Hardy-Ramanujan Journal
Vol.19 (1996) 2-56

RAMANUJAN'S LATTICE POINT PROBLEM, PRIME NUMBER THEORY AND OTHER REMARKS

BY

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(TO PROFESSOR D.R. HEATH-BROWN ON HIS FORTY-THIRD BIRTHDAY)

§ 1. INTRODUCTION. The present paper consists of four useful main remarks which are not worth publishing separately, but we hope that taken together they are of sufficient interest. The first concerns the problem of S. Ramanujan (see Chapter V of [G.H.H]) of finding an asymptotic formula (with a good error term) for the number of integers of the form $2^u 3^v$ less than n where u and v are non-negative integers. This problem has been solved satisfactorily (in view of the work of K.F. Roth [K.F.R] and the more

Received by the editors on 15.12.1994.

recent results of [N.I.F] and here reference may be made to a paper by A. Baker and G. Wusthölz [A.B.,G.W] for latest contribution and explicit results with good economical constants) by G.H. Hardy and J.E. Littlewood (see Chapter IX, page 105 of [J.F.K]). Thus in a way we update the information on this problem of S. Ramanujan.

The second is a contribution to the explicit formula in prime number theory. This essentially removes the factor $(\log x)^2$ in the error term and so "improves" an old classical result of E. Landau [E.L]. There are also contributions to density estimates in the neighbourhood $|\sigma - \frac{3}{4}| \leq \frac{D \log\log T}{\log T}$ ($D > 0$ arbitrary constant) of $\frac{3}{4}$. We improve Ingham's result to

$$N(\sigma, T) \ll_D (T \log T \log\log T)^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^2 (\log\log T)^{-1}$$

and Huxley's result to

$$N(\sigma, T) \ll_D T^{\frac{(5\sigma-3)(1-\sigma)}{\sigma^2+\sigma-1}} (\log T)^{\frac{39}{5}} (\log\log T)^{\frac{8}{5}},$$

(see Appendix and Postscript); in fact the first estimate is valid even in $|\sigma - \frac{3}{4}| \leq \frac{1}{4} - \frac{1}{D}$); which are not very impressive, but we need them in our results. Thus for example we are able to prove things like $\left(\psi(x) = \sum_{m \geq 1} \sum_{p^m \leq x} \log p\right)$

$$\psi(x+h) - \psi(x) \sim h, h = x^{\frac{7}{12}} (\log x)^{8\frac{23}{24}+\varepsilon},$$

$$\frac{1}{X} \int_X^{2X} (\psi(x+H) - \psi(x) - H)^2 dx = O(H^2 (\log X)^{-\varepsilon_1}), H = X^{\frac{1}{6}} (\log X)^{10\frac{5}{8}+\varepsilon}$$

($\varepsilon_1 > 0$ is a constant depending on ε) and

$$\frac{1}{X} \int_X^{2X} (\psi(x+H) - \psi(x) - H)^2 dx = O(H^2 (\log X)^{-1-\varepsilon_1}), (\varepsilon_1 \text{ as before})$$

$$H = X^{\frac{1}{6}}(\log X)^{12\frac{19}{24}+\varepsilon}.$$

(See also Remarks 1,2 and 3 at the end of the proof of Theorem 4 of the post-script for improvements). These have applications to Diophantine approximations and there are other results which we will establish in § 3. For the earlier results in this direction (due to A. Ivić, Y. Motohashi, G. Harman) see the book of A. Ivić [A.I]. Harman's results are better in some ways and our results are better from some other points of view. It must be mentioned that the result involving h is not better than that of D.R. Heath-Brown, (see [D.R.H-B]) who proves more powerful results by using his new method which is deeper. In fact by his method he proves things like $\psi(x+h)-\psi(x)\sim h$ even when $h = x^{\frac{7}{12}-\varepsilon(x)}$ where $\varepsilon(x)$ is any function of x which tends to zero as $x\to\infty$.

There has been another set of deep ideas to deal with the difference between consecutive primes. These ideas founded by H. Iwaniec and M. Jutila [H.I,M.J] have been developed in several papers by D.R. Heath-Brown, H. Iwaniec, J. Pintz (for these see [A.I]). The latest result is due to S.-t. Lou and Qi Yao which states that with $h = x^{\frac{1}{2}+\frac{1}{22}+\varepsilon}$ we have

$$\pi(x+h)-\pi(x) \gg h(\log x)^{-1}.$$

(see [S.-t.L, Q.Y]). But what we have presented here is the limit of the Hoheisel-Ingham-Selberg method (one of the new things in our results being an improvement of the error term in Landau's explicit formula). A full generalisation of the Hoheisel- Ingham-Selberg method with an ingenious contribution by Hooley and Huxley was given by K. Ramachandra in $[KR]_1$. This is continued in [K.R, A.Sa, K.S] (to appear) by K. Ramachandra, A. Sankaranarayanan and K. Srinivas. We quote two samples in the reference to the three authors mentioned. They are

$$\sum_{x \leq n \leq x+h} \mu(n) = O(h\, exp(-c(log\ x)^{\frac{1}{6}})), h = x^{\frac{7}{12} + \frac{d}{loglog\ x}}$$

and

$$\frac{1}{X}\int_X^{2X} \mid \sum_{x \leq n \leq x+H} \mu(n) \mid^2 dx = O(H^2\, exp(-c'(log\ X)^{\frac{1}{6}}), H = X^{\frac{1}{6} + \frac{d'}{loglog\ X}}$$

(here $c > 0, c' > 0, d > 0, d' > 0$ are constants). These results use localised versions of some results of J.E. Littlewood and A. Selberg, due to K. Ramachandra and A. Sankaranarayanan.

The third is a simple proof that $\sum_{n \leq x} \mu(n) = O(x\, exp(-c(log\ x)^a))$ with constants c and a satisfying $c > 0$ and $0 < a < 1$, implies that $(1-\beta)^{-1} = O((log(\gamma+2))^{\frac{1}{a}-1})$ for all zeros $\beta + i\gamma, \gamma > 0$, of $\zeta(s)$. There is a lengthy proof of this in [L.B-D]. Our proof is based on ideas which we owe to H.L. Montgomery [H.L.M]. For the proof of the well-known result that the upper bounds for $(1-\beta)^{-1}$ imply the corresponding upper bounds for $\sum_{n \leq x} \mu(n)$ (see $[K.R]_1$. Here Lemmas 5 and 6 on pages 313-329 give a method of obtaining upper bounds for $\mid \zeta(s) \mid^{-1}$ and etc. which are necessary to prove this). In [L.B-D] there is a simple proof due to A.E. Ingham in the appendix by E.

Bombieri of obtaining bounds for $\psi(x)-x$ starting from bounds for $\sum_{n\le x}\mu(n)$. It will be nice to obtain a simple proof of the other way implication.

The fourth and the last is what we call $(2-\delta)$-hypothesis and its consequence that for real non-principal characters $\chi(mod\ k)$ we have $L(1,\chi)\gg(log\ k)^{-1}$. (For a result of Rodosskij in this direction see [H.-E.R] p.101). We now state

$(2-\delta)$-*Hypothesis* (A). *Let* $k\ge 2,(\ell,k)=1$. *Then given any constant* $\delta>0$, *there exists a constant* $D>0$ *such that for all* $X\ge k^D$ *we have,*

$$\sum_{X\le p\le 2X, p\equiv \ell(mod\ k)} 1 \le \frac{2-\delta}{\varphi(k)}\sum_{X\le p\le 2X} 1.$$

The sieve method of A. Selberg (for references see [K.P], [H.H, H.-E.R] and [H.-E.R]) gives $2+\delta$ in place of $2-\delta$. Thus A. Selberg's result misses the $(2-\delta)$-hypothesis by a narrow margin. There is another method due to H.L. Montgomery and R.C. Vaughan (see [H.L.M] and also the chapter on Brun-Titchmarsh theorem in [H.E.R]) of dealing with this problem. But this method (although more powerful) also misses the $(2-\delta)$-hypothesis by roughly the same narrow margin. Actually the following hypothesis is a consequence of the $(2-\delta)$-hypothesis. This hypothesis suffices to prove the lower bound for $L(1,\chi)$ stated above.

$(2-\delta)$-*hypothesis* (B). *We have*

$$\sum_{\ell=1,(\ell,k)=1}^{k}\left(\sum_{X\le p\le 2X, p\equiv \ell(mod k)} 1\right)^2 \le \frac{2-\delta}{\varphi(k)}\left(\sum_{X\le p\le 2X} 1\right)^2$$

under the same conditions on δ, D and X as before.

§ 2. **RAMANUJAN'S LATTICE POINT PROBLEM.** Ramanujan's lattice point problem $2^u 3^v \leq n$ (which is clearly equivalent to $0 \leq u \log 2 + v \log 3 \leq \log n$) asserts that the number of lattice points $(u, v)(u \geq 0, v \geq 0)$ is

$$\frac{(\log n)^2}{2 \log 2 \log 3} + \frac{\log n}{2 \log 2} + \frac{\log n}{2 \log 3} + o(\log n).$$

Ramanujan appears to have had no proof of this. In Chapter V of [G.H.H]), Hardy considers the problem of lattice points (u, v) satisfying $0 \leq u\omega + v\omega' \leq \eta$ where ω, ω' are positive real constants such that $\theta = \omega'/\omega$ is irrational and proves that the number of such lattice points is (as $\eta \to \infty$)

$$\frac{1}{2}\left(\frac{\eta^2}{\omega\omega'} + \frac{\eta}{\omega} + \frac{\eta}{\omega'}\right) + E(\eta)$$

where $E(\eta) = o(\eta)$. In fact G.H. Hardy and J.E. Littlewood proved some finer theorems on an assumption on the convergents $\frac{p_m}{q_m}$ to the simple continued fraction expansion of θ. *More specifically let* $q_{m+1} = O(q_m^{\alpha_0})$ *where* α_0 *is a constant satisfying* $1 \leq \alpha_0 \ll 1$. Then their theorem (see page 105 of [J.F.K], here the dominant term is $\frac{1}{2}(\eta^2\omega^{-1}\omega'^{-1} - \eta\omega^{-1} - \eta\omega'^{-1})$ since the lattice points on $u = 0$ and also those on $v = 0$ are excluded) runs as follows

THEOREM 2.1. *If* $\alpha_0 = 1$ *then* $E(\eta) = O(\log \eta)$; *otherwise* $E(\eta) = O_\epsilon\left(\eta^{1-\alpha_0^{-1}+\epsilon}\right)$ *for every* $\epsilon > 0$.

