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ON RIEMANN ZETA-FUNCTION AND ALLIED QUESTIONS-II

BY

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§ 1. **INTRODUCTION.** It is best to begin with two definitions. We write $s = \sigma + it$ unless otherwise stated.

DEFINITION 1. *Let* $\{a_n\}$ $(n = 1, 2, 3, \cdots)$ *with* $a_1 = 1$ *be any sequence of complex numbers* (a_n *may depend on two parameters* T *and* H *with* $T \geq H \geq$ *a large positive constant to appear later) which vanish for all but finitely many* n. *Then* $F_0(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *is called a Titchmarsh polynomial.*

DEFINITION 2. *Let* $\{a_n\}$ $(n = 1, 2, 3, \cdots))$ *with* $a_1 = 1$ *be any sequence of complex numbers* (a_n *may depend on two parameters* T *and* H, *with* $T \geq H \geq$ *a large positive constant, to follow). Suppose* $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *is convergent somewhere in the complex plane (depending on* T *and* H *to follow) and can be continued analytically in* $(\sigma > 0, T - H_1 \leq t \leq T + H + H_1)$, *where* $H_1(\leq \frac{1}{2}T)$ *is a suitable function of* T, *and there* $| F(s) | \leq T^{A \log H}$ *where* $A \geq 1$ *is any suitable constant. Then* $F(s)$ *is called an infinite Titchmarsh polynomial.*

REMARK. In definitions 1 and 2 we may replace $n^{-s} (n \geq 1)$ by λ_n^{-s} where

$\{\lambda_n\}$ is any sequence of real numbers with $\lambda_1 = 1$ and $\frac{1}{A} \leq \lambda_{n+1} - \lambda_n \leq A (n = 1, 2, 3, \cdots)$, $A \geq 1$ being a constant. We are thus led to "Generalised Titchmarsh polynomial" and "Infinite generalised Titchmarsh polynomial".

CONJECTURE 1. *There exist absolute numerical constants $c_1 > 0$ and $c_2 > 0$ such that for any Titchmarsh polynomial $F_0(s)$, we have*

$$\frac{1}{H} \int_T^{T+H} | F_0(it) |^2 \, dt \geq c_1 \sum_{n \leq c_2 H} | a_n |^2 . \tag{1}$$

CONJECTURE 2. *There exist constants $c_1 > 0$ and $c_2 > 0$ (which may depend at most on A) such that for any infinite Titchmarsh polynomial $F(s)$, we have*

$$\frac{1}{H} \int_T^{T+H} | F(it) |^2 \, dt \geq \frac{1}{2} c_1 \sum_{n \leq c_2 H} | a_n |^2 \tag{2}$$

REMARK 1. By LHS of (2) we mean the limit of the mean value of $| F(s) |^2$ as $\sigma \to 0$ from the right. We do not know how to prove Conjecture 1. We believe that it implies the stronger Conjecture 2. But we cannot prove this implication.

REMARK 2. For many important applications it suffices to suppose that, in Conjectures 1 and 2, $| a_n | \leq (nH)^A$, where $A \geq 1$ is some constant. But we cannot prove these conjectures nor the implication referred to in Remark 1 above even under this restriction.

We first prove (in § 3) the following theorem.

THEOREM 1. *If $H_1 > 0$ is a large positive constant times loglog T and the infinite Titchmarsh polynomial $F(s)$ converges absolutely at Re $s = 2$ and can be continued analytically in $(\sigma \geq -\delta, T - H_1 \leq t \leq T + H + H_1)$ and there maximum of $| F(s) | \leq T^{A \log H}$, then Conjecture 1 implies*

$$\frac{1}{H} \int_T^{T+H} | F(it) |^2 \, dt \geq \frac{1}{2} c_1 \sum_{n \leq c_2 H} | a_n |^2 . \tag{3}$$

REMARK. The condition *Re* $s = 2$ is unimportant. We can work with any constant in place of 2.

Next (in § 4 and § 5) we prove the following theorem. We define $d_k(n)$ (for complex k) by $(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n)n^{-s}, Re\ s \geq 2$.

