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IS THE BINOMIAL COEFFICIENT $\binom{2n}{n}$ SQUAREFREE?

BY

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Abstract

In this paper, we prove the Erdös conjecture that the binomial coefficient $\binom{2n}{n}$ is never squarefree, for all $n > 4$.

In this paper, we prove the Erdös conjecture that the binomial coefficient $\binom{2n}{n}$ is never squarefree, for all $n > 4$.

Sarközy [1] has proved that $\binom{2n}{n}$ is never squarefree, for n large enough. In his proof he has utilised Jutila's estimates [2] for $\sum_{p\leq x} e(2\pi i\theta/p)$ to estimate $\sum_{p\leq x} \log p e^{2\pi i\theta/p}$. Instead, by applying Vaughan's identity and then using the theory of exponent pairs, we are able to get better estimates. By explicitly calculating the constants involved, we can state that the result is true for $n \geq 2^{8000}$.

The case $4 < n < 2^{8000}$, is attacked by simple direct methods.

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NOTATION

- e(x) stands for $e^{2\pi i x}$.
- $\{x\}$ = fractional part of $x = x - [x]$.
- C will denote an absolute constant whose absolute value is ≤ 1, not nessarily the same at each ocurrence.
- We will write $\binom{2n}{n} = (s(n))^2 q(n)$, where q(n) is squarefree.

THEOREM

For $n \geq 2^{8000}$, $\binom{2n}{n}$ is never square free.

PROOF

In the prime factorisation of $\binom{2n}{n}$, let r_p denote the exponent of the prime p. Then

$$r_p = \sum_{i=1}^{n} \left(\left[\frac{2n}{p^i} \right] - \left[\frac{n}{p^i} \right] \right) \tag{1}$$

where $T = [log_p 2n]$

Note that,

$$\left[\frac{2n}{p^i} \right] - 2 \left[\frac{n}{p^i} \right] = \begin{cases} 1 & \text{if } \left\{ \frac{n}{p^i} \right\} \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

So if for some p, $\left\{ \frac{n}{p} \right\}$ and $\left\{ \frac{n}{p^2} \right\} \geq \frac{1}{2}$, then $r_p \geq 2$. Then

$$s(n) \geq \prod_{p} p$$

where the product runs over primes for which $\{n/p\}$ $\{n/p^2\} \geq 1/2$. Thus

$$\log s(n) \geq \sum_{p} \log p \tag{3}$$

where the sum runs over primes for which $\{n/p\}$ and $\{n/p^2\} \geq 1/2$. Therefore

$$\log s(n) \geq \sum_{\substack{\sqrt{n} < p \leq \sqrt{2n} \\ \{n/p\} \geq 1/2}}{}^{c} \log p \tag{4}$$

To incorporate the condition $\{n/p\} \geq 1/2$ into the summation, we use the following lemma of Vinogradov [3].

LEMMA Let ρ, η, Δ , be real numbers satisfying

$$0 < \Delta < \frac{1}{2} \tag{5}$$

$$\Delta \leq \rho - \eta \leq 1 - \Delta \tag{6}$$

Then there exists a periodic fumction $\psi(x)$, with period 1, satisfying

- (i) $\psi(x) = 1$ in the interval $\eta + \Delta/2 \leq x \leq \rho - \Delta/2$.
- (ii) $\psi(x) = 0$ in the interval $\rho + \Delta/2 \leq x \leq 1 + \eta - \Delta/2$
- (iii) $0 \leq \psi(x) \leq 1$ in the remainder of the interval $\eta - \Delta/2 \leq x \leq 1 + \eta - \Delta/2$.
- (iv) $\psi(x)$ has a Fourier expansion of the form,

$$\psi(x) = (\rho - \eta) + \sum_{m=1}^{\infty} (a_m \cos(2\pi m x) + b_m \sin(2\pi x))$$

where

$$|a_m| \leq \frac{2}{\pi m}, \quad |a_m| \leq 2(\rho - \eta), \quad |a_m| < \frac{2}{\pi^2 m^2 \Delta}$$

and

$$|b_m| \leq \frac{2}{\pi m}, \quad |b_m| \leq 2(\rho - \eta), \quad |b_m| < \frac{2}{\pi^2 m^2 \Delta}$$

We use this lemma, with $\Delta = \epsilon, \eta = 1/2 + \epsilon/2, \rho = 1 - \epsilon/2$ .Then (5) and (6) force $\epsilon \leq 1/4$ and we get a periodic function F such that