COROLLARY. (i) *If θ is a quadratic irrationality then* $E(\eta) = O(\log \eta)$.

(ii) *If θ is any algebraic irrationality of degree ≥ 3, then $E(\eta) = O_\varepsilon(\eta^\varepsilon)$ for every $\varepsilon > 0$.*

(iii) *If $\theta = (\log 3)(\log 2)^{-1}$, then $E(\eta) = O_\varepsilon\left(\eta^{1-\alpha_0^{-1}+\varepsilon}\right)$ where $\alpha_0 = 2^{40}\log 3$ and $\varepsilon > 0$ is arbitrary.*

REMARK 1. The O-constant in (ii) is not effective. In (i) and (iii) it is effective.

REMARK 2. Instead of $(\log 3)(\log 2)^{-1}$ in (iii) we can take any irrational $\theta = (\log a)(\log b)^{-1}$ where a and b are two positive integers such that θ is irrational. Then α_0 will depend on a and b.

PROOF OF THE COROLLARY. Note that $\mid \theta - \frac{p_m}{q_m} \mid \ll q_m^{-1} q_{m+1}^{-1}$ and so

$$q_{m+1} \ll \mid q_m\theta - p_m \mid^{-1} .$$

Thus we need a lower bound for $\mid q_m\theta - p_m \mid$ of the type $\gg q_m^{-\alpha_0}$. This may not be satisfied by all real irrationalities θ. But by well-known results on quadratic irrationalities we know that this holds with $\alpha_0 = 1$. Thus (i) follows. By a famous result of K.F. Roth [K.F.R] this is true for all algebraic irrationalities θ of degree ≥ 3 and α_0 can be taken to be any constant > 1. This proves (ii). In Ramanujan's case N.I. Feldman [N.I.F] has shown that α_0 exists. However by the explicit results of A. Baker and G. Wasthölz [A.B, G.W] it follows that we can take $\alpha_0 = 2^{40}\log 3$. This proves (iii).

§ 3. PRIME NUMBER THEORY (EXPLICIT FORMULA,

DENSITY RESULTS AND APPLICATIONS). The main object of this section is to prove the following theorem and to apply it to study the difference between consecutive primes which in turn we will apply to a problem on Diophantine approximations. (For density results see sections A.2 and A.3 of the appendix).

THEOREM 3.1. *Let* $T \geq 10, x \geq 10, \frac{x}{T} \geq 10$. *Then*

$$\psi(x) = x - \frac{1}{T}\int_T^{2T}\left(\sum_{\beta\geq 0, |\gamma|\leq \tau}\frac{x^\rho}{\rho}\right)d\tau + O\left(\frac{\log x}{\log \frac{x}{T}}\cdot\frac{x}{T}\right),$$

where $\vartheta(x) = \sum_{p\leq x}\log p, \psi(x) = \sum_{m\geq 1}\vartheta(x^{\frac{1}{m}})$, $\psi_0(x) = \frac{1}{2}(\psi(x+0)+\psi(x-0))$, *and* ρ *runs over all the zeros of* $\zeta(s)$ *with the restrictions indicated. The constant implied by the O-symbol is absolute. We note that* $\frac{x}{T}(\log \frac{x}{T})^{-1}$ *exceeds a positive constant.*

We now draw an immediate corollary. The first part of the corollary seems to be new. The second part is a well-known result due to H. Cramer.

COROLLARY. *Let* $10 \leq f \leq x^{\frac{1}{4}}$, *and* $h = x^{\frac{1}{2}}(\log x)f$. *Then on* R.H (Riemann hypothesis) *we have*

$$\psi(x+h) - \psi(x) - h = O(hf^{-\frac{1}{2}}),$$

and so on R.H,

$$p_{n+1} - p_n \ll p_n^{\frac{1}{2}}\log p_n.$$

REMARK 1. All that we need for the proof of this theorem is Vinogradov's zero-free region, Euler product and the functional equation (it may also be noted that functional equation is not essential). Thus it can be extended to more general situations (like zeta and L-functions of algebraic number fields) where these are available. We can also establish analogues of this theorem to error estimations of A. Weil's explicit formulae, [A.W], (see also [S.L]). These will be treated elsewhere.

REMARK 2. Trivially if $\frac{x}{T} \gg x^{\varepsilon}$ for some fixed $\varepsilon > 0$ then the O-term is $O(\frac{x}{T})$. Otherwise it can still be replaced by $O(\frac{XC(X)}{T})$ (where $C(X)(\geq 1)$ is any constant or a function tending to infinity we assume $C(x) \asymp C(X)$ provided $\frac{X}{T} \geq \log X$, if we are content with $O(X(C(X))^{-1})$ exceptions of integers $[x]$ in $X \leq x \leq 2X$. This can be seen as follows. Clearly our proof of this theorem shows that what we want are upper bounds for (we assume $h_0 \geq \log X$ and write $\Lambda(n) = \log p$ if $n = p^m (m \geq 1)$ and zero otherwise) $S_1(x) = \sum_{|n-x| \leq h_0} \Lambda(n)$ and $S_2(x) = \sum_{h_0 \leq |n-x| \leq \frac{x}{2}} \Lambda(n)(n-x)^{-2}$. Clearly $S_2(x)$ is of the same order of magnitude as

$$S_3(x) = \frac{1}{h_0^2} \sum_{h_0 \leq |n-x| \leq 2h_0} \Lambda(n) + \frac{1}{(2h_0)^2} \sum_{2h_0 \leq |n-x| \leq 4h_0} \Lambda(n) + \cdots$$

where the RHS has an obvious termination. The imperfection which arises due to the application of Brun's sieve can be corrected as follows (of course with $O(\cdots)$ exceptions mentioned above). Note that $S_1(x)$ and $S_2(x)$ are non-negative. It can be easily seen that by prime number theorem we have

$$\sum_{X\le[x]\le 2X} S_1([x]) \ll Xh_0$$

since there are cancellations on LHS. Hence $S_1(x) \le C(X)h_0$ with at most $O(X(C(X))^{-1})$ exceptions. Similarly $S_2(x) \le C(X)h_0^{-1}$ with at most the same number of exceptions. We put $h_0 = xT^{-1}$ and recover the bound $O((C(X))^{-1}XT^{-1})$ in the theorem with the number of exceptions just referred to. Note that for $1 \le h \le x$ we have

$$\sum_{X\le[x]\le 2X} \sum_{m\ge 2}(\vartheta((x+h)^{\frac{1}{m}})-\vartheta(x^{\frac{1}{m}})) \le \sum_{X\le[x]\le 2X} \sum_{m\ge 2}\frac{\log x}{m} \sum_{x^{\frac{1}{m}}\le n\le (x+h)^{\frac{1}{m}}} 1 \ll hX$$

These ideas lead to the following (conditional) theorem of A. Selberg [A.Se]. (We quote only an impressive special case).

THEOREM 3.2. (A. SELBERG). *Let* $\Phi(x) = (\log x)^2 \log\log\log x$. (*Assume Riemann's hypothesis* i.e. $\beta \ge 0$ *implies* $\beta = \frac{1}{2}$). *Then in the interval* $X \le [x] \le 2X$ *we have*

$$\pi(x+\Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x}$$

for all integers $[x]$ *with* $o(X)$ *exceptions.*

REMARK 3. Our later arguments (used by us to prove the unconditional Theorem 3.5) show how to prove the theorem of A. Selberg. We leave the details for the reader.

To prove Theorem 3.1 we need a few lemmas which are of independent interest. We begin with Lemma 1, which is well-known.

LEMMA 1. *For $y > 0$ and $c > 0$ we have,*

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{y^s}{s}ds = \delta(y) + O\left(min\left(\frac{y^c}{T\mid log\ y\mid}, y^c\right)\right)$$

where $\delta(y) = 0, \frac{1}{2}$ or 1 according as $y < 1, y = 1$ or $y > 1$.

REMARK. The O-constant can be proved to be absolute. But we do not need this fact.

PROOF. The proof is standard. (If $y > 1$ move the line of integration to $\sigma = -\infty$. If $y = 1$ move it to $\sigma = -T$. If $y < 1$ move it to $\sigma = \infty$). Note that we can always move it to $\sigma = \pm T$.

LEMMA 2. *Let $y > 0, c > 0$. Then*

$$\frac{1}{T}\int_{T}^{2T}\left(\frac{1}{2\pi i}\int_{c-i\tau}^{c+i\tau}\frac{y^s}{s}ds - \delta(y)\right)d\tau = O\left(min\left(\frac{y^c}{T^2(log\ y)^2}, y^c\right)\right).$$

PROOF. We have by Lemma 1,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}ds = \delta(y),$$

provided $y \neq 1$. (It is not difficult to uphold this even for $y = 1$). If $y = 1$ consider

$$\frac{1}{2\pi i}\left(\int_{c-i\infty}^{c-i\tau}\frac{ds}{s} + \int_{c+i\tau}^{c+i\infty}\frac{ds}{s}\right)$$

$$= \frac{1}{2\pi i}\left(\int_{-\infty}^{-\tau}\frac{idt}{c+it} + \int_{\tau}^{\infty}\frac{idt}{c+it}\right) = \frac{1}{2\pi}\int_{\tau}^{\infty}\left(\frac{1}{c-it} + \frac{1}{c+it}\right)dt = O(\frac{1}{\tau}).$$

Next put $s_1 = c + i\tau$ and integrating by parts we have

$$I(\tau) \equiv \int_{s_1}^{c+i\infty}\frac{y^s}{s}ds = \frac{y^s}{s\ log\ y}\Big]_{s_1}^{\infty} + \int_{s_1}^{\infty}\frac{y^s}{s^2\ log\ y}ds = -\frac{y^{s_1}}{s_1\ log\ y} + O\left(\frac{y^c}{\tau^2(log\ y)^2}\right).$$

The O-estimate follows by observing that on $\sigma = c, t \geq \tau, \frac{1}{s^2} = \frac{1}{(i\tau)^2} + O(\tau^{-3})$, and using the fact that τ^{-2} is monotonic. Integrating by parts again and using

$$\frac{1}{T}\int_T^{2T} I(\tau)d\tau = \left(\frac{1}{T}\int_T^{\infty}\cdots - \int_{2T}^{\infty}\cdots\right) = O\left(\frac{y^c}{T^2(\log y)^2}\right),$$

the lemma follows.

