)$. We expand the product $\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)$ in two different ways by substituting the q-product of generating function for triangular numbers, i.e., $z=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^n)}$ and observe that $z^k=\sum_{n=0}^{\infty}\delta_k(n)q^n$ to get the required result. More precisely, a simple expansion of $\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)$ gives that

$$\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)=1+\sum_{k=1}^{\infty}\frac{(-1)^k4^k}{\pi^{2k}}\left(\sum_{\substack{n_1n_2,\ldots,n_k\\ n_i-\text{odd \& distinct}}}\frac{1}{n_1^2n_2^2\cdots n_k^2}\right)z^{2k}$$
$$=1+\sum_{k=1}^{\infty}\frac{(-1)^k4^k\zeta_k^{odd}(2)}{\pi^{2k}}z^{2k}.$$

Now, we substitute $z^{2k}=\sum_{n=0}^{\infty}\delta_{2k}(n)q^n$ and see that the coefficient of q^n is

$$1+\sum_{k=1}^{\infty}\frac{(-1)^k4^k\zeta_k^{odd}(2)}{\pi^{2k}}\delta_{2k}(n). \tag{2.19}$$

On the other hand, first we substitute $z^2=\sum_{n=0}^{\infty}\delta_2(n)q^n$ in $\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)$ and then we expand the product to obtain the coefficients of q^n. The coefficients of q^n in this process is given by

$$\sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell\\ m_i-odd\&distinct}}\frac{(-4)^\ell}{\pi^{2\ell}}\frac{\delta_2(s_1)\delta_2(s_2)\cdots\delta_2(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{\substack{m\neq m_1,m_2,\ldots,m_\ell\\ m-\text{odd}}}\left(1-\frac{1}{m^2\pi^2}\right). \tag{2.20}$$

Hence, comparing the q^n- coefficients of the expansions of the product $\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)$ (in two ways) by substituting $z=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}$, we obtain the required result. The first equality is obvious from the Taylor series expansion of $\cos z$.

Following the same arguments and using the product and sum expression for $\sin^2 z$ and $\cos^2 z$ respectively (given in (1.9)), we have the following results.

Theorem 2.3. *For a given positive integer n, we have the following identities for $r_k(n)$, $\delta_k(n)$ and $\mathcal{N}_k^a(n)$ (for a fixed $a \geq 3$) respectively.*

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k-1}\binom{2k}{2r+1}\right)\frac{(-1)^{k-1}}{(2k)!}r_{2k}(n)\\&=\sum_{f=1}^{n}r_2(f){\sum}^*\left(\prod_{1\leq e\leq \ell}\left(-\frac{2}{\pi^2m_e^2}r_2(s_e)+\frac{1}{\pi^4m_e^4}r_2(s_e)\right)\prod_{m\neq m_1,m_2,\ldots,m_\ell}\left(1-\frac{2}{m^2\pi^2}+\frac{1}{m^4\pi^4}\right)\right),\end{aligned}\tag{2.21}$$

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k-1}\binom{2k}{2r+1}\right)\frac{(-1)^{k-1}}{(2k)!}\delta_{2k}(n)\\&=\sum_{f=1}^{n}\delta_2(f){\sum}^*\left(\prod_{1\leq e\leq \ell}\left(-\frac{2}{\pi^2m_e^2}\delta_2(s_e)+\frac{1}{\pi^4m_e^4}\delta_2(s_e)\right)\prod_{m\neq m_1,m_2,\ldots,m_\ell}\left(1-\frac{2}{m^2\pi^2}+\frac{1}{m^4\pi^4}\right)\right),\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k-1}\binom{2k}{2r+1}\right)\frac{(-1)^{k-1}}{(2k)!}\mathcal{N}_{2k}^a(n)\\&=\sum_{f=1}^{n}\mathcal{N}_2^a(f){\sum}^*\left(\prod_{1\leq e\leq \ell}\left(-\frac{2}{\pi^2m_e^2}\mathcal{N}_2^a(s_e)+\frac{1}{\pi^4m_e^4}\mathcal{N}_2^a(s_e)\right)\prod_{m\neq m_1,m_2,\ldots,m_\ell}\left(1-\frac{2}{m^2\pi^2}+\frac{1}{m^4\pi^4}\right)\right).\end{aligned}\tag{2.23}$$

where ${\sum}^*$ *denotes the double summation* $\displaystyle\sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n-f)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell\\ m_i-distinct}}$.