(i)$F(x) = 1$ in $1/2 + \epsilon \leq x \leq 1 - \epsilon$, (ii)$F(x) = 0$ in $0 \leq x \leq 1/2$, (iii)$0 \leq x \leq 1$ in the rest of the interval $[0, 1]$ and $F(x + 1) = F(x)$. (iv)The Fourier series of F(x) can be rewritten as

$$F(x) = (1/2 - \epsilon) + \sum_{\substack{m = -\infty \\ m \neq 0}}^{\infty} d_m e(mx)$$

where

$$|d_m| = \frac{1}{2} \left| a_{|m|} + i b_{|m|} \right| = \frac{1}{2} \left(a_{|m|}^2 + b_{|m|}^2 \right)^{1/2} \leq \frac{1}{2} \frac{2^{3/2}}{\pi |m|} < \frac{1}{2|m|} \tag{7}$$

and

$$|d_m| = \frac{1}{2} \left(a_{|m|}^2 + b_{|m|}^2 \right)^{1/2} \leq \frac{1}{2} \frac{2^{3/2}}{\pi^2 m^2 \epsilon} < \frac{1}{6 m^2 \epsilon} \tag{8}$$

So from (4) and the lemma

$$\log s(n) \geq \sum_{\sqrt{n}<p\leq\sqrt{2n}} F(n/p)\log p$$

$$= \sum_{\sqrt{n}<p\leq\sqrt{2n}} (1/2-\epsilon)\log p + \sum_{\sqrt{n}<p\leq\sqrt{2n}} \left(\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} d_m e(\frac{mn}{p}) \right) \log p$$

$$= (1/2-\epsilon) \sum_{\sqrt{n}<p\leq\sqrt{2n}} \log p + \sum_{\sqrt{n}<r\leq\sqrt{2n}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} d_m e(\frac{mn}{r})\Lambda(r)$$

$$- \sum_{\substack{\sqrt{n}<p^i\leq\sqrt{2n} \\ i\geq 2}} \left(\sum_{m\neq 0} d_m e(\frac{mn}{p})\log p \right) \qquad (9)$$

Since $|\sum_{m\neq 0} d_m e(\frac{mn}{p})| \leq 1$,the modulus of the last term is

$$\leq \sum_{i\geq 2} \sum_{n^{\frac{1}{2i}}<p\leq(2n)^{\frac{1}{2i}}} \log p$$

$$\leq \frac{\log 2n}{\log 2} \sum_{n^{1/4}<p\leq(2n)^{1/4}} \log p$$

$$\leq \frac{\log 2n}{\log n} \left(\theta((2n)^{1/4}) - \theta((n)^{1/4})\right) \qquad (10)$$

Vaughan's identity [4] states that

$$\Lambda(r) = \sum_{\substack{xy = r \\ x \le Z}} \mu(x)\log y - \sum_{xy=r} C_x - \sum_{\substack{xy = r \\ x,y \ge Z}} \tau_x \Lambda(y)$$

where Z can be chosen to be any integer $\le r$,

$$C_x = \sum_{\substack{kl = x \\ k,l \le Z}} \Lambda(k)\mu(l) \quad \text{and} \quad \tau_x = \sum_{\substack{d|x \\ d \le Z}} \mu(d)$$

Using the above identity, the second term in R.H.S of (9) is

$$= S_1 - S_2 - S_3 \tag{11}$$

where

$$S_1 = \sum_{\sqrt{n}<r\le\sqrt{2n}} \sum_{\substack{m = -\infty \\ m \ne 0}}^{\infty} d_m \sum_{\substack{xy = r \\ x \le Z}} \mu(x) \log y e(\frac{mn}{xy})$$

$$S_2 = \sum_{\sqrt{n}<r\le\sqrt{2n}} \sum_{\substack{m = -\infty \\ m \ne 0}}^{\infty} d_m \sum_{xy=r} C_x e(\frac{mn}{xy})$$

$$S_3 = \sum_{\sqrt{n}<r\le\sqrt{2n}} \sum_{\substack{m = -\infty \\ m \ne 0}}^{\infty} d_m \sum_{\substack{xy = r \\ x,y \ge Z}} \tau_x \Lambda(y) e(\frac{mn}{xy})$$

To estimate S_1, S_2, S_3 we will use the theory of exponent pairs.

DEFINITION

Consider sums of the form

$$S = \sum_{B<t\leq B+h} e(f(t))$$

where $B > 1$, $1 < h \leq B$ and $f \in C^r[B, 2B]$, for some $r \geq 5$. Suppose

$$k_1 A \leq |f'(t)| \leq l_1 A$$

and

$$k_r AB^{1-r} \leq |f^r(t)| \leq l_r AB^{1-r} \tag{12}$$

where $A > 1/2$. If (χ, λ), $0 \leq \chi \leq \frac{1}{2} \leq \lambda \leq 1$ is such that for any f satisfying the above conditions

$$|S| \ll A^{\chi} B^{\lambda}$$

then (χ, λ) is said to be an exponent pair. The constant implied in the above inequality will depend on the constants k_r, l_r and the exponent pair (χ, λ). We will denote it by $c(f, \chi, \lambda)$. $(0, 1)$ is trivially an exponent pair.

Also, if (χ, λ) is an exponent pair then,

Rule A $\left(\frac{\chi}{2\chi+2}, \frac{1}{2} + \frac{\lambda}{2\chi+2}\right)$ is an exponent pair;

Rule B $(\lambda - \frac{1}{2}, \chi + \frac{1}{2})$ is an exponent pair;

Rule C the convex combination $(k\chi_1 + (1-k)\chi_2, k\lambda_1 + (1-k)\lambda_2)$, $0 \leq k \leq 1$, of any two exponent pairs (χ_1, λ_1) and (χ_2, λ_2) is an exponent pair.