LEMMA 3. *Let* $c = 1 + (\log x)^{-1}$. *Then*

$$\frac{1}{T}\int_T^{2T}\left(\frac{1}{2\pi i}\int_{c-i\tau}^{c+i\tau}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}ds - \psi_0(x)\right)d\tau = O\left(\frac{\log x}{\log\frac{x}{T}}\cdot\frac{x}{T}\right)$$

provided $T \geq 10, x \geq 10, \frac{x}{T} \geq 10$.

PROOF. By Lemma 2, LHS is

$$\ll \sum_{n=1}^{\infty}\Lambda(n)\min\left\{(\tfrac{x}{n})^c, (\tfrac{x}{n})^c T^{-2}(\log \tfrac{x}{n})^{-2}\right\}$$

$$\ll \sum_{|x-n|\leq h_0}\Lambda(n) + \frac{x^2}{T^2}\sum_{\frac{x}{2}\leq n\leq\frac{3x}{2}, |x-n|\geq h_0}\Lambda(n)(x-n)^{-2}$$

$$+\frac{x}{T^2}\sum_{n\leq\frac{x}{2}}\frac{\Lambda(n)}{n(\log\frac{x}{n})^2} + \frac{x}{T^2}\sum_{n\geq\frac{3x}{2}}\frac{\Lambda(n)}{n(\log\frac{x}{n})^2},$$

(by using $|\log\frac{x}{n}| = -\log(1-(1-\frac{x}{n})) > 1 - \frac{x}{n}$ for $x < n$ and a similar treatment for $x > n$)

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4 \text{ say}.$$

To estimate $\sum_1$ we use Brun's result (see [K.P]) in the form $\pi(x+h_0) - \pi(x) = O\left(\frac{h_0}{\log(h_0+1)}\right)$ where $\pi(x) = \sum_{p\leq x} 1$. To estimate $\sum_2$ we split it up into $h_0 < |x-n| \leq 2h_0, 2h_0 < |x-n| \leq 4h_0 \cdots$ and so on and to estimate each subsum we use Brun's result again. To estimate $\sum_3$ we split it into $\frac{x}{4} < n \leq \frac{x}{2}, \frac{x}{8} < n \leq \frac{x}{4} \cdots$ and so on and in each use $\pi(x) = O(x(\log x)^{-1})$. To estimate $\sum_4$ we adopt a similar procedure. Thus we obtain

$$\sum_1 + \sum_2 + \sum_3 + \sum_4 \ll h_0 \log x(\log h_0)^{-1} + \frac{x^2}{h_0 T^2} \log x(\log h_0)^{-1} + \frac{x}{T^2} + \frac{x}{T^2},$$

choosing $h_0 = \frac{x}{T}$ we obtain the lemma.

LEMMA 4. *There exists a constant* $c_1 > 0$, *such that* $\zeta(s) \neq 0$ *for* $\sigma \geq 1 - c_1(\log T)^{-\frac{2}{3}-\varepsilon}, |t| \leq \frac{5}{2}T$, *and we have* $\frac{\zeta'(s)}{\zeta(s)} = O((\log T)^{\frac{2}{3}+\varepsilon'})$. *Here* $\varepsilon, \varepsilon'$ *are arbitrary positive constants, of which* $\varepsilon' > \varepsilon$.

PROOF. This is a famous result due to I.M. Vinogradov. For example see [K.R]$_1$.

LEMMA 5. *We have, with* $\sigma_0 = 1 - c_1(\log T)^{-\frac{2}{3}-\varepsilon}$, and $10 \leq T < \frac{x}{10}$,

$$\left| \int_{c+i\tau}^{\sigma_0+i\tau} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s} ds \right| = O\left(\frac{x}{T}(\log x)^{-\frac{1}{3}+\varepsilon'}\right)$$

PROOF. The proof follows from Lemma 4.

LEMMA 6. *In Lemma 3 the integrand with respect to integration by* τ *can be replaced by*

$$\frac{1}{2\pi i} \int_{\sigma_0-i\tau}^{\sigma_0+i\tau} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s} ds + O\left(\frac{x}{T}(\log x)^{-\frac{1}{3}+\varepsilon'}\right).$$

Furthermore the error in changing τ *to any real number* $\tau' = \tau + O(1)$ *is* $O(xT^{-1}(\log x)^{-100})$.

PROOF. The first assertion follows by applying Cauchy's theorem of residues and using Lemma 5. The second follows by Lemma 4 and the expression for σ_0.

LEMMA 7. *The integral in Lemma 6 is the same as*

$$\sum_{\beta \geq 0, |\gamma| \leq \tau} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log x)^{-98}\right).$$

PROOF. It is well-known that for every integer $m \geq 10$ there are lines $t = t_m (m \leq t_m \leq m+1)$ on which we have $\zeta'(s)(\zeta(s))^{-1} = O((\log t)^2)$, see [A.E.I]. We use Lemma 6 and Cauchy's theorem of residues as in [A.E.I] and use that $\sum_{\beta<0, |\gamma| \leq \tau'} x^\rho \rho^{-1} = O(1)$. This and the fact that $\zeta'(0)(\zeta(0))^{-1}$ is a constant prove Lemma 7.

PROOF OF THEOREM 3.1. Lemma 1 to 7 complete the proof of Theorem 3.1.

The following theorem (the first part due to I.M. Vinogradov, see [A.A.K, S.M.V], the second part due to A.E. Ingham, see [E.C.T]; the third part due to M.N. Huxley (see [A.I] also [M.N.H]) who improved on a fundamental theorem of H.L. Montgomery, see [H.L.M]) is essential in proving the unconditional Theorems 3.4 and 3.5 to follow.

THEOREM 3.3. *Let* $0 \leq v \leq 1$ *and let* $N(v,T))$ *denote the number of zeros (counted with multiplicity) of* $\zeta(s)$ *in* $\sigma \geq v, 0 \leq t \leq T)$*. Then*

$N(\sigma, T) = 0$ for $\sigma \geq 1 - c_2(\log T)^{-\frac{2}{3}}(\log\log T)^{-\frac{1}{3}}$, ($c_2 > 0$ is a constant),

$$N(\sigma,T) \ll T^{A_1(\sigma)(1-\sigma)}(\log T)^5, \tfrac{1}{2} \le \sigma \le 1,$$

$$N(\sigma,T) \ll T^{A_2(\sigma)(1-\sigma)}(\log T)^9, \tfrac{3}{4} \le \sigma \le 1,$$

where $A_1(\sigma) = 3(2-\sigma)^{-1}$ *and* $A_2(\sigma) = (5\sigma - 3)(\sigma^2 + \sigma - 1)^{-1}$. *Also* $N(0, T+1) - N(0,T) = O(\log T)$, *and*

$$N(\sigma,T) \ll T^{2(1-\sigma)}\log T (0 \le \sigma \le \frac{1}{2}).$$

NOTE. In the second and the third assertions of Theorem 3.3 we have replaced the *log* factors in the range $|\sigma - \frac{3}{4}| \le \frac{D \log\log T}{\log T}$ (D any positive constant) by $(\log T)^{\frac{19}{5}}$ and $(\log T)^8(\log\log T)^{\frac{4}{5}}$ respectively. See § A2 and § A3 in the appendix. These improvements are essential for proving Theorems 3.4 and 3.5. As stated in the introduction these results can be improved further (see the Appendix and the Post-script).

REMARK. Note that $A_1(\sigma)$ is increasing and $A_2(\sigma)$ is decreasing and also that $A_1(\frac{3}{4}) = A_2(\frac{3}{4}) = \frac{12}{5}$.

THEOREM 3.4. *Let* $h = x^{\frac{7}{12}}(\log x)^B$ *where* $B(> 8\frac{23}{24})$ *is a constant. Then*

$$\vartheta(x+h) - \vartheta(x) = h + O(h(\log x)^{-\varepsilon})$$

where $\varepsilon(>0)$ *is a constant depending on* B.

REMARK 1. Note that $\psi(x) = \sum_{m\ge1} \sum_{p^m \le x} \log p, \vartheta(x) = \sum_{p \le x} \log p$, and that for $1 \le h \le x$, we have

$$\sum_{m\ge2} \sum_{x \le p^m \le x+h} \log p \ll \sum_{m\ge2} \frac{\log x}{m}\left(\left(\tfrac{x+h}{m}\right)^{\frac{1}{m}} - \left(\tfrac{x}{m}\right)^{\frac{1}{m}} + 1\right)$$

$$\ll \log x \sum_{m\geq 2} \frac{1}{m}\left(\frac{h}{m}\left(\frac{x}{m}\right)^{\frac{1}{m}-1}+1\right) \ll \log x \log\log x + hx^{-\frac{1}{2}}.$$

REMARK 2. Actually our proof gives a better result with $(\log x)^B$ replaced by $(\log x)^{8\frac{23}{24}}$ times a certain "small function of x" which goes to infinity with x.

REMARK 3. See sections A2 and A3 of the appendix at the end for improvements of Theorem 3.3.