Theorem 2.4. *For a given positive integer n, we have the following identities for $r_k(n)$, $\delta_k(n)$ and $\mathcal{N}_k^a(n)$ (for a fixed $a \geq 3$) respectively.*

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k}\binom{2k}{2r}\right)\frac{(-1)^{k}}{(2k)!}r_{2k}(n)\\&={\sum}^{\#}\prod_{1\leq e\leq \ell}\left(-\frac{2}{\pi^2m_e^2}r_2(s_e)+\frac{1}{\pi^4m_e^4}r_2(s_e)\right)\prod_{\substack{m\neq m_1,m_2,\ldots,m_\ell\\ m-\text{odd}}}\left(1-\frac{2}{m^2\pi^2}+\frac{1}{m^4\pi^4}\right),\end{aligned}\tag{2.24}$$

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k}\binom{2k}{2r}\right)\frac{(-1)^{k}}{(2k)!}\delta_{2k}(n)\\&={\sum}^{\#}\prod_{1\leq e\leq \ell}\left(-\frac{2}{\pi^2m_e^2}\delta_2(s_e)+\frac{1}{\pi^4m_e^4}\delta_2(s_e)\right)\prod_{\substack{m\neq m_1,m_2,\ldots,m_\ell\\ m-\text{odd}}}\left(1-\frac{2}{m^2\pi^2}+\frac{1}{m^4\pi^4}\right).\end{aligned}\tag{2.25}$$

and

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k}\binom{2k}{2r}\right)\frac{(-1)^{k}}{(2k)!}\mathcal{N}_{2k}^a(n)\\&={\sum}^{\#}\left(\prod_{1\leq e\leq \ell}\left(-\frac{2}{\pi^2m_e^2}\mathcal{N}_2^a(s_e)+\frac{1}{\pi^4m_e^4}\mathcal{N}_2^a(s_e)\right)\prod_{\substack{m\neq m_1,m_2,\ldots,m_\ell\\ m-\text{odd}}}\left(1-\frac{2}{m^2\pi^2}+\frac{1}{m^4\pi^4}\right)\right).\end{aligned}\tag{2.26}$$

where ${\sum}^{\#}$ *denotes the summation* $\displaystyle\sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell\\ m_i-\text{odd \& distinct}}}$.

3. Identities for k-colored partition functions:

Let n be a positive integer. For each $k \geq 1$, let $p_k(n)$ denote the number of k-colored partition of n. For $k = 1$, it is the usual partition function $p(n)$, first considered by Ramanujan. We define $\overline{p_k}(n)$, as the counting function for the number of k-colored over-partition of a positive integer n, and $pd_k(n)$ denotes the number of k-colored partition of n into distinct parts, and $po_k(n)$ denotes the number of k-colored partition of n into odd parts. The generating function for these k-colored partition functions is given in terms of q-product as follows.

$$\begin{aligned}
\sum_{n=0}^{\infty} p_k(n) q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k} = (q;q)_\infty^{-k}, \\
\sum_{n=0}^{\infty} \overline{p_k}(n) q^n &= \prod_{n=1}^{\infty} \frac{(1+q^n)^k}{(1-q^n)^k} = \frac{(-q;q)_\infty^k}{(q;q)_\infty^k}, \\
\sum_{n=0}^{\infty} pd_k(n) q^n &= \prod_{n=1}^{\infty} (1+q^n)^k = (-q;q)_\infty^k, \\
\text{and} \qquad \sum_{n=0}^{\infty} po_k(n) q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})^k} = (q^{-1};q^2)_\infty^k.
\end{aligned} \tag{3.27}$$

For details on the partition functions and their combinatorial interpretations, we refer to [And98]. Similar identities obtained in Theorem 2.1 and Theorem 2.2, can be given for these k-colored partition functions. The proof of these identities take place by substituting $z = (-q;q)_\infty$ (for $pd_k(n)$), $z = (q^{-1};q^2)_\infty$ (for $po_k(n)$), $z = \frac{1}{(q;q)_\infty}$ (for $p_k(n)$) and $z = \frac{(-q;q)_\infty}{(q;q)_\infty}$ (for $\overline{p_k}(n)$) in the expression of $\sin z$ and $\cos z$ and following the same arguments as in the proofs of Theorem 2.1 and Theorem 2.2. Below, we state these identities only for $pd_k(n)$, and identities for others can be stated analogously.