Detailed proof for the above statement can be found in chapter 2 of A.Ivic's book [5]. For our purpose we need to calculate the constant $c(f,\chi,\lambda)$

LEMMA 1

Let f satisfy the conditions (12) with $A \geq 1$. Then

$$|S| \leq \left\{ \frac{8(l_1 - k_1)}{\sqrt{k_2}} + \frac{16}{\sqrt{k_2}} + \frac{4}{\pi}(l_1 - k_1) + 6 \right\} A^{1/2} B^{1/2} \tag{13}$$

PROOF

We will assume without loss of generality that f' is monotonically decreasing (if not, we consider $\overline{S}$) and that $f'(B) = \beta, f'(B+h) = \alpha$ .Let η be any real number such that $0 < \eta < 1$. Suppose

$$\eta - 1 < \alpha < \eta \tag{14}$$

By Euler's summation formula

$$S = \sum_{B<n\leq B+h} e(f(n))$$

$$= -\int_B^{B+h} e(f(t))d(\chi(t)) + \int_B^{B+h} e(f(t))dt, \text{ where } \chi(t) = t - [t] - \frac{1}{2}$$

$$S = -e(f(t))(\chi(t))|_B^{B+h} + \int_B^{B+h} \chi(t)e(f(t))2\pi i f'(t)dt + \int_B^{B+h} e(f(t))dt$$

Using the Fourier expansion for $\chi(t)$,

$$S = c + \int_B^{B+h} e(f(t))dt - 2i \sum_{m=1}^{\infty} \frac{1}{m} \int_B^{B+h} \sin(2\pi m) e(f(t)) f'(t) dt \quad (15)$$

$$= c + \int_B^{B+h} e(f(t))dt + \sum_{m=1}^{\infty} \frac{1}{m} \int_B^{B+h} \left(e(-m) - e(m)\right) e(f(t)) f'(t) dt$$

$$= c + \int_B^{B+h} e(f(t))dt + \sum_{m=1}^{\infty} \frac{1}{2\pi i m} \int_B^{B+h} \frac{f'(t)}{f'(t) - m} d(e(f(t) - mt))$$

$$- \sum_{m=1}^{\infty} \frac{1}{2\pi i m} \int_B^{B+h} \frac{f'(t)}{f'(t) + m} d(e(f(t) + mt))$$

With f' the function $\frac{f'(t)}{f'(t)+m}$ is also monotonically decreasing. So, by the mean value theorem for integrals, the absolute value of the last term is

$$\leq \sum_{m=1}^{\infty} \frac{1}{2\pi m} \frac{\beta}{\beta + m} \left| \int_B^{\xi} d(e(f(t) + mt)) \right| \quad (\text{ for some } \xi \leq B + h\ )$$

$$\leq \frac{1}{2\pi} \left(\sum_{m=1}^{\infty} \frac{2\beta}{m(\beta + m)} \right)$$

$$\leq \frac{1}{\pi} \left(\sum_{m \leq \beta} \frac{1}{m} + \sum_{m > \beta} \frac{\beta}{m(\beta + m)} \right)$$

The above expression $\leq \frac{2\zeta(2)}{\pi}$ if $\beta \leq 2$. Otherwise it is

$$\leq \frac{1}{\pi} \left(\int_1^{[\beta]+1} \frac{2}{t} dt + \beta \int_{[\beta]}^{\infty} \frac{dt}{(t-1)^2} \right)$$

$$\leq \frac{1}{\pi}\left(2\log(1+\beta)+\frac{\beta}{[\beta]-1}\right) \leq \frac{1}{\pi}(2\log(1+\beta)+3)$$

So, in any case, it is less than $\frac{2\log(1+\beta)}{\pi}+\frac{\pi}{3}$ (16)

$$\left|\sum_{m\geq\beta+\eta}\frac{1}{2\pi i m}\int_B^{B+h}\frac{f'(t)}{f'(t)-m}d(e(f(t)-mt))\right| \leq \frac{1}{2\pi}\left(\sum_{m\geq\beta+\eta}\frac{2\beta}{m(m-\beta)}\right)$$

$$\leq \frac{1}{\pi}\left(\sum_{\beta+\eta\leq m\leq 2\beta}\frac{\beta}{m(m-\beta)}+\sum_{2\beta<m<\infty}\frac{\beta}{m(m-\beta)}\right)$$

$$\leq \frac{1}{\pi}\left(\sum_{\beta+\eta\leq m\leq 2\beta}\frac{1}{m-\beta}+\sum_{2\beta<m<\infty}\frac{\beta}{m(m-\beta)}\right)$$

$$\leq \frac{1}{\pi}\left(\frac{1}{\eta}+\int_{\beta+\eta}^{[2\beta]+1}\frac{dt}{t-1}+\beta\int_{[2\beta]}^{\infty}\frac{dt}{t^2}\right)$$