PROOF. From Theorem 3.1, we have

$$\psi(x+h)-\psi(x) = h + O\left(\int_0^h \sum_{\beta\geq 0, |\gamma|\leq 2T} (x+v)^{\beta-1} dv\right) + O\left(\frac{\log x}{\log \frac{x}{T}}\cdot\frac{x}{T}\right).$$

We put $T = x^{\frac{5}{12}}(\log x)^{-B+\epsilon}$ and we find that the second O-term is $O(h(\log x)^{-\epsilon}$

Now uniformly in $0 \leq v \leq h$, we have

$$\sum_{\beta\geq 0, |\gamma|\leq 2T} (x+v)^{\beta-1} \ll \sum_{\beta\geq 0, |\gamma|\leq 2T} x^{\beta-1} = -\int_0^1 x^{\sigma-1} dN(\sigma, 2T)$$

$$= -N(\sigma,2T)x^{\sigma-1}\Big]_{\sigma=0}^{\sigma=1} + \log x \int_0^1 N(\sigma,2T)x^{\sigma-1}d\sigma$$

$$\ll x^{-1}T \log T + O(\log x(M_1 + M_2 + M_3(\log\log x)(\log x)^{-1}))$$

where

$$M_1 = \max_{0\leq\sigma\leq\frac{3}{4}-\frac{D\log\log T}{\log T}} (N(\sigma,2T)x^{\sigma-1}),$$

$$M_2 = \max_{\frac{3}{4}+\frac{D\log\log T}{\log T}\leq\sigma\leq 1} (N(\sigma,2T)x^{\sigma-1})$$

and

$$M_3 = \max_{|\sigma-\frac{3}{4}|\leq\frac{D\log\log T}{\log T}} (N(\sigma,2T)x^{\sigma-1}).$$

Here $D > 0$ is a large constant. By choosing D suitably we can get $M_1 + M_2 = O(h(\log x)^{-1000})$, a result easy to verify. All that remains to be proved is $M_3 = O(h(\log x)^{-\varepsilon})$. This and another estimate for the quantity M_3' (which occurs in the proof of Theorem 3.5)) will be established at the end of proof of Theorem 3.5.

THEOREM 3.5. *Let $H = X^{\frac{1}{6}}(\log X)^{B'}$ where $B'(> 0)$ is a constant. If $B' > 10\frac{5}{8}$ then*

$$\frac{1}{X}\int_X^{2X}(\vartheta(x+H)-\vartheta(x)-H)^2dx \ll H^2(\log x)^{-\varepsilon}$$

where $\varepsilon(> 0)$ is a constant depending only on B'. Also if $B' > 12\frac{19}{24}$ we have

$$\frac{1}{X}\int_X^{2X}(\vartheta(x+H)-\vartheta(x)-H)^2dx \ll H^2(\log X)^{-1-\varepsilon}$$

where $\varepsilon(> 0)$ is a constant depending only on B'.

PROOF. In view of Remark 1 below Theorem 3.4 it suffices to prove the theorem with ψ in place of ϑ. By Theorem 3.1, there holds (uniformly in $\frac{X}{2} \le x \le \frac{5X}{2}$) the inequality (hereafter we suppress the condition $\beta \ge 0$ in the sum over ρ)

$$|\,\psi(x+H)-\psi(x)-H\,| \ll \frac{1}{T}\int_T^{2T}\Big|\sum_{|\gamma|\le\tau}\frac{(x+H)^\rho - x^\rho}{\rho}\Big|\,d\tau + \frac{X}{T}\frac{\log X}{\log\frac{X}{T}}$$

and so

$$|\,\psi(x+H)-\psi(x)-H\,|^2 \ll \frac{1}{T}\int_T^{2T}\Big|\sum_{|\gamma|\le\tau}\frac{(x+H)^\rho - x^\rho}{\rho}\Big|^2 d\tau + \left(\frac{X}{T}\frac{\log X}{\log\frac{X}{T}}\right)^2$$

Note that

$$|\sum_{|\gamma|\leq\tau}\cdots|^2\leq 9(|\Sigma_1|^2+|\Sigma_2|^2+|\Sigma_3|^2)$$

where Σ_1, Σ_2 and Σ_3 are the sums restricted to

$$\beta\leq\frac{3}{4}-\frac{D\ loglog\ T}{log\ T}, \beta\geq\frac{3}{4}+\frac{D\ loglog\ T}{log\ T} \text{ and } |\beta-\frac{3}{4}|\leq\frac{D\ loglog\ T}{log\ T}$$

respectively. We treat the integral

$$\frac{1}{T}\int_T^{2T}|\Sigma_3|^2\,d\tau$$

and the other two mean values involving $|\Sigma_1|^2$ and $|\Sigma_2|^2$ can be treated similarly. In fact the last two mean values together make up only $O(H^2(log\ X)^{-1000})$ as can be verified by following the method of treatment of the one involving $|\Sigma_3|^2$. For this it will be advantageous to use the upper bound (note that the integrand is non-negative),

$$\frac{1}{X}\int_X^{2X}\cdots\leq\frac{2}{X}\int_0^{\frac{X}{2}}\left(\frac{1}{X}\int_{X-f}^{2X+f}\frac{1}{T}\int_T^{2T}|\sum_3|^2\,d\tau dx\right)df=J \text{ say.}$$

This method of averaging has been first used by K. Ramachandra in [K.R]$_2$. Put $T=X^{\frac{5}{6}}(log\ X)^{-B''}$ where $B''<B'$ is a constant. We now split up Σ_3 into $O(loglog\ T)$ abutting β-intervals $I=I(\sigma)$ of length $(log\ T)^{-1}, \sigma$ denoting the left hand end point. Thus

$$|\Sigma_3|^2\ll(loglog\ T)\sum_I|\sum_{\beta\in I}\frac{(x+H)^\rho-x^\rho}{\rho}|^2$$

$$\ll(loglog\ T)H\sum_I\int_0^H|\sum_{\beta\in I}(x+v)^{\rho-1}|^2\,dv.$$

Again (remember that $I=I(\sigma)$)

$$\frac{1}{X}\int_{X-f}^{2X+f} \mid \sum_{\beta\in I}(x+v)^{\rho-1} \mid^2 dx$$

$$= \frac{1}{X}\sum_{\rho_1\in I}\sum_{\rho_2\in I}\frac{(2X+f)^{\rho_1+\bar{\rho}_2-1}-(X-f)^{\rho_1+\bar{\rho}_2-1}}{\rho_1+\bar{\rho}_2-1}.$$

Note that the innermost summand here is $O(X^{2\sigma-1})$. We use this bound for all pairs (ρ_1,ρ_2) of zeros (which figure) which satisfy $\mid \rho_1+\bar{\rho}_2-1 \mid\leq 10$ and we find the total contribution from these pairs to be $O(X^{2\sigma-2}N(\sigma,2T)log\ T)$. For zeros with $\mid \rho_1+\bar{\rho}_2-1 \mid> 10$ we use the average with respect to the additional parameter f and obtain the bound

$$\sum_{|\rho_1+\bar{\rho}_2-1|>10} X^{2\sigma-2} \mid \rho_1+\bar{\rho}_2-1 \mid^{-2} \ll X^{2\sigma-2}N(\sigma,2T)log\ T.$$

With these explanations we see that

$$\frac{1}{X}\int_X^{2X}(\psi(x+H)-\psi(x)-H)^2dx \ll \frac{X^2}{T^2}+H^2(log\ T)(loglog\ T)^2M_3'$$

where $M_3' = max(N(\sigma,2T)X^{2\sigma-2})$, the maximum being taken over $\mid \sigma-\frac{3}{4} \mid\leq 100D(loglog\ X)(log\ X)^{-1}$. Note that $T = X^{\frac{5}{6}}(log\ X)^{-B''}$ and so $\frac{X^2}{T^2} = X^{\frac{1}{3}}(log\ X)^{2B''}, (B''<B')$. We will now prove that $M_3 = O((log\ x)^{-\varepsilon})$, if $B > 8\frac{23}{24}$, as promised already and also $M_3' = O((log\ X)^{-1-\varepsilon})$ if $B' > 10\frac{5}{8}$ and further $M_3' = O((log\ X)^{-2-\varepsilon}$ if $B' > 12\frac{19}{24}$. Here ε is a certain fixed positive constant depending only on B'.

We begin with the study of M_3. We have by Theorem 3.3, (taken with A.2 and A.3 of appendix)

$$M_3 \ll M_3^{(1)}(log\ x)^{\frac{19}{5}} + M_3^{(2)}(log\ x)^8(loglog\ x)^{\frac{4}{5}}$$

where $M_3^{(1)}$ is the maximum of $(T^{A_1(\sigma)}x^{-1})^{1-\sigma}$ in $\frac{3}{4}-\frac{100D\,loglog\,x}{log\,x}\leq\sigma\leq\frac{3}{4}+\frac{D_1 loglog\,x}{log\,x}$ and $M_3^{(2)}$ is the maximum of $(T^{A_2(\sigma)}x^{-1})^{1-\sigma}$ in $\frac{3}{4}+\frac{D_1\,loglog\,x}{log\,x}\leq\sigma\leq\frac{3}{4}+\frac{D\,loglog\,x}{log\,x}$, where $D_1>0$ is a constant. We choose D_1 such that $M_3^{(1)}$ and $M_3^{(2)}$ have nearly the same bound. Clearly (note that we have chosen $T=x^{\frac{5}{12}}(log\,x)^{-B+\varepsilon}$) we have

$$M_3^{(1)}(log\,x)^{\frac{19}{5}-\frac{3}{5}(B-\varepsilon)}\ll\left(\max_{\dots}\left(x^{\frac{5}{12}A_1(\sigma)-1}\right)^{1-\sigma}\right)(log\,x)^{\frac{19}{5}-\frac{3}{5}(B-\varepsilon)}$$

and

$$M_3^{(2)}(log\,x)^{8-\frac{3}{5}(B-\varepsilon)}(loglog\,x)^{\frac{4}{5}}$$
$$\ll\left(\max\left(x^{\frac{5}{12}A_2(\sigma)-1}\right)^{1-\sigma}\right)(log\,x)^{8-\frac{3}{5}(B-\varepsilon)}(loglog\,x)^{\frac{4}{5}}.$$

Write $\sigma=\frac{3}{4}+\lambda$ where $|\lambda|\leq\frac{100\,D\,loglog\,x}{log\,x}$. (Note that $log\,T\ll log\,x\ll log\,T$). We have

$$(\frac{5}{12}A_1(\sigma)\ 1)(1-\sigma)=(\frac{5}{4(2-\sigma)}-1)(1-\sigma)=(\frac{5}{5-4\lambda}-1)(\frac{1}{4}-\lambda)=\frac{\lambda}{5}+O(\lambda^2).$$