THEOREM 2. *Let T and H exceed large positive constants and let k be any complex number with $|k| \leq \log H$ and $Re\ k \geq 0$. Then conjecture 1 implies*

$$\frac{1}{H}\int_{T}^{T+H} |(\zeta(\frac{1}{2}+it))^k|^2\, dt \geq \frac{1}{2}c_1 \sum_{n\leq c_2 H} |d_k(n)|^2 n^{-1}. \qquad (4)$$

Here we have assumed that $\zeta(s) \neq 0$ in $(\sigma > \frac{1}{2}, T - H_1 \leq t \leq T + H + H_1)$ where H_1 is a large positive constant multiple of $\log T \log H(\log\log T)^{-1}$.

REMARK 1. We are unable, so far, to put Theorem 2 in the language mean square of infinite Titchmarsh polynomials. Also we believe that in Theorem 2 we can manage to take H_1 to be a positive constant multiple of *loglog* T. We have been unable to prove this.

REMARK 2. Theorem 2 goes through for the zeta and L-functions of algebraic number fields (in place of $\zeta(s)$). In fact it has analogues for $D(s)$ (to be introduced in § 6, and there with $\lambda_n = n$) in place of $\zeta(s)$. For example we can select $\alpha = \frac{1}{2}$.

NOTATION. We use $c_0, c_0', c_1, c_2, \cdots$ to denote positive constants which depend at most on A whenever it occurs. The O symbols used extensively by Hardy and Littlewood and the symbols $\gg$ and $\ll$ of I.M. Vinogradov have the usual meaning. Sometimes we indicate the constant parameters on which these symbols imply by writing them below these symbols.

§ 2. **APPLICATIONS OF THEOREMS 1 AND 2.** It is not hard to deduce the following two Theorems 3 and 4 as corollaries to Theorems 1 and 2 respectively. (We concentrate on results mainly with the condition that H does not exceed a large positive constant times *loglog* T. We have dealt with the case H exceeding a large positive constant times *loglog* T very satisfactorily elsewhere (without the help of Conjectures 1 or 2) in our previous papers (see the paper I[4] of this series of papers for references).

THEOREM 3. *Conjecture 1 implies*

$$\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k}\,dt \geq \frac{1}{2}c_1 \sum_{n\leq c_2H} (d_k(n))^2 n^{-1} \tag{5}$$

for all integers $k \geq 1$ *and so we have first that LHS of (5) is* $\gg_k (\log H)^{k^2}$ *and next*

$$\max_{T\leq t\leq T+H} |\zeta(\frac{1}{2}+it)| > Exp\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right). \tag{6}$$

THEOREM 4. *Put* $k = k_0e^{i\theta}$ *where* $\cos\theta \geq 0$ *and* $k_0 \geq 1$ *is any integer. Then under the conditions of Theorem 2, LHS of (4) is* $\gg_k (\log H)^{|k^2|}$ *and next with* $z = e^{i\theta}$ *we have*

$$\max_{T\leq t\leq T+H} |(\zeta(\frac{1}{2}+it))^z| > Exp\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right), \tag{7}$$

LHS *trivially* ∞ *if* $\cos\theta < 0$, *of course on* RH.

REMARK 1. It is well-known that for complex k and all $x \geq 2$, we have $\sum_{n\leq x} |d_k(n)|^2 n^{-1} \gg_k (\log x)^{|k^2|}$. The inequalities (6) and (7) with some positive constant c_0 in place of $\frac{3}{4}$ follow from (5) and (4). All that we have to prove is that for all x exceeding a large positive constant, we have

$$\max_{|k|\leq \log x}\left(\sum_{n\leq x} |d_k(n)|^2 n^{-1}\right)^{|2k|^{-1}} > Exp\left(c_0\sqrt{\frac{\log x}{\log\log x}}\right). \tag{8}$$

This is proved by us in [2]. But in [1] R. Balasubramanian has shown by an ingenious method that the logarithms of both sides of (8) are asymptotic, as $x \to \infty$, with $c_0 = 0.75\cdots$.

REMARK 2. In Theorems 3 and 4 we have stated results on the critical line. But we can also state these for any σ with $\frac{1}{2} < \sigma < 1$. But the result corresponding to (8) is

$$\max_{|k|\leq \log x}\left(\sum_{n\leq x} |d_k(n)|^2 n^{-2\sigma}\right)^{|2k|^{-1}} > Exp\left(c_0'\frac{(\log x)^{1-\sigma}}{\log\log x}\right) \tag{9}$$

(where $c_0' > 0$ is constant) and so the results although valid for short intervals are not so good as those of H.L. Montgomery [3] who treats only long intervals like $0 \leq t \leq T$. His method fails for short intervals. Actually in [2] we have proved that the logarithm of LHS of (9) is $O((log\ x)^{1-\sigma}(loglog\ x)^{-1})$.