The above expression is

$$\leq \frac{1}{\pi}\left(\frac{1}{\eta}+\log(2\beta+1)+3\right) \text{ if } \beta\geq 3$$

$$\leq \frac{1}{\pi}\left(\frac{1}{\eta}+\log(2\beta+1)+3\zeta(2)\right) \text{ if } \beta<3$$

So in any case the bound is $\frac{1}{\pi}\left(\frac{1}{\eta}+\log 92\beta+1)+3\zeta(2)\right)$. (17)

Also

$$\sum_{1\le m\le \beta+\eta} \frac{1}{m}\int_B^{B+h} f'(t)e(f(t)-mt)dt$$

$$= \frac{1}{2\pi i}\sum_{1\le m\le \beta+\eta} \frac{1}{m}\int_B^{B+h} d(e(f(t)-mt) + \sum_{1\le m\le \beta+\eta}\int_B^{B+h} e(f(t)-mt)dt$$

$$= \frac{c}{\pi}(\log(\beta+\eta+1)) + \sum_{1\le m\le \beta+\eta}\int_B^{B+h} e(f(t)-mt)dt$$

So we have

$$S = \sum_{0\le m\le \beta+\eta}\int_B^{B+h} e(f(t)-mt)dt + c\left(1+\frac{1}{\pi}(2\log(1+\beta)+3\right)$$

$$+\frac{c}{\pi}\left(\frac{1}{\eta}+\log(2\beta+1)+3\zeta(2)\right) + \frac{c}{\pi}\log(\beta+\eta+1)$$

$$= \sum_{0\le m\le \beta+\eta}\int_B^{B+h} e(f(t)-mt)dt + c\left(1+\frac{3}{\pi}+\frac{1}{\eta\pi}+\frac{3\zeta(2)}{\pi}+\frac{4}{\pi}\log(\beta+\eta+1)\right) \tag{18}$$

Suppose, contrary to (14), $\alpha \notin (\eta-1,\eta)$ Then if $\eta-1<\alpha-k\le\eta$, define $h(t)=f(t)-kt$ .Then h satisfies the conditions asssumed in the first paragraph, and the above proof goes through for h(t).

But

$$S = |\sum_{B<n\le B+h} e(h(n))|$$

$$= \sum_{\alpha'-\eta\leq m\leq\beta'+\eta} \left(\int_B^{B+h} e(h(t)-mt)dt\right) + c\left(1+\frac{3}{\pi}+\frac{1}{\eta\pi}+\frac{3\zeta(2)}{\pi}+\frac{4}{\pi}\log(2\beta'+\eta+\right.$$

where $\alpha' = \alpha - k, \beta' = \beta - k$.

So

$$S \leq \left|\sum_{\alpha-\eta<m<\leq\beta+\eta}\int_B^{B+h} e(f(t)-mt)dt\right| +$$

$$+C\left(1+\tfrac{3}{\pi}+\tfrac{1}{\eta\pi}+\tfrac{3\zeta(2)}{\pi}+\tfrac{4}{\pi}\log(2(\beta-\alpha)+3\eta+1)\right)$$

If $|f''(t)|$ has a lower bound m, then

$$\left|\int_B^{B+h} e(f(t))dt\right| \leq 8/\sqrt{m} \qquad \text{(by Lemma 2.2, [5])}$$

Also, choosing $\eta = 1/3$, and noting $|(\beta-\alpha)| \leq (l_1 - k_1)A$,

$$S \leq \frac{8(\beta-\alpha+2)}{\sqrt{k_2AB^{-1}}} + c\left(4+\frac{4}{\pi}\log(2(\beta-\alpha)+2)\right)$$

$$\leq \frac{8(l_1-k_1)A}{\sqrt{k_2AB^{-1}}} + \frac{16}{\sqrt{k_2AB^{-1}}} + c\left(4+\frac{4}{\pi}\log 2+\frac{4}{\pi}\log(\beta-\alpha-1)\right)$$

$$\leq \frac{8(l_1-k_1)}{\sqrt{k_2}}A^{1/2}B^{1/2} + \frac{16}{\sqrt{k_2}}A^{1/2}B^{1/2} + c\left(6+\frac{4}{\pi}(l_1-k_1)A^{1/2}B^{1/2}\right)$$

$$\leq A^{1/2}B^{1/2}\left(\frac{8(l_1-k_1)}{\sqrt{k_2}}+\frac{16}{\sqrt{k_2}}+\frac{4}{\pi}(l_1-k_1)+6\right)$$

LEMMA 2

If $|S| \le C(f,\chi,\lambda)A^{\chi}B^{\lambda}$ for all f satisfying conditions (12) then

$$|S| \le c\left(f, \frac{\chi}{2\chi+2}, \frac{1}{2} + \frac{\lambda}{2\chi+2}\right) A^{\frac{\chi}{2\chi+2}} B^{\frac{1}{2}+\frac{\lambda}{2\chi+2}}$$

where the constant in the line above is

$$max\left(c(f,1/2,1/2), \left[4\left\{\left(1+\frac{16}{\sqrt{k_3 l_2}}+\frac{7}{4l_2}\right)+(2l_2)^{\chi}c(g,\chi,\lambda)\right\}+1\right]^{1/2}\right)$$

where $g = f'/2l_2$.