Hence the exponent of x is an increasing function of λ and so

$$M_3^{(1)}(log\,x)^{\frac{19}{5}-\frac{3}{5}(B-\varepsilon)}\ll x^{\frac{\lambda_1}{5}}(log\,x)^{\frac{19}{5}-\frac{3}{5}(B-\varepsilon)}=(log\,x)^{\frac{D_1}{5}+\frac{19}{5}-\frac{3}{5}(B-\varepsilon)}$$

where $\lambda_1=\frac{D_1 loglog\,x}{log\,x}$. Also

$$\left(\tfrac{5}{12}A_2(\sigma)-1\right)(1-\sigma)=\left(\tfrac{5}{12}\cdot\tfrac{5\sigma-3}{\sigma^2+\sigma-1}-1\right)(1-\sigma)$$
$$=\left(\tfrac{5}{12}\cdot\tfrac{5(\frac{3}{4}+\lambda)-3}{(\frac{3}{4}+\lambda)^2+(\frac{3}{4}+\lambda)-1}-1\right)(\tfrac{1}{4}-\lambda)$$
$$=\left\{\tfrac{5}{12}(5\lambda+\tfrac{3}{4})(\tfrac{9}{16}+\tfrac{3\lambda}{2}+\lambda^2+\tfrac{3}{4}+\lambda-1)^{-1}-1\right\}(\tfrac{1}{4}-\lambda)$$

$$= \left\{\tfrac{5}{12}(5\lambda + \tfrac{3}{4}) - \tfrac{9}{16} - \tfrac{3\lambda}{2} - \lambda^2 - \tfrac{3}{4} - \lambda + 1\right\}\left\{\tfrac{9}{16} + \tfrac{3}{4} - 1 + \tfrac{5\lambda}{2} + \lambda^2\right\}^{-1}$$

$$\times(\tfrac{1}{4} - \lambda)$$

$$= \lambda\left\{\tfrac{25}{12} - \tfrac{18}{12} - \tfrac{12}{12} - \lambda\right\}\left\{\tfrac{5}{16} + \tfrac{5\lambda}{2} + \lambda^2\right\}^{-1}(\tfrac{1}{4} - \lambda)$$

$$= \lambda\left\{-\tfrac{5}{12} - \lambda\right\}\left\{\tfrac{16}{5}\right\}\left\{1 + 8\lambda + \tfrac{16\lambda^2}{5}\right\}^{-1}(\tfrac{1}{4} - \lambda)$$

$$= \lambda\{-\tfrac{5}{12} - \lambda\}(\tfrac{16}{5})(1 - 8\lambda + O(\lambda^2))(\tfrac{1}{4} - \lambda)$$

$$= -\tfrac{\lambda}{3} + O(\lambda^2).$$

Hence the exponent of x is a decreasing function of λ and so

$$M_3^{(2)}(\log x)^{8-\frac{3}{5}(B-\varepsilon)}(\log\log x)^{\frac{4}{5}} \ll x^{-\frac{\lambda_1}{3}}(\log x)^{8-\frac{3}{5}(B-\varepsilon)}(\log\log x)^{\frac{4}{5}}$$

$$= (\log x)^{-\frac{D_1}{3}+8-\frac{3}{5}(B-\varepsilon)}(\log\log x)^{\frac{4}{5}}.$$

Choose D_1 such that $\frac{D_1}{3} + \frac{D_1}{5} = 8 - \frac{19}{5} = \frac{21}{5}$ i.e. $\frac{D_1}{5} = \frac{63}{40}$. Hence

$$M_3^{(1)}(\cdots) + M_3^{(2)}(\cdots) \ll (\log x)^{\frac{63}{40}+\frac{19}{5}-\frac{3}{5}(B-\varepsilon)}.$$

Now $\frac{63}{40} + \frac{19}{5} - \frac{3}{5}(B - \varepsilon) < 0$ for some $\varepsilon > 0$ if $B > 8\frac{23}{24}$. (This completes the proof of Theorem 3.4). In a similar manner we recall that $T = X^{\frac{5}{6}}(\log X)^{-B''}$. So we have

$$M_3' \ll M_3'^{(1)}(\log X)^{\frac{19}{5}-\frac{3}{5}B''} + M_3'^{(2)}(\log X)^{8-\frac{3}{5}B''}(\log\log X)^{\frac{4}{5}}$$

where $M_3'^{(1)}$ = maximum of $\left(X^{\frac{5}{6}A_1(\sigma)-2}\right)^{1-\sigma}$ taken over $\frac{3}{4} - \frac{100D\log\log X}{\log X} \leq \sigma \leq \frac{3}{4} + \frac{D_1\log\log X}{\log X}$ and $M_3'^{(2)}$ = maximum of $\left(X^{\frac{5}{6}A_2(\sigma)-2}\right)^{1-\sigma}$ taken over

$\frac{3}{4} + \frac{D_1 loglog\, X}{log X} \leq \sigma \leq \frac{3}{4} + \frac{100D\, loglog\, X}{log\, X}$ (The constant D_1 appearing in the inequality just mentioned should not be confused with the earlier one). Hence by the previous calculations involving λ, we have

$$M_3' \ll (X^{2\frac{\lambda_1}{5}}(log\, X)^{\frac{19}{5}} + X^{-2\frac{\lambda_1}{3}}(log\, X)^8)(log\, X)^{-\frac{3}{5}B''}(loglog\, X)^{\frac{4}{5}}$$

$$\left(\text{where } \lambda_1 = \frac{D_1 loglog\, X}{log\, X}\right)$$

$$\ll (log\, X)^{\frac{2D_1}{5}+\frac{19}{5}-\frac{3}{5}B''}(loglog\, X)^{\frac{4}{5}}$$

provided we choose D_1 such that $\frac{2D_1}{5} + \frac{19}{5} = -\frac{2D_1}{3} + 8$ i.e. $\frac{2D_1}{5}(1+\frac{5}{3}) = \frac{21}{5}$ i.e. $\frac{2D_1}{5} = \frac{63}{40}$. Thus $M_3' \ll (log\, X)^{\frac{63}{40}+\frac{19}{5}-\frac{3}{5}B''}$ ($\ll (log\, X)^{-1-\varepsilon}$ provided $B' > B'' > 10\frac{5}{8}$), and this proves the first part of Theorem 3.5). Also for the second part of Theorem 3.5 (we need $B'' < 2B' - 1 - \varepsilon$) and

$$M_3' \ll (log\, X)^{\frac{63}{40}+\frac{19}{5}-\frac{3}{5}B''}\left(\ll (log\, X)^{-2-\varepsilon} \text{ provided } B' > B'' + \frac{1}{2} > 12\frac{19}{24}\right).$$

This proves Theorem 3.5 completely.

COROLLARIES TO THEOREMS 3.2, 3.4 AND 3.5. Let $\alpha > 0$ and $\beta > 0$ be any two constants. Then

(i) *For every prime p there exists a prime q* such that $0 < \alpha p - \beta q < p^{\frac{7}{12}}(log\, p)^{8\frac{23}{24}+\varepsilon}$.

(ii) *There are infinitely many pairs (p,q) of primes p,q such that* $0 < \alpha p - \beta q < p^{\frac{1}{6}}(log\, p)^{12\frac{19}{24}+\varepsilon}$.

(iii) *On Riemann hypothesis (R.H) there are infinitely many pairs* (p, q) *of primes* p, q *such that* $0 < \alpha p - \beta q < (\log p)^{100}$.

REMARK 1. (R.H) implies Lindelöf hypothesis (L.H. which states that for every fixed $\varepsilon > 0$ we have $t^{-\varepsilon}\zeta(\frac{1}{2} + it) \to 0$ as $t \to \infty$) gives p^{ε} in place of $(\log p)^{100}$. (100 can certainly be improved).

REMARK 2. We leave the deduction of these corollaries to the reader as an exercise. These corollaries are not the best known. For latest results see a forthcoming paper announced in § A.1 of the appendix.

REMARK 3. If we want the same results as precise as (iii) they are available unconditionally, but they prove the existence of some α, β out of some sets of pairs. Hence they can be considered not the main part of this paper. For this reason only one such result will be briefly mentioned in § A.1 of the appendix at the end of this paper.

§ 4. **A REMARK THAT MONTGOMERY MISSED.** In this section we sketch (using ideas of H.L. Montgomery [H.L.M]) a short and simple proof that $\sum_{n \le x} \mu(n) = O(x\ exp(-c(\log x)^{a}))$ (where $c(> 0)$ and a $(0 < a < 1)$ are constants) implies $(1-\beta)^{-1} \ll (\log \gamma_1)^{\frac{1}{a}-1}$ (with $\gamma_1 = \gamma + 100$) for any zero $\rho = \beta + i\gamma(\gamma \ge 0, 1 \ge \beta \ge \frac{99}{100})$ of $\zeta(s)$. Put $M_X(s) = \sum_{n \le X} \mu(n)n^{-s}$, $F(s) = \zeta(s)M_X(s)$. We have $F(s) = 1 + \sum_{n \ge X} a_n n^{-s}$ for $Re\ s > 1$, where $| a_n | \le d(n)$, $d(n)$ being the number of divisors (≥ 1) of n. We first prove a lemma.

LEMMA. *We have (under the hypothesis made on $M_X(s)$) the estimate*

$$|M_X(\frac{3}{4}+it)| \ll (|t|+10)X^{\frac{1}{4}} exp(-\frac{1}{2}c(log\ X)^a).$$

PROOF. We have with $s = \frac{3}{4} + it$ and $M(u) = M_u(0)$,

$$M_X(s) = \int_{1-0}^{X+0} u^{-s} dM(u) = \frac{M(u)}{u^s}\Big]_{1-0}^{X+0} + s\int_1^X \frac{M(u)}{u^{s+1}}du$$

$$= O(X^{\frac{1}{4}} exp(-c(log\ X)^a) + O((|t|+10)\int_1^X u^{\frac{1}{4}} exp(-c(log\ u)^a)\frac{du}{u})$$

$$= O(X^{\frac{1}{4}}(|t|+10) exp(-c(log\ X)^a) log\ X)$$

since $\max_{1\le u\le X}(u^{\frac{1}{4}} exp(-c(log\ u)^a)) = O(X^{\frac{1}{4}} exp(-c(log\ X)^a))$. Thus the lemma is proved.