Theorem 3.1. *For a given positive integer n, we have the following identities for $pd_k(n)$ involving the special values of multiple zeta function $\zeta_k(2)$ given by*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} pd_{2k}(n) = \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} pd_{2k}(n) \\
&= \sum_{\{s_1,s_2,\ldots,s_\ell\} \in S(n)} \sum_{\substack{m_1,m_2,\ldots,m_\ell \\ m_i - distinct}} \frac{(-1)^\ell}{\pi^{2\ell}} \frac{pd_2(s_1) pd_2(s_2) \cdots pd_2(s_\ell)}{m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1,m_2,\ldots,m_\ell} \left(1 - \frac{1}{m^2\pi^2}\right).
\end{aligned} \tag{3.28}$$

These identities also valid if $pd_k(n)$ is replaced by one of these k-colored partition functions among $p_k(n)$, $\overline{p_k}(n)$ and $po_k(n)$.

Theorem 3.2. *For a given positive integer n, we have the following identities for $pd_k(n)$ involving the special values of multiple zeta function $\zeta_k^{odd}(2)$ given by*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} pd_{2k}(n) = \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} pd_{2k}(n) \\
&= \sum_{\{s_1,s_2,\ldots,s_\ell\} \in S(n)} \sum_{\substack{m_1,m_2,\ldots,m_\ell \\ m_i - \text{odd \& distinct}}} \frac{(-4)^\ell}{\pi^{2\ell}} \frac{pd_2(s_1) pd_2(s_2) \cdots pd_2(s_\ell)}{m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1,m_2,\ldots,m_\ell \\ m - \text{odd}}} \left(1 - \frac{1}{m^2\pi^2}\right).
\end{aligned} \tag{3.29}$$

These identities also valid if $pd_k(n)$ is replaced by one of these k-colored partition functions among $p_k(n)$, $\overline{p_k}(n)$ and $po_k(n)$.

Remark 3.1. *The identities appearing in the Theorem 2.3 and Theorem 2.4 can be given for each of these k-colored partition functions $p_k(n)$, $\overline{p_k}(n)$, $pd_k(n)$ and $po_k(n)$ by replacing the arithmetical function $r_k(n)$ by one of the above k-colored partition functions.*

3.A. k-colored identities associated to Rogers-Ramanujan partition functions:

For a positive integer n, Let $[\lambda_1, \lambda_2, \cdots, \lambda_r]$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$, $r \in \mathbb{N}$ and each $\lambda_i \geq \lambda_{i+1} > 0$, denote a partition of n. We define the following partition functions given by

$$\begin{aligned} RR(n) &:= \#\{\text{partiton } [\lambda_1, \lambda_2, \ldots, \lambda_r] \text{ of } n \mid \lambda_i - \lambda_{i+1} \geq 2 \text{ for each i}\} \quad \text{and} \\ RS(n) &:= \#\{\text{partiton } [\lambda_1, \lambda_2, \ldots, \lambda_r] \text{ of } n \mid \lambda_i - \lambda_{i+1} \geq 2, \text{ and } \lambda_i \geq 2 \text{ for each i}\}. \end{aligned} \tag{3.30}$$

The generating function for these is given by

$$\sum_{n=0}^{\infty} RR(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q,q)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} RS(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q,q)_n}. \tag{3.31}$$