Proof

We proceed, as in the proof of Lemma (2.8) of [5], except that we calculate the constants explicitly at every stage, to obtain the above result.

Now we can estimate S_1, S_2, S_3.

$$|S_1| \le \sum_{\substack{m=-\infty \\ m \ne 0}}^{\infty} |d_m| \left| \sum_{x \le Z} \mu(x) \sum_{\frac{\sqrt{n}}{x} < y \le \frac{\sqrt{2n}}{x}} \log y e(\frac{mn}{xy}) \right|$$

$$\le \sum_{\substack{m=-\infty \\ m \ne 0}}^{\infty} |d_m| \sum_{x \le Z} \left| \sum_{\frac{\sqrt{n}}{x} < y \le \frac{\sqrt{2n}}{x}} \log y e(\frac{mn}{xy}) \right|$$

Since $\log y$ is monotonic, the modulus of the innermost sum is

$$\le \log \frac{\sqrt{2n}}{x} \left| \sum_{\xi < y \le \frac{\sqrt{2n}}{x}} e(\frac{mn}{xy}) \right|$$

for some $\xi \in (\frac{\sqrt{n}}{x}, \frac{\sqrt{2n}}{x})$

So we need to estimate an exponential sum $\sum_{B<t\leq B+h} e(f(t))$ with $B = \xi \leq \frac{\sqrt{2n}}{x}$, $f(y) = \frac{mn}{xy}$,$A = |m|x$, where f satisfies conditions (12) with

$$k_1 = \frac{1}{2}, k_2 = \frac{1}{\sqrt{2}}, k_3 = \frac{3}{2}, k_4 = \frac{6}{\sqrt{2}}$$

$$l_1 = 1, l_2 = 2, l_3 = 6, l_4 = 24$$

So

$$|S_1| \leq \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |d_m| \sum_{x\leq Z} \log \frac{\sqrt{2nx}}{C}(f,\chi,\lambda)(|m|x)^{\chi}(\frac{\sqrt{2n}}{x})^{\lambda}$$

for any exponent pair (χ, λ).

$$|S_1| \leq 2^{\lambda/2}\frac{\log(2n)}{2}n^{\lambda/2}\left(\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |d_m||m|^{\chi}\right) c(f.\chi,\lambda)\left(\sum_{x\leq Z} x^{\chi-\lambda}\right)$$

$$|S_1| \leq C(f,\chi,\lambda)2^{\frac{\lambda}{2}-1}\left(\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |d_m||m|^{\chi}\log(2n)\right) n^{\frac{\lambda}{2}}\frac{(Z+2)^{\chi-\lambda+1}}{(\chi-\lambda+1)} \tag{19}$$

where the parameter Z can be chosen to be any number $\leq \sqrt{2n}$.

Similarly

$$|S_2| \leq \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |d_m| \sum_{d\leq Z} \left| \sum_{l\leq Z} \Lambda(l) \sum_{\sqrt{n}<dly\leq\sqrt{2n}} e(\frac{mn}{xy}) \right|$$

Again, apply the theory of exponent pairs to estimate the exponential sum. Here $A = |m|dl$, $B = \frac{\sqrt{n}}{dl}$, $f(y) = \frac{mn}{dly}$ and the constants l_i, K_i are the same as in S_1.

$$|S_2| \leq \left(\frac{\log 2n}{2}\right) \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |d_m| \sum_{d,l\leq Z} c(f,\chi,\lambda)(|m|dl)^{\chi_1}(\frac{\sqrt{n}}{dl})^{\lambda_1}$$

$$\leq \frac{c(f,\chi,\lambda)}{2} \log 2n \left(\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |d_m||m|^{\chi_1} \right) \left(\sum_{d\leq Z} d^{\chi_1-\lambda_1} \right) \left(\sum_{l\leq Z} l^{\chi_1-\lambda_1} \right)$$

$$\leq \frac{c(f,\chi,\lambda)}{2} \left(\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |d_m||m|^{\chi_1} \right) (\log 2n)\, n^{\lambda_1} \frac{(Z+2)^{2(\chi_1-\lambda_1+1)}}{(\chi_1-\lambda_1+1)^2} \qquad (21)$$

To estimate S_3, we write it as

$$S_3 = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |d_m|(T_1+T_2)$$

where

$$T_1 = \sum_{\substack{x,y\geq Z \\ \sqrt{n}<xy\leq\sqrt{2n} \\ x\geq y}} \tau_x \wedge(y) e(\tfrac{mn}{xy}) \text{ and }$$

$$T_2 = \sum_{\substack{x,y\geq Z \\ \sqrt{n}<xy\leq\sqrt{2n} \\ x<y}} \tau_x \wedge(y) e(\tfrac{mn}{xy})$$