We next write with $X \ge 10, w = u + iv, Y = \frac{X}{(log X)^2}$, the identity

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(\rho+w)Y^w\Gamma(w)dw = exp(-\frac{1}{Y}) + \sum_{n\ge X} a_n n^{-\rho} e^{-\frac{n}{Y}}$$

where clearly RHS is $1+o(1)$ uniformly in ρ. In the integral in LHS we move the line of integration to u given by $\beta + u = \frac{3}{4}$. Since our choice of X will be subject to $X \le exp((log(|\gamma|+10))^{a^{-1}}D)$ (where $D(>0)$ is a constant) and $|\Gamma(1-\rho)| \ll e^{-|\gamma|}$ the contribution from the pole of $\zeta(s+w)$ at $w = 1-\rho$ is $o(1)$. The integral has no pole at $w = 0$ since $\zeta(\rho) = 0$. The integral on the line $u = \frac{3}{4} - \beta$ is (by the lemma above)

$$\ll \int_{-\infty}^{\infty}(|\gamma+v|+10)^2\left(\frac{X}{(log\ X)^2}\right)^{\frac{3}{4}-\beta} X^{\frac{1}{4}} exp(-\frac{1}{2}c(log\ X)^a)\times$$

$$\times\ exp(-|v|)dv$$

$$\ll \gamma_1^2 X^{1-\beta} exp(-\tfrac{1}{4}c(log\ X)^a)$$

$$\ll \gamma_1^2 X^{\delta(log\ \gamma_1)^{1-\frac{1}{a}}} exp(-\tfrac{1}{4}c(log\ X)^a) \text{ if } 1-\beta \le \delta(log\ \gamma_1)^{1-\frac{1}{a}}.$$

We put $X = exp((log\ \gamma_1)^{\frac{1}{a}} D)$ and obtain finally

$$\begin{aligned} 1+o(1) &\ll \gamma_1^2\ exp(\delta D\ log\ \gamma_1) exp(-\tfrac{1}{4}cD^a log\ \gamma_1) \\ &= \gamma_1^{2+\delta D-\frac{1}{4}cD^a} = o(1) \text{ if } \delta = \tfrac{1}{D} \text{ and } D \text{ is large.} \end{aligned}$$

This contradiction proves the required result.

§ 5. THE PROOF THAT $(2-\delta)(B)$ HYPOTHESIS IMPLIES $L(1,\chi) \gg (log\ k)^{-1}$.

We begin with a lemma.

LEMMA. *The $2-\delta$ hypothesis* (B) *implies that*

$$\sum_{X \le p \le 2X} (1+\chi(p)) \gg \delta \sum_{X \le p \le 2X} 1 \gg \delta X (log\ X)^{-1}$$

for any real character $\chi(mod\ k)$.

PROOF. It suffices to prove that if $1 \le \ell_1 < \ell_2 < \cdots < \ell_r \le k$ are $r = \frac{1}{2}\varphi(k)$ residue classes *mod k* coprime to k, then

$$\sum_{j=1}^{r} \sum_{X \le p \le 2X, p \equiv \ell_j (mod\ k)} 1 \le \left(1 - \frac{\delta}{100}\right) \sum_{X \le p \le 2X} 1.$$

LHS is $\le \sqrt{r} \cdot \sqrt{\frac{2-\delta}{\varphi(k)}} \sum_{X \le p \le 2X} 1$ by Hölder's inequality. We note that $r = \frac{1}{2}\varphi(k)$ and so $\left(r\ \frac{2-\delta}{\varphi(k)}\right)^{\frac{1}{2}} = (1-\frac{\delta}{2})^{\frac{1}{2}} \le 1 - \frac{\delta}{100}$. Hence we apply this observation to the $\frac{1}{2}\varphi(k)$ residue classes ℓ_j for which $\chi(\ell_j) = -1$. It follows that the total number of primes p with $X \le p \le 2X$ for which $\chi(p) = 1$ is $\gg \delta \sum_{X \le p < 2X} 1$. This proves the lemma.

Next we define $G(s) = \zeta(s)L(s,\chi) = \sum_{n=1}^{\infty} a_n n^{-s}$ and verify that $a_n \geq 0$ and $a_p = 1 + \chi(p) \geq 1$ for all primes p with $\chi(p) = 0$ or 1. Thus

$$\sum_{X \leq n \leq 2X} a_n \geq \sum_{X \leq p \leq 2X} a_p \gg \frac{\delta X}{\log X} \text{ when } X = k^{c(\delta)} \text{ where } c(\delta)(>0)$$

is a constant which depends only on δ. Hence

$$\frac{1}{2\pi i}\int_{u=2} G(1+w)((2X)^w - X^w)\Gamma(w)dw = \sum_{n=1}^{\infty} n^{-1}a_n\left(e^{-\frac{n}{2X}} - e^{-\frac{n}{X}}\right)$$

$$\geq \sum_{X \leq p \leq 2X} p^{-1}a_p\left(e^{-\frac{p}{2X}} - e^{-\frac{p}{X}}\right) \gg \frac{\delta}{\log X} \gg (\log k)^{-1}.$$

But in the integral involving $G(w)$ we move the line of integration to $u = -\frac{1}{4}$. Note the estimates $|\zeta(\frac{3}{4}+it)| \leq |t| + 10$ and $|L(\frac{3}{4}+it)| \leq 100k(|t|+10)$ and that the pole at $w = 0$ of $\zeta(1+w)$ contributes the residue $L(1,\chi)\log 2$. This completes the proof that

$$\frac{1}{\log k} \ll \frac{1}{\log X} \ll L(1,\chi)+$$

$$+\int_{-\infty}^{\infty} |G(\frac{3}{4}+iv)||(2^{-\frac{1}{4}+iv}-1)|X^{-\frac{1}{4}}|\Gamma(-\frac{1}{4}+iv)|dv = L(1,\chi)+O(\frac{1}{(\log k)^2})$$

by choosing a large $c(\delta)$ say required by the $2-\delta$ hypothesis. We have used $|\Gamma(-\frac{1}{4}+iv)| \ll exp(-|v|)$. (Note that a result of the type $(2-\delta)$ hypothesis is welcome even with $\delta = k^{-\frac{1}{4}}$ (provided all the constants are explicit), or a result with a constant $\delta > 0$ and $X = exp(k^{\frac{1}{4}})$. This would imply an effective result $L(1,\chi) \gg k^{-\frac{1}{4}}(\log k)^{-1}$).

REMARK. In connection with this section we may note the result (due to C.L. Siegel [C.L.S]) that $L(1,\chi) > C(\varepsilon)k^{-\varepsilon}$ holds for every $\varepsilon > 0$ and a suitable constant $C(\varepsilon)(> 0)$, depending only on ε. But unfortunately when $\varepsilon < \frac{1}{2}, C(\varepsilon)$ is ineffective (i.e. it cannot be calculated). See also [K.R]$_3$ for a simple proof of Siegel's theorem.

APPENDIX

§ **A.1** The following result (due to K. Ramachandra) is Theorem 1 of his paper [KR]$_4$.

Let ε be any positive constant < 1 and N any natural number $> 2\varepsilon^{-1}$. Let $\alpha_1, \cdots, \alpha_N$ be any given positive real numbers no two of which are equal. Then there exist two of the numbers α_j say β and γ, such that the inequality

$$0 < | \beta p - \gamma q | < p^{\varepsilon}$$

holds for infinitely many prime pairs (p, q). The proof is based on Selberg Sieve. These and other improvements will form the subject matter of a forthcoming paper (problems and results on $\alpha p - \beta q$) by us where the latest results on this subject and also the history of the subject are contained.

§ **A.2** The object of this section is to prove the following theorem.

THEOREM. *Let* $\frac{1}{2} + \delta \leq \sigma \leq 1 - \delta$, *where* $\delta > 0$ *is any small constant. Then*

$$N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^{\frac{4+\sigma}{2-\sigma}}.$$

REMARK. By a more complicated proof we may cover the range $\frac{1}{2} \leq \sigma \leq 1$.

COROLLARY. *In the neighbourhood* $| \sigma - \frac{3}{4} | \leq \frac{D \log\log T}{\log T}$ *of* $\frac{3}{4}$, *we can take the power of* $\log T$ *to be* $\frac{4+\sigma}{2-\sigma}(\sigma = \frac{3}{4}) = \frac{19}{5}$.

We now sketch the proof of the theorem just mentioned.