REMARK 3. It may be noted that the maximums in (8) and (9) even though we do not impose $|\ k\ | \leq log\ x$ are really attained in the range $|\ k\ | \leq log\ x$ and precisely when $|\ k\ |$ exceeds a positive constant power of $log\ x$ (see [2]). Hence for Theorems 3 and 4 Conjectures 1 and 2 with $c_1(log\ H)^{-100}$ in place of c_1 are enough.

REMARK 4. On the line $\sigma = 1$ there are excellent results (see [5]).

§ 3. **PROOF OF THEOREM 1.** Let $s = it$ and let t exceed a large positive constant. For $X \geq 1$ we define $A_X(s)$ by

$$A_X(s) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s+w)X^2 Exp\left(\left(sin\ \frac{w}{100}\right)^2\right)\frac{dw}{w} \tag{10}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n^{it}}\Delta\left(\frac{X}{n}\right) \tag{11}$$

where for real $\chi > 0, \Delta(\chi)$ is defined by

$$\Delta(\chi) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \chi^w Exp\left(\left(sin\ \frac{w}{100}\right)^2\right)\frac{dw}{w}. \tag{12}$$

Here $w = u + iv$ is a complex variable and $|\ \chi^w\ | = \chi^u$. By moving the line of integration to $u = 4$ and $u = -4$ respectively we have

$$\Delta(\chi) = O(\chi^4) = 1 + O(\chi^{-4}) \tag{13}$$

and so the contribution to $A_X(s)$ from very large n is very small. Notice that our condition on a_n in definition 2 implies that $|\ a_n\ |$ does not exceed a fixed power (depending on T and H) of n. Hence the expression (11) is a Titchmarsh polynomial plus a term whose absolute value is very small. We truncate the integral for $A_X(s)$ suitably and we have (with $Y \leq T^{\frac{1}{2}}$)

$$A_X(s) = \frac{1}{2\pi i}\int_{u=2,|v|\leq Y} \cdots + O\left(T^{2A\ log\ H}X^2\left(ExpExp\left(\frac{Y}{400}\right)\right)^{-1}\right). \tag{14}$$

Here moving the line of integration to $u = -\delta$, we have,

$$A_X(s) = F(s) + O(T^{2A\ \log\ H} X^2(\cdots)^{-1}) + O\left(\int_{u=-\delta} T^{2A\ \log\ H} X^{-\delta} \frac{|\,dw\,|}{|\,w^2\,|}\right). \tag{15}$$

We note that H is not more than a positive constant times *loglog* T and so by first choosing X to be a very large positive constant power of $T^{A\ \log\ H}$ and then Y to be a large positive constant times *loglog* T, we see that

$$A_X(s) = F(s)\ +\ \text{a very small term} \tag{16}$$

Thus on using Conjecture 1 we have

$$\frac{1}{H}\int_T^{T+H} |\,F(it)\,|^2\,dt \geq \frac{3}{4}H^{-1}\int_T^{T+H} |\,A_X(s)\,|^2\,dt \geq \frac{1}{2}c_1 \sum_{n\leq c_2 H} |\,a_n\,|^2\,. \tag{17}$$

This proves Theorem 1 completely.

§ 4. **FIRST PART OF THE PROOF OF THEOREM 2.** Let now $s = \frac{1}{2} + \delta + it$ ($t > 0$ large, $\delta > 0$ now not a constant) and $X \geq 1$. With $B_X(s)$ now defined by

$$B_X(s) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} ((\zeta(s+w))^k X^w Exp\left(\left(sin\ \frac{w}{100}\right)^2\right)\frac{dw}{w} \tag{18}$$

$$= \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}\Delta\left(\frac{X}{n}\right), \tag{19}$$