In $T_1, Y^2 \leq xy \leq \sqrt{2n}$. That is $y \leq (2n)^{1/4}$. The range of $Y = [Z,(2n)^{1/4}]$ can be split into 1 intervals (Z,2Z), (2Z, 4Z) etc., where $1 = \left[\frac{\log(2n)^{1/4}}{2Z}\right]$. So a typical interval is (U, U+U') where $U' \leq U, Z \leq U, U+U' \leq (2n)^{1/4}$. Note that

$$U \leq Y \leq 2U \Rightarrow \frac{\sqrt{n}}{2U} \leq x \leq \frac{\sqrt{2n}}{U}.$$

By Cauchy-Schwarz's inequality.

$$T_1^2 \leq 1^2 \left[\sum_{U<Y\leq U+U'} (\wedge(y))^2\right] \left[\sum_{U<y\leq U+U'} |\sum_{\substack{x\geq y \\ \sqrt{n}<xy\leq\sqrt{2n}}} \tau_x e(\frac{mn}{xy})|^2\right]$$

we have

$$\begin{aligned}
W_1 &= \sum_{U<Y\leq U+U'} |\sum_{\substack{x\geq y \\ \sqrt{n}<xy\leq\sqrt{2n}}} \tau_x e(\tfrac{mn}{xy})|^2 \\
&= \sum_{U<Y\leq U+U'} \left(\sum_{\substack{x\geq y \\ \sqrt{n}<xy\leq\sqrt{2n}}} \tau_x e(\tfrac{mn}{xy})\right)\left(\sum_{\substack{t\geq y \\ \sqrt{n}<ty\leq\sqrt{2n}}} \tau_t e(-\tfrac{mn}{xy})\right) \\
&\leq \sum_{\frac{\sqrt{n}}{2U}<t\leq\frac{\sqrt{2n}}{U}} \sum_{\frac{\sqrt{n}}{2U}<x\leq\frac{\sqrt{2n}}{U}} |\tau_x \tau_t| \left|\sum_{\substack{U<y\leq U+U \\ \sqrt{n}<xy,ty\leq\sqrt{2n}}} e(\tfrac{mn}{xy} - \tfrac{mn}{ty})\right|
\end{aligned}$$

Here $f(y) = \frac{mn}{y}(\frac{1}{t}\,\frac{1}{x})$, $A = \frac{|m|n|t-x|}{U^2tx}$, $B \le 2U$ (whenever $t \neq x$) and the constants

$$k_1 = \tfrac{1}{4},\, k_2 = \tfrac{1}{4},\, k_3 = \tfrac{3}{8},\, k_4 = \tfrac{12}{16}$$

$$\ell_1 = 1,\, \ell_2 = 2,\, \ell_3 = 6,\, \ell_4 = 24$$

So

$$W_1 \le \Bigg[\sum_{\frac{\sqrt{n}}{2U} < t \le \frac{\sqrt{2n}}{U}} d(t) \sum_{\substack{\frac{\sqrt{n}}{2U} < x \le \frac{\sqrt{2n}}{U} \\ x > t}} d(x) \left(\frac{|m| n (x-t)}{Utx} \right)^{\chi_2} (2U)^{\lambda_2} +$$

$$+ \sum_{\frac{\sqrt{n}}{2U} < t \le \frac{\sqrt{2n}}{U}} d^2(t) U + + \sum_{\frac{\sqrt{n}}{2U} < t < \frac{\sqrt{2n}}{U}} d(t) \sum_{\substack{\frac{\sqrt{n}}{2U} < x \le \frac{\sqrt{2n}}{U} \\ x > t}} d(x) \left(\frac{|m| n (t-x)}{Utx} \right)^{\chi_2} (2U)^{\lambda_2} \Bigg]$$

where (χ_2, λ_2) is an exponent pair. So

$$W_1 \le 2|m|^{\chi_2} U^{\lambda_2 - 2\chi_2} n^{\chi_2} \left(\sum_{\frac{\sqrt{n}}{2U} < t \le \frac{\sqrt{2n}}{U}} \frac{d(t)}{t^{\chi_2}} \right) \left(\sum_{\substack{x > t \\ \frac{\sqrt{n}}{2U} < x \le \frac{\sqrt{2n}}{U}}} d(x) \right)$$

$$+ U \sum_{\frac{\sqrt{n}}{2U} < t \le \frac{\sqrt{2n}}{U}} d^2(t) +$$

$$+ 2|m|^{\chi_2} U^{\lambda_2 - 2\chi_2} n^{\chi_2} \left(\sum_{\frac{\sqrt{n}}{2U} < t \le \frac{\sqrt{2n}}{U}} d(t) \right) \left(\sum_{\substack{x < t \\ \frac{\sqrt{n}}{2U} < x \le \frac{\sqrt{2n}}{U}}} \frac{d(x)}{x^{\chi_2}} \right)$$

$$\le \left(4n^{\chi_2} U^{\lambda_2 - 2\chi_2} \left(\frac{\left(\frac{\sqrt{2n}}{U} + 2 \right)^{1-\chi_2}}{1-\chi_2} \log \frac{\sqrt{2n}}{U} \right) \right)$$

$$\left(\left(\frac{\sqrt{2n}}{U}+2\right)\log\frac{\sqrt{2n}}{U}+U\frac{\sqrt{2n}}{u}\left(\log 2\frac{\sqrt{2n}}{U}\right)^3\right)$$

$$\leq |m|^{\chi_2}\left[\frac{4}{1-\chi_2}\frac{\log^2(2n)}{4}(n)^{\chi_2}U^{-2+\lambda_2-\chi_2}(\sqrt{2n}+2U)^{2-\chi_2}+\sqrt{2n}(\log\sqrt{2n})^3\right]$$