PROOF. Let $\rho = \beta + i\gamma$ be any zero of $\zeta(s)$ where $\frac{1}{2} + \delta \leq \beta \leq 1 - \delta, T \leq \gamma \leq 2T$. Put $F(s) = \zeta(s)M_T(s) - 1$, where $M_T(s) = \sum_{n \leq T}(\mu(n)n^{-s})$. It is not hard to prove that

$$\int_{\frac{1}{2}T}^{3T} | F(\frac{1}{2} + it) |^{\frac{4}{3}} dt \ll T(log\ T)^2 \text{ and } \int_{\frac{1}{2}T}^{3T} | F(1 + it) |^2 dt \ll (log\ T)^4.$$

We apply Cauchy's theorem and obtain (with $w = u + iv$)

$$1 = | F(\rho) | \leq | \frac{1}{2\pi i} \int F(\rho + w) e^{w^2} X^w \frac{dw}{w} |$$

where the contour is the rectangle bounded by the lines $u = \frac{1}{2} - \beta, u = 1 - \beta, v = -\frac{1}{2}T, v = T$, provided X lies between two constant powers of T. We may ignore the horizontal line contributions and we obtain

$$1 \ll X^{\frac{1}{2}-\sigma} \int_{u=\frac{1}{2}-\beta} | F(\rho + w) e^{w^2} dw | + X^{1-\sigma} \int_{u=1-\beta} | F(\rho + w) e^{w^2} dw |$$

where we have assumed that β belongs to the interval $\left(\sigma - \frac{1}{log\ T}, \sigma + \frac{1}{log\ T}\right)$. Thus by Hölder's inequality we have

$$\begin{aligned} 1 &\ll X^{\frac{1}{2}-\sigma} \left(\int_{u=\frac{1}{2}-\beta} | F(\rho + w) |^{\frac{4}{3}} | e^{w^2} || dw |\right)^{\frac{3}{4}} + \\ &\quad + X^{1-\sigma} \left(\int_{u=1-\beta} | F(\rho + w) |^2 | e^{w^2} || dw |\right)^{\frac{1}{2}} \\ &\ll X^{\frac{1}{2}-\sigma} I_{\frac{1}{2}}^{\frac{3}{4}} + X^{1-\sigma} I_1^{\frac{1}{2}} \end{aligned}$$

(where $I_{\frac{1}{2}}$ and I_1 here have the obvious meaning),

$$\ll \left(I_{\frac{1}{2}}^{\frac{3}{4}(1-\sigma)} \ I_1^{\frac{1}{2}(\sigma-\frac{1}{2})} \right)^2$$

(by a suitable choice of X, the conditions on X being statisfied if we increase $I_{\frac{1}{2}}$ and I_1 by T^{-1000}). Note that

$$\sum_\rho I_{\frac{1}{2}} \ll \sum_\rho \int |F(\frac{1}{2}+iv')|^{\frac{4}{3}} e^{-(\gamma-v')^2}dv' \ll T(\log T)^3,$$

(by a change of variable). Similarly $\sum_\rho I_1 \ll (\log T)^5$.

We fix V_1 and $V_{\frac{1}{2}}$ by the condition that $V_{\frac{1}{2}}^{\frac{3}{2}(1-\sigma)} \times V_1^{\sigma-\frac{1}{2}}$ equals a small constant. Also

$$\sum\nolimits_1 \equiv \sum_{\rho, I_{\frac{1}{2}} \geq V_{\frac{1}{2}}} 1 \ll T(\log T)^3 V_{\frac{1}{2}}^{-1} \text{ and } \sum\nolimits_2 \equiv \sum_{\rho, I_1 \geq V_1} 1 \ll (\log T)^5 V_1^{-1}$$

Thus the number of zeros in question is

$$\ll (\log T)^3 \left(\frac{T}{V_{\frac{1}{2}}} + \frac{(\log T)^2}{V_1}\right) \ll (\log T)^3 \left(\frac{T}{V_{\frac{1}{2}}} + (\log T)^2 V_{\frac{1}{2}}^{\frac{3-3\sigma}{2\sigma-1}}\right)$$

$$\ll (\log T)^3 \left(\frac{T}{V^{2\sigma-1}} + (\log T)^2 V^{3-3\sigma}\right)$$

(by a change of notation)

$$\ll (\log T)^3 \left(T^{3-3\sigma}(\log T)^{2(2\sigma-1)}\right)^{(2-\sigma)^{-1}}$$

(by a proper choice of V)

$$\ll (\log T)^3 T^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^{\frac{4\sigma-2}{2-\sigma}} \ll T^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^{\frac{4+\sigma}{2-\sigma}}.$$

(This is the bound for the number of zeros with $\left(\sigma - \frac{1}{\log T} \leq \beta \leq \sigma + \frac{1}{\log T}\right.$, $T \leq \gamma \leq 2T$). Here replacing σ by $\sigma + \frac{1}{\log T}, \sigma + \frac{2}{\log T}, \sigma + \frac{3}{\log T}, \cdots$ (The greatest not exceeding $1 - \frac{1}{2}\delta$ we can use the Ingham bound for $\sigma \geq 1 - \frac{1}{2}\delta$) and adding we obtain

$$N(\sigma, 2T) - N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^{\frac{4+\sigma}{2-\sigma}},$$

and so the same bound holds for $N(\sigma, 2T)$. Replacing T by $\frac{1}{2}T$ we obtain the theorem.

§ A.3. It is the object of the section to prove the following theorem.

THEOREM. *For $\frac{3}{4} \leq \sigma \leq 1$, we have, for $T \geq 20$,*

$$N(\sigma, T) \ll T^{\frac{(5\sigma-3)}{(\sigma^2+\sigma-1)}(1-\sigma)}(log\ T)^8(loglog\ T)^{\frac{4}{5}}.$$

We first state a fundamental result of H.L. Montgomery (Theorem 8.4 with $\theta = 0$ of [H.L.M] taken with the last result of appendix II therein). We state it as Lemma 1. We recall the usual terminology. Let S^* be a set of complex numbers. (We denote the number of elements of S^* by $|S^*|$). It is said to be well-spaced if the differences of the imaginary parts of any two of them is ≥ 1 in absolute value.

LEMMA 1. (H.L. MONTGOMERY). *Let $N \geq 1$ be an integer and $S(s) = \sum_{n=N}^{2N} a_n n^{-s}$ where $a_N, \cdots, a_{2N}$ are any complex numbers. Put*

$$M_0(T) = \max_{|t| \leq T} \int_{-\infty}^{\infty} |\zeta(it+iv)| e^{-|v|} dv.$$

Suppose S^* is a finite well spaced set of complex numbers each of which has real part $\geq \sigma_0$. Then

$$\sum_{s\in S^*} |S(s)|^2 \ll (N + |S^*| M_0(4T)) \sum_{n=N}^{2N} |a_n|^2 n^{-2\sigma_0}$$

where T is the difference of the maximum and the minimum of the imaginary parts of the complex numbers $s \in S^*$.

REMARK. Lemma 1 follows on noting the fact that

$$|((2N)^{iv} - N^{iv})\Gamma(iv)| \ll e^{-|v|}.$$

LEMMA 2. (R. BALASUBRAMANIAN, A. IVIĆ and K. RAMACHANDRA). *For* $|t| \geq 2$, *we have*

$$\int_t^{t+1} |\zeta(1+i\tau)|\, d\tau \ll (\log\ |t|)^{\frac{1}{4}}.$$

PROOF. Follows from Theorem 1 of [R.B, A.I, K.R].

LEMMA 3. *With the notation of Lemma1, we have for* $T \geq 2$,

$$\sum_{s\in S^*} |S(s)|^2 \ll (N + |S^*| T^{\frac{1}{2}} (\log T)^{\frac{1}{4}}) \sum_{n=N}^{2N} |a_n|^2 n^{-2\sigma_0}.$$

PROOF. Follows from the functional equation for $\zeta(s)$ and Lemma 2.

LEMMA 4. *With the notation of Lemma 1, we have*

$$|\{s \mid s \in S^*, |S(s)| \geq V\}| \ll GNV^{-2} + TG^3NV^{-6}(\log T)^{\frac{1}{2}}$$

where $G = \sum_{n=N}^{2N} |a_n|^2 n^{-2\sigma_0}$ *and* $N \geq (\log T)^{\frac{1}{2}}$.

PROOF. Define T_0 by $T_0^{\frac{1}{2}}(\log T)^{\frac{1}{4}}G = \varepsilon V^2$ where $\varepsilon(>0)$ is a small constant. Then the number of numbers s of S^* contained in any t interval of length T_0 is $\ll GNV^{-2}$ by Lemma 3. We need $T_0 \geq 2$. However if this is not satisfied, we have $(GV^{-2}(\log T)^{\frac{1}{4}})^{-2}$ is small and so $G^3V^{-6}(\log T)^{\frac{3}{4}}$ is big and hence

$$TG^3NV^{-6}(\log T)^{\frac{1}{2}} \gg TN(\log T)^{\frac{1}{4}}$$

which renders the required estimate trivial. Hence, we may ignore this condition and we have in any case the bound

$$(1+\frac{T}{T_0})GNV^{-2} \ll GNV^{-2} + TG^3NV^{-6}(\log T)^{\frac{1}{2}}.$$

From now on, we introduce the following notation. For real numbers M, N with $M \geq 1, N \geq 1$, $S_M^N(\cdots)$ will mean that $\sum_{M \leq n \leq N} a_n n^{-s}$ where a_n are any complex numbers. In applications the a_n will be obviously more precise.

LEMMA 5. *We have for any constant $A > 0$,*

$$\sum_{s \in S^*} | S^N_{\frac{N}{(\log N)^A}}(\cdots) |^2 \ll \log\log N \sum_{\frac{N}{(\log N)^A} \leq n \leq N} | a_n |^2 n^{-2\sigma_0} \left(n + T^{\frac{1}{2}}(\log T)^{\frac{1}{4}} \mid S^* \mid\right).$$

PROOF. We break up the sum $S^N_{\frac{N}{(\log N)^A}}(\cdots)$ into maximum number of disjoint sums of the type $S_U^{2U}(\cdots)$ (U running over powers of 2 i.e. $2^n (n \geq 0)$, $\frac{N}{(\log N)^A} \leq U \leq N$. One or two may be a part there of, but we can define some a_n's to be 0. (Several times, we use such a splitting without stating it explicitly). We then use

$$| S^N_{\frac{N}{(\log N)^A}}(\cdots) |^2 \ll (\log\log N) \sum_U | S_U^{2U}(\cdots) |^2$$

and apply Lemma 3.

LEMMA 6. *We have*

$$\sum_{s \in S^*} | S_1^N(\cdots) |^2 \ll \log N \sum_{n=1}^{N} | a_n |^2 n^{-2\sigma_0} \left(n + T^{\frac{1}{2}}(\log T)^{\frac{1}{4}} \mid S^* \mid\right).$$

PROOF. Similar to that of Lemma 5.

LEMMA 7. *Let $d(n)$ denote the number of positive integers dividing n. Then for $n \geq 2$, we have*

$$\sum_{n\leq x}(d(n))^2 \ll x(\log x)^3.$$

PROOF. This is well-known.