Let $R1(n)$ denote the numbers of partition $[\lambda_1, \lambda_2, \ldots, \lambda_r]$ of n such that each $\lambda_i \equiv \pm 1 (\text{mod } 5)$, and $R2(n)$ denote the numbers of partition $[\lambda_1, \lambda_2, \ldots, \lambda_r]$ of n such that each $\lambda_i \equiv \pm 2 (\text{mod } 5)$, respectively. The generating function of these is given by

$$\begin{aligned} \sum_{n=0}^{\infty} R1(n)q^n &= \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-4})(1-q^{5m-1})} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \\ \text{and} \quad \sum_{n=0}^{\infty} R2(n)q^n &= \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-2})(1-q^{5m-3})} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \end{aligned} \tag{3.32}$$

respectively. Then, Rogers-Ramanujan identities say that

$$RR(n) = R1(n) \qquad \text{and} \qquad RS(n) = R2(n). \tag{3.33}$$

To prove the Rogers-Ramanujan identities, it is equivalent to show the following

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q,q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q,q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{3.34}$$

respectively. For details, we refer to [And98]. Now, we define the following k-colored Rogers-Ramanujan type partitioned functions as follows: $R1_k(n)$ denote the numbers of partition $[\lambda_1, \lambda_2, \ldots, \lambda_r]$ of n such that each $\lambda_i \equiv \pm 1 (\text{mod } 5)$ and its part is colored by at most k-colours, and $R2_k(n)$ denote the numbers of partition $[\lambda_1, \lambda_2, \ldots, \lambda_r]$ of n such that each $\lambda_i \equiv \pm 2 (\text{mod } 5)$ and its part is colored by at most k-colours, respectively. The generating functions are given by

$$\sum_{n=0}^{\infty} R1_k(n)q^n = \frac{1}{(q;q^5)_\infty^k (q^4;q^5)_\infty^k} \qquad \text{and} \qquad \sum_{n=0}^{\infty} R2_k(n)q^n = \frac{1}{(q^2;q^5)_\infty^k (q^3;q^5)_\infty^k}, \tag{3.35}$$

respectively. Here, we state our result without proof as it follows exactly the same arguments as in the proofs of Theorem 2.1 and Theorem 2.2, respectively.

Theorem 3.3. *For a given positive integer n, we have the following identities for $R1_k(n)$ involving the special values of multiple zeta function $\zeta_k(2)$ given by*

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} R1_{2k}(n) = \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} R1_{2k}(n) \\ &= \sum_{\{s_1,s_2,\ldots,s_\ell\} \in S(n)} \sum_{\substack{m_1,m_2,\ldots,m_\ell \\ m_i-distinct}} \frac{(-1)^\ell}{\pi^{2\ell}} \frac{R1_2(s_1)R1_2(s_2)\cdots R1_2(s_\ell)}{m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1,m_2,\ldots,m_\ell} \left(1 - \frac{1}{m^2\pi^2}\right). \end{aligned} \tag{3.36}$$

These identities also valid if $R1_k(n)$ is replaced by $R2_k(n)$.

Theorem 3.4. *For a given positive integer n, we have the following identities for $R1_k(n)$ involving the special values of multiple zeta function $\zeta_k^{odd}(2)$ given by*

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}R1_{2k}(n)=\sum_{k=0}^{\infty}\frac{(-1)^k4^k\zeta_k^{odd}(2)}{\pi^{2k}}R1_{2k}(n)$$
$$=\sum_{\{s_1,s_2,\ldots,s_\ell\}\in S(n)}\ \sum_{\substack{m_1,m_2,\ldots,m_\ell\\ m_i-\text{odd }\ \&\ \text{ distinct}}}\frac{(-4)^\ell}{\pi^{2\ell}}\frac{R1_2(s_1)R1_2(s_2)\cdots R1_2(s_\ell)}{m_1^2m_2^2\cdots m_\ell^2}\prod_{\substack{m\neq m_1,m_2,\ldots,m_\ell\\ m-\text{odd}}}\left(1-\frac{1}{m^2\pi^2}\right). \tag{3.37}$$

These identities also valid if $R1_k(n)$ is replaced by $R2_k(n)$.

Remark 3.2. *The identities appearing in the Theorem 2.3 and Theorem 2.4 can be given for $R1_k(n)$ and $R2_k(n)$.*

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