Using $\sum_{U<y\leq U+U'}(\wedge(y))^2 \leq (\log(2n)^{1/4})^2U$, and the estimates for $\sum_{t\leq Z} d^2(t)$ etc we obtain

$$T_1^2 \leq |m|^{\chi_2}\left[\frac{\log^6(2n)}{(1-\chi_2)}n^{\chi_2}U^{-1+\lambda_2}(\sqrt{2n}+2U)^{2-\chi_2}+\log(2n)^3.U.\sqrt{(2n)}\right] \tag{22}$$

Similarly, in T_2' the range of $x=[Z,4\sqrt{2n}]$ is split into 1 intervals. Again, applying the Cauchy-Schwarz's inequality we get

$$T_2^2 \leq 1^2\left(\sum_{V\leq x\leq V+V'}(\tau_x(x))^2\right)\left(\sum_{V\leq x\leq V+V'}|\sum_{\substack{y\geq x\\ \sqrt{n}<xy\leq\sqrt{2n}}}\wedge(y)\,e(\frac{mn}{xy})|^2\right)$$

Then

$$W_2=\sum_{V\leq x\leq V+V'}\left(\sum_{\substack{y\geq x\\ \sqrt{n}<xy\leq\sqrt{2n}}}\wedge(y)\,e(\frac{mn}{xy})\right)\left(\sum_{\substack{y\geq t\\ \sqrt{n}<ty\leq\sqrt{2n}}}\wedge(y)\,e(\frac{-mn}{ty})\right)$$

$$\leq \sum_{\frac{\sqrt{n}}{2V}<y\leq\frac{\sqrt{2n}}{V}}\ \sum_{\frac{\sqrt{n}}{2V}<t\leq\frac{\sqrt{2n}}{V}}|\wedge(y)\wedge(t)|\left|\sum_{V<x\leq V+V'}{}' e\left(\frac{mn}{x}(\frac{1}{y}-\frac{1}{t})\right)\right|$$

$$\leq (\log(\sqrt{2n}))^2 \sum_{\frac{\sqrt{n}}{2V}<y\leq\frac{\sqrt{2n}}{V}} \sum_{\frac{\sqrt{n}}{2V}<t\leq\frac{\sqrt{2n}}{V}} \left| \sum_{\substack{V<x\leq V+V' \\ \sqrt{n}<xy,xt\leq\sqrt{2n}}} e\left(\frac{mn}{x}(\frac{1}{y}-\frac{1}{t})\right) \right|$$

$$\leq \frac{(\log 2n)^2}{4} \left(\sum_{\frac{\sqrt{n}}{2V}<y\leq\frac{\sqrt{2n}}{V}} \sum_{\substack{y>t \\ \frac{\sqrt{n}}{2V}<t\leq\frac{\sqrt{2n}}{V}}} \left(\frac{|m|n(y-t)}{vyt}\right)^{\chi_2} (2V)^{\lambda_2} + \right.$$

$$\left. + \sum_{\substack{y=t \\ \frac{\sqrt{n}}{2V}<t\leq\frac{\sqrt{2n}}{V}}} 1 + \sum_{\frac{\sqrt{n}}{2V}<y\leq\frac{\sqrt{2n}}{V}} \sum_{\substack{t>y \\ \frac{\sqrt{n}}{2V}<t\leq\frac{\sqrt{2n}}{V}}} \left(\frac{|m|n(y-t)}{vyt}\right)^{\chi_2} (2V)^{\lambda_2} \right)$$

$$\leq \frac{(\log 2n)^2}{4} \left(2 \sum_{\frac{\sqrt{n}}{2V}<y\leq\frac{\sqrt{2n}}{V}} 2V^{\lambda_2-2\chi_2}|m|^{\chi_2} n^{\chi_2} \sum_{t\leq y} \frac{1}{t^{\chi_2}} + \sqrt{n} \right)$$

$$\leq \frac{(\log 2n)^2}{4} \left(4V^{\lambda_2-2\chi_2}|m|^{\chi_2} n^{\chi_2} \sum_{\frac{\sqrt{n}}{2V}<y\leq\frac{\sqrt{2n}}{V}} \left(\frac{\sqrt{2n}}{V}+2\right)^{1-\chi_2} + \sqrt{n} \right)$$

Therefore,

$$T_2^2 \leq |m|^{\chi_2} \left(\frac{4}{128} (\log 32n)^3 (\log 2n)^4 V^{-1+\lambda_2-\chi_2} n(2\sqrt{2n})(\sqrt{2n}+2V)^{1-\lambda_2} \right.$$

$$\left. + \frac{(\log 32n)^3(\log 2n)^3}{128} \sqrt{n}V \right) \tag{23}$$

by using the fact $\sum_{x\leq t} \tau_x^2 \leq t(\log 2t)^3$.