We have as before

$$\Delta(\chi) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \chi^w\ Exp((sin\ \frac{w}{100})^2)\frac{dw}{w}, (\chi > 0) \tag{20}$$

and so

$$\Delta(\chi) = O(\chi^4) = 1 + O(\chi^{-4}). \tag{21}$$

Under the conditions of Theorem 2 we have for

$$z = \sigma + i\tau, (\sigma > \frac{1}{2}, |\,t-\tau\,| \leq c_3 logloglog\ T),$$

$$T - H_1 + c_3 logloglog\ T \leq \tau \leq T + H + H_1 - c_3\ logloglog\ T,$$

$$-c_4 \frac{log\ T}{loglog\ T} max \left(1, log \left(\frac{c_5}{(\sigma - \frac{1}{2}) loglog\ T}\right)\right) \leq log \mid \zeta(z) \mid \leq c_6 \frac{log\ T}{loglog\ T} \tag{22}$$

and

$$\mid arg\ \zeta(z) \mid \leq c_7 \frac{log\ T}{loglog\ T}. \tag{23}$$

For these results see [8] and also [6]. These results are due to K. Ramachandra and A. Sankaranarayanan. For convenient reference we state in § 6 a very general result of theirs for use by others for further work if necessary. Note that if $k = k_1 + ik_2$ (k_1, k_2 real and $k_1 \geq 0$) we have

$$\begin{aligned} \mid \zeta(z))^k \mid &= \mid Exp((k_1 + ik_2)(log \mid \zeta(z) \mid + i\ arg\ \zeta(z))) \mid \\ &= \mid \zeta(z) \mid^{k_1} Exp(-k_2\ arg\ \zeta(z)) \qquad (24) \\ &\leq Exp\left(c_8 \frac{log\ T\ log\ H}{loglog\ T}\right) \qquad (25) \end{aligned}$$

As before we move the line of integration to $u = -\delta$ and we have

$$B_X(s) = (\zeta(s))^k + O\left(T^{\frac{c_9 log\ H}{loglog\ T}} X^2 (\cdots)^{-1}\right) + O\left(\int_{u=-\delta, |v| \leq Y} T^{\frac{c_9 log\ H}{loglog\ T}} X^{-\delta} \frac{\mid dw \mid}{\mid w \mid}\right) \tag{26}$$

We note that H is not more than a constant times $loglog\ T$ and so by first choosing X by

$$X^{\frac{\delta}{2}} = T^{\frac{c_9\ log\ H}{loglog T}} \tag{27}$$

and then choosing Y such that

$$T^{\frac{c_9\ log\ H}{loglog T}} \cdot T^{\frac{4}{\delta} \frac{c_9\ log\ H}{loglog T}} \left(ExpExp\ \frac{Y}{100}\right)^{-1} \tag{28}$$

is very small we find that

$$B_X(s) = (\zeta(s))^k + \text{a small quantity} + O\left(X^{-\frac{\delta}{2}}\left(log\ \frac{1}{\delta} + log\ Y\right)\right). \tag{29}$$

The last O-term in (29) is small if $T^{-\frac{c_9\ log\ H}{loglog\ T}}(log\ \frac{1}{\delta} + log\ T)$ is small since $1 \leq Y \leq T^{\frac{1}{2}}$. In the next section we show that we can shoose δ to be a large positive constant times $T^{-\frac{\log H}{\log\log T}}$ in order to prove Theorem 2. Thus we have to choose Y to be greater than a large positive constant multiple of

$$\log\log\left(T^{\frac{8}{\delta}c_9\frac{\log H}{\log\log T}}\right) = \log\left(\frac{8}{\delta}\;c_9\;\frac{\log H\;\log T}{\log\log T}\right).$$

Thus (with $s = \frac{1}{2} + \delta + it$) we have

$$\begin{aligned}\frac{1}{H}\int_T^{T+H} \mid (\zeta(s))^k \mid^2 dt &\gg \frac{99}{100}c_1 \sum_{n\leq c_2 H} \mid d_k(n) \mid^2 n^{-1-2\delta} \\ &\geq \frac{3}{4}c_1 \sum_{n\leq c_2 H} \mid d_k(n) \mid^2 n^{-1}. \end{aligned} \tag{30}$$

§ 5. **DIFFERENCE OF TWO INTEGRALS.** The rest of the work consists in showing that D defined by

$$D = \frac{1}{H}\int_T^{T+H}\left(\mid (\zeta(\frac{1}{2}+it))^k \mid^2 - \mid (\zeta(\frac{1}{2}+\delta+it))^k \mid^2\right) dt, \tag{31}$$

has a small absolute value provided δ is small, but not too small since we want $\log \frac{1}{\delta}$ to be small. This would prove that

$$\frac{1}{H}\int_T^{T+H} \mid (\zeta(\frac{1}{2}+it))^k \mid^2 dt \geq \frac{1}{2}c_1 \sum_{n\leq c_2 H} \mid d_k(n) \mid^2 n^{-1}, \tag{32}$$

which is precisely Theorem 2.

By (24) we have

$$\begin{aligned}\mid D \mid &\leq \frac{1}{H}\int_T^{T+H} \mid \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \frac{\partial}{\partial\sigma}(\zeta(\sigma+it))^{2k} d\sigma \mid dt \\ &\leq \frac{\mid 2k \mid}{H} Exp\left(2c_8\frac{\log T\;\log H}{\log\log T}\right)\int_T^{T+H}\int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \mid \frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} \mid d\sigma dt. \end{aligned} \tag{33}$$

To estimate the double integral in (33) we need a few lemmas. We begin with

LEMMA 1. *Let $f(z)$ be analytic in $\mid z - z_0 \mid \leq R$ and on the boundary of this disc let $Re\; f(z) \leq U$. Then*

$$f'(z_0) = O((U + \mid f(z_0) \mid)R^{-1}) \tag{34}$$

where the implied constant is absolute.

PROOF. The lemma is well-known (see page 8, Theorem 2.4.1 of [7]).

LEMMA 2. *We have, with $s = \sigma + it$ as in (33),*

$$\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = O\left(\log T + |\log \zeta(s)| + \sum_{\rho \in D_0} |\rho - s|^{-1}\right) \tag{35}$$

where ρ runs over the zeros in the disc D_0 given by $|z - s| \leq R = \frac{1}{100}$.

PROOF. Put $z_0 = s$ and

$$f_0(z) = \frac{\zeta(z)}{\prod_{\rho \in D_0}\left(1 - \frac{z-s}{\rho-s}\right)}$$

we have

$$\max_{z \in D_0} |f_0(z)| \leq \max_{z \in D_1} |f_0(z)|.$$

where D_1 is the disc $|z - s| \leq \frac{1}{4}$. Hence by applying Lemma 1 to $f(z) = \log f_o(z)$ we obtain the lemma.

LEMMA 3. *We have*

$$\frac{1}{H}\int_T^{T+H}\int_{\frac{1}{2}}^{\frac{1}{2}+\delta} (\log T + |\arg \zeta(s)|) d\sigma dt = O(\delta \log T). \tag{36}$$

PROOF. Follows from the result of Ramachandra and Sankaranarayanan mentioned in (23).

LEMMA 4. *We have*

$$\frac{1}{H}\int_T^{T+H}\int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \max\left(1, \log \frac{c_5}{(\sigma - \frac{1}{2})\log\log T}\right) d\sigma dt = O(\delta \log\log T + \delta^{\frac{1}{2}}). \tag{37}$$

PROOF. Follows from

$$\int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \log\left(\frac{1}{\sigma - \frac{1}{2}}\right) d\sigma \leq \sqrt{\delta}\left(\int_{\frac{1}{2}}^{\frac{1}{2}+\delta}\left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^2 d\sigma\right)^{\frac{1}{2}}$$

LEMMA 5. *We have*

$$\frac{1}{H}\int_T^{T+H}\int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \mid log \mid \zeta(s) \mid\mid d\sigma dt = O\left(\left(\delta \; loglog \; T + \delta^{\frac{1}{2}}\right)\frac{log \; T}{loglog \; T}\right). \tag{38}$$

PROOF. Follows from Lemma 4 and the result of Ramachandra and Sankaranarayanan mentioned in (23).