Therefore,

$S_3 \leq \sum_m |dm|(T_1+T_2)$ where $T_1 \& T_2$ are as in (22) and (23).

S_1, S_2, S_3 are of the order $(\log n) n^{\lambda/2} Z^{\chi-\lambda+1}$, $(\log n)\, n^{\lambda_1} Z^{2(\chi_1-\lambda_1+1)}$ and $(\log n)^{7/2} n^{1/2+\chi_2/4} Z^{-1/2(\chi_2-\lambda_2+1)}$ respectively. To minimize $S_1+S_2+S_3$ we choose $Z = n^{7/40}(\log 2n)^{7/2}, (\chi,\lambda) = (1/2,1/2), (\chi_1,\lambda_1) = (1/14,11/14)\,(\chi_2,\lambda_2) = (1/14,11/14)$. Also let $\epsilon = 1/8$ and let $n \leq 2^{8000}$. Then we can get an upper bound for S_1, S_2, S_3 as follows.

$C(f,1/2,1/2)$ is calculated using lemma 1. The exponent pair $(1/14,11/14)$ is got by applying the Van der Corput's process (Rule A) twice to $(1/2,1/2)$. So $C(f,1/14,11/14)$ can be calculated by using lemma 2. We get $C(f,1/2,1/2) = 31$ in term S_1 and $C(f,1/14,11/14) = 31$ in term S_2 $C(f,1/14,11/14) = 51$ in term S_3.

Substituting all parameters,

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} |dm| m^k \leq 6.3, 0 \leq k \leq 1/2.(\text{using}7\&8) \text{ and}$$

$$S_1 = O\left((\log 2n)^{9/2} n^{17/40}\right)$$

$$S_2 = O\left((\log 2n)^{3} n^{69/140}\right)$$

$$S_3 = O\left((\log 2n)^{3} n^{69/140}\right)$$

In fact when $n \leq 2^{8000}$

$$S_1 \leq 10^{-100} n^{1/2}, S_2 \leq .0013\, n^{1/2}, S_3 \leq .00001\, n^{1/2}$$

$$SoS_1 + S_2 + S_3 \leq .0014\, n^{1/2}$$

From (9), (10), (11) we get

$$\log s(n) \geq (1/2-\epsilon)(\theta\sqrt{2n}) - \theta(\sqrt{n}) - (|S_1|+|S_2|+|S_3|)$$
$$- \frac{\log(2n)}{\log 2}((\theta(2n)^{1/4}) - \theta((n)^{1/4}))$$

$.99m \leq \theta(m) \leq 1.002m$ for all $m \geq 1319007$ [6]. So $\log s(n) \geq 3/8(.39786)n^{1/2} - (.0014)n^{1/2} - 10^{-400}n^{1/2} > 0$ for all $n \geq 2^{8000}$.

Theorem 2

Let $n = a_{r+1}P^r + \ldots a_2 P + a$, where $0 \leq a_r < P, P a$ prime. If atleast two of the $a_i's$ are $\geq \frac{P+1}{2}$, then P^2 divides $\binom{2n}{n}$.

Proof

If $a_i \geq \frac{P+1}{2}$, then $\{\frac{n}{P^i}\} = \frac{a_i P^{i-1} + \ldots a_1}{P^i} \geq \frac{(P+1)P^{i-1}}{2P^i} \geq \frac{1}{2}$. Then it follows from (1) and (2) that $r_p \geq 2$, whenever two of the $a_i's \geq \frac{P+1}{2}$.

Take P = 2, and consider the binary expansion of any n. The above theorem says that except for n of form $2^j, 2^2|\binom{2n}{n}$.

So, we just have to verify that for $n = 2^j, 2 < j \leq 8000$ some $P^2|\binom{2n}{n}$. This we did, with the help of a computer. We have written a program to find out the last few digits of 2^j, when written in base P, and to check if at least two of these digits are greater than or equal to $\frac{P+1}{2}$. Thus, by applying Theorem 2, we have been able to verify that for all $2^j, 2 < j \leq 8000$, there is a prime $P < 100$, such that $P^2|\binom{2n}{n}$ (except for j = 4). $3^2|\binom{2n}{n}$ $\binom{32}{16}$, even though 2^4 does not satisfy the hypothesis of Theorem 2.

This proves the conjecture.

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P.S. When this paper was in the course of publication the following paper has appeared.

J.W. SANDER, *Prime power divisors of multinomial coefficients and Artin's conjecture*, J. Number Theory, 46 (1994), 372-384.

Here the author points out that G. Velammal, A. Granville and O. Ramaré have proved independent of each other the conjecture of P. Erdös solved by G. Velammal in the present paper. The referee to Velammal's paper has also pointed out this.

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