LEMMA 6. *Let ρ be a zero of the Riemann zeta-function with Re $\rho \leq \frac{1}{2}$. Then*

$$\frac{1}{H}\int_T^{T+H}\int_{\frac{1}{2}}^{\frac{1}{2}+\delta}\frac{d\sigma dt}{\mid \rho - s \mid} = O(\delta^{\frac{1}{2}} log \; T). \tag{39}$$

PROOF. Let $\rho = \beta + i\gamma$ with $\beta \leq \frac{1}{2}$. Then $\mid \rho - s \mid^{-1} \leq \mid \frac{1}{2} - s + i\gamma \mid^{-1}$ and hence

$$\int_T^{T+H} \mid \frac{1}{2} - \sigma - it + i\gamma \mid^{-1} dt = \int_{|t-\gamma|\leq\sigma-\frac{1}{2}} \cdots + \int_{T\geq|t-\gamma|\geq\sigma-\frac{1}{2}} \cdots$$

$$= O(1) + O(log \; T) + O(log \frac{1}{\sigma - \frac{1}{2}})$$

and hence the lemma follows.

LEMMA 7. *The number of zeros ρ counted with $T \leq Im \; \rho \leq T+1$ and $0 \leq Im \; \rho \leq 1$, is $O(log \; T)$.*

PROOF. This is well-known. See for example Theorem 3.1.1 on page 11 of [7].

LEMMA 8. *We have*

$$\frac{1}{H}\int_T^{T+H}\int_{\frac{1}{2}}^{\frac{1}{2}+\delta}\left(\sum_{\rho\in D_0} \mid \rho - s \mid^{-1}\right) d\sigma dt = O(\delta^{\frac{1}{3}}(log \; T)^3). \tag{40}$$

PROOF. Of course the sum over ρ depends on s. But we extend the sum over all the zeros of $\zeta(s)$ in $T-1 \leq t \leq T+H+1$. Now we can interchange the sum over ρ and the double integral. The number of zeros is now $O(H \; log \; T)$. The lemma now follows from Lemma 6.

LEMMA 8. *We have*

$$D = O\left(\delta^{\frac{1}{2}}(\log T)^4 Exp\left(2c_8\frac{\log T \log H}{\log\log T}\right)\right) \tag{41}$$

PROOF. Follows from (33) and Lemmas 2 to 7.

LEMMA 9. *$|D|$ is very small if δ is taken to be a large positive constant power of $Exp\left(-\frac{\log T \log H}{\log\log T}\right)$.*

PROOF. Follows from Lemma 8.

Theorem 2 now follows from the results of § 4 and § 5.

§ 6. A RESULT OF K. RAMACHANDRA AND A. SANKARANARAYANAN. While stating Lemma 1 of § 2 on page 392 of [6] the condition (4) of that paper is not necessary. It was meant for other purposes. Accordingly we state this once again.

THEOREM 5. (K. RAMACHANDRA AND A. SANKARANARAYANAN). *Let*

$$D(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \tag{42}$$

where $a_1 = 1 = \lambda_1, \frac{1}{A} \leq \lambda_{n+1} - \lambda_n \leq A$ ($A \geq 1$ is a constant) $\{\lambda_n\}$ is any sequence of real numbers and $\{a_n\}$ is any sequence of complex numbers with $|a_n| \leq n^A$. Let $\alpha > \delta$ ($\delta > 0$ a constant) and let $R(H,\alpha)$ denote the rectangle $(\sigma \geq \alpha, T_1 - H \leq t \leq T_1 + H)$. Let $D(s)$ be continuable analytically in $R(H, \alpha - \delta)$ and there $\max |D(s)| \leq T^A$ (where $A_5 \log\log\log T \leq H \leq \frac{1}{2}T$) and T_1 is any number lying between $T - H$ and $2T + H$. Let $D(s) \neq 0$ in $R(H,\alpha)$. Then for $t = T_1, s = \sigma + it$ in $R(H,\alpha)$ we have uniformly for $\sigma \geq \alpha, t = T_1$,

$$-A_1\frac{\log T}{\log\log T}\max\left[1, \log\left(\frac{A_2}{(\sigma-\alpha)\log\log T}\right)\right] \leq \log|D(s)| \leq A_3\frac{\log T}{\log\log T} \tag{43}$$

and

$$|\arg D(s)| \leq A_4\frac{\log T}{\log\log T}. \tag{44}$$

Here $A_1, A_2, \cdots, A_5$ are positive constants depending only on δ and A.

Note. It is enough to assume $D(s) \neq 0$ in $(\sigma > \alpha, T_1 - H \leq t \leq T_1 + H)$.

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