LEMMA 8. *With the notation of Lemma 1, we have for $N \geq (\log T)^{\frac{1}{2}}$, $Y \geq 1$,*

$$|\{s \mid s \in S^*, \mid S(s) \mid\geq V\}| \ll N^{2-2\sigma_0}V^{-2}(\log N)^3\frac{Y^{4j}}{N^{4j}}$$
$$+TN^{4-6\sigma_0}V^{-6}(\log T)^{\frac{1}{2}}(\log N)^9(\frac{Y}{N})^{12j}.$$

provided that $\mid a_n \mid\ll d(n)(\frac{Y}{n})^{2j}$ $(j = 0$ *or* $1)$.

PROOF. Follows from Lemmas 4 and 7.

The next three lemmas deal with the set $|\{s \mid s \in S^*, \mid S_1^N(s) \mid\geq V\}|$ where $(S_1^N(s) = \sum_{n=1}^{N} a_n n^{-s}), \mid a_n \mid\leq 1$ and $\sigma_0 = \frac{1}{2}$. Clearly this set is contained in the union of the two sets

$$S_1^* = \left\{s \mid s \in S^*, \mid S_1^{\frac{N}{(\log N)^A}}(\cdots) \mid\geq \frac{1}{2}V\right\}$$

and

$$S_2^* = \left\{s \mid s \in S^* \mid S^N_{\frac{N}{(\log N)^A}}(\cdots) \mid\geq \frac{1}{2}V\right\}.$$

As a preparation to get bounds for $\mid S_1^* \mid$ and $\mid S_2^* \mid$, we restate Lemmas 5 and 6. R.H.S of Lemma 5 (with $S^* = S_2^*$) is

$$\ll N \log\log N + (\log\log N)^2T^{\frac{1}{2}}(\log T)^{\frac{1}{4}} \mid S^* \mid.$$

LEMMA 9. *We have*

$$| S_2^* | \ll N(\log\log N)V^{-2} + TN(\log\log N)^5(\log T)^{\frac{1}{2}}V^{-6},$$

provided $N \geq (\log T)$.

PROOF. Define T_0 by $(\log\log N)^2(\log T)^{\frac{1}{4}}T_0^{\frac{1}{2}} = \varepsilon V^2$ for a small constant $\varepsilon(> 0)$. i.e., $T_0 = (\varepsilon V^2(\log\log N)^{-2}(\log T)^{-\frac{1}{4}})^2$. If $T_0 \leq 4$ then $V^2 \leq 2\varepsilon^{-1}(\log\log N)^2(\log T)^{\frac{1}{4}}$ and so $V^{-6} \geq (2\varepsilon^{-1})^{-3}(\log\log N)^{-6}(\log T)^{-\frac{3}{4}}$. Thus RHS in the lemma is $\gg TN(\log\log N)^{-1}(\log T)^{-\frac{1}{4}}$ which is a trivial upper bound for $| S_2^* |$. Thus in any case,

$$\begin{aligned} | S_2^* | &\ll N(\log\log N)V^{-2}(1 + \tfrac{T}{T_0}) \\ &\ll N(\log\log N)V^{-2} + TN(\log\log N)^5(\log T)^{\frac{1}{2}}V^{-6}. \end{aligned}$$

Thus the lemma is proved.

NOTE. R.H.S of Lemma 6 is $\ll N \log N + T^{\frac{1}{2}}(\log N)^2(\log T)^{\frac{1}{4}} | S^* |$. We recall that we now apply the lemma to $S^* = S_1^*$.

LEMMA 10. *We have*

$$| S_1^* | \ll (N(\log N)V^{-2} + TN(\log N)^5(\log T)^{\frac{1}{2}}V^{-6})(\log N)^{-A}$$

provided $N \geq \log T$.

PROOF. Define T_0 by $(\log N)^2(\log T)^{\frac{1}{4}}T_0^{\frac{1}{2}} = \varepsilon V^2$ where $\varepsilon(> 0)$ is a small constant i.e. $T_0 = (\varepsilon V^2(\log N)^{-2}(\log T)^{-\frac{1}{4}})^2$. Thus

$$N(\log N)V^{-2}\left(1 + \frac{T}{T_0}\right) \ll N(\log N)V^{-2} + TN(\log N)^5(\log T)^{\frac{1}{2}}V^{-6}.$$

If $T_0 \leq 4$, we have $\varepsilon V^2(\log N)^{-2}(\log T)^{-\frac{1}{4}} \leq 2$ i.e. $V^{-6} \gg (\log N)^6(\log T)^{-\frac{3}{4}}$ and so the term

$$TN(\log N)^5(\log T)^{\frac{1}{2}}V^{-6} \gg TN(\log N)^{11}(\log T)^{-\frac{1}{4}}$$

leads to a trivial estimate for $|S_1^*|$.

LEMMA 11. *We have, for $N \geq \log T$,*

$$|\{s \mid s \in S^*, | S_1^N(\cdots)| \geq V\}| \ll N(\log\log N)V^{-2} + N(\log\log N)^5(\log T)^{\frac{1}{2}}V^{-6}.$$

PROOF. Combining Lemmas 9 and 10, we obtain the lemma by choosing $A = 100$.

From now on, we introduce four positive parameters X, V, V_1, U which depend on T in such a way that their logarithms lie between two constant multiples of $\log T$. Moreover $U \geq X$. We collect together the results of Halász-Turán-Montgomery theory which we use in later sections.

LEMMA 12. *We have*

$$\left|\left\{s \mid Re\ s = \tfrac{1}{2}, s \in S^*, \left|\sum_{n\leq X}\mu(n)n^{-s}\right| \geq V_1\right\}\right|$$

$$\ll XV_1^{-2}\log\log T + TX(\log\log T)^5(\log T)^{\frac{1}{2}}V_1^{-6}.$$

Also, Lemma 8 gives (with $Y > 0$),

$$\left|\left\{s \mid Re\ s \geq \sigma, s \in S^*, \left|\sum_{n=U}^{2U}\frac{a_n b_n}{n^s}\right| \geq V\right\}\right|$$

$$\ll U^{2-2\sigma}(\log T)^3V^{-2}(\tfrac{Y}{U})^{4j} + TU^{4-6\sigma}(\log T)^{9\frac{1}{2}}V^{-6}(\tfrac{Y}{U})^{12j}$$

provided $|a_n| \ll d(n), |b_n| \ll (\frac{Y}{n})^{2j} (j = 0$ or $1)$.

We are now in a position to prove the main theorem of this section i.e. to improve the Huxley's constant 9 (Theorem 3.3). We denote by $\rho = \beta + i\gamma$, a typical zero of $\zeta(s)$ in $\{\frac{3}{4} - \delta \leq \sigma \leq 1, T \leq t \leq 2T\}$ with $|\beta - \sigma| \leq (\log T)^{-1}$ where T is assumed to exceed a large absolute constant. σ will be fixed and the O-constants will be independent of σ.

STEP 1. *We put* $R(w) = exp\left(\left(sin\frac{w}{100}\right)^2\right), M_X(s) = \sum_{n \leq X} \mu(n) n^{-s}$, $F(s) = \zeta(s) M_X(s) - 1 = \sum_{n \geq X} a_n n^{-s}$ *(the series being absolutely convergent in* $\sigma > 1$*) where* $|a_n| \leq d(n)$. *We have for* $Y \geq 1$ *and a complex variable* $w = u + iv$,

$$\sum_{n \geq X} a_n n^{-s} \Delta(\frac{Y}{n}) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s+w) Y^w R(w) \frac{dw}{w}$$

where $\Delta(\frac{Y}{n}) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (\frac{Y}{n})^w R(w) \frac{dw}{w}$. *From the representation of* $\Delta(\frac{Y}{n})$, *it follows by moving the line of integration to* $u = -2$ *(resp. by taking absolute value of the integrand on* $u = 2$ *itself) that* $\Delta(\frac{Y}{n}) = 1 + O((\frac{n}{Y})^2)$ *or* $O((\frac{Y}{n})^2)$ *according as* $n \leq Y$ *or* $n \geq Y$.

STEP II. *Let* ρ *be a zero in question. Obviously we have* $F(\rho) = -1$ *and so by first truncating the integral by neglecting* $|v| \geq c \; loglog \; T$ *(*c*, a large constant) moving the line of integration to* u *given by* $\beta + u = \frac{1}{2}$, *we obtain*

$$\sum_{n \geq X} a_n n^{-\rho} \Delta(\frac{Y}{n}) = -1 + o(1) + O\left(Y^{\frac{1}{2}-\sigma} \int_{|v| \leq c \; loglog \; T} (|F(\frac{1}{2} + i\gamma + iv)| + 1) e^{-|v|} dv\right).$$

The integral in the O*-estimate is*

$$\ll \left(\max_{|v| \leq c \; loglog \; T} |M_X(\tfrac{1}{2} + i\gamma + iv)|\right) \left(\int_{|v| \leq c \; loglog \; T} (|\zeta(\tfrac{1}{2} + i\gamma + iv)| + 1) \, e^{-|v|}\right.$$

$$= M_X(\tfrac{1}{2} + i\gamma')I(\gamma)(say).$$

STEP III. *Noting that* $|\gamma - \gamma'| \leq c \, loglog\, T, N(0, T+1) - N(0, T) \ll log\, T$, *we see (by using the first part of lemma) that the number of zeros* ρ *with* $|M_X(\tfrac{1}{2} + i\gamma')| \geq V_1$ *is*

$$\ll \left(XV_1^{-2} loglog\, T + TXV_1^{-6}(log\, T)^{\frac{1}{2}}(loglog\, T)^5\right) log\, T\; loglog\, T.$$

STEP IV. *By first applying Hölders inequality and then extending the range of integration to* $v = \frac{T}{2}$ *to* $v = \frac{5T}{2}$ *and changing the variable from* v *to* $v + \gamma$ *and observing that*

$$\begin{aligned} \sum_\rho e^{-|\gamma+v|} &\ll log\, T, \\ \sum_\rho (I(\gamma))^4 &\ll \sum_\rho \int_{\frac{T}{2}}^{\frac{5T}{2}} |\zeta(\tfrac{1}{2} + iv)|^4 e^{-|\gamma+v|} dv \\ &\ll T(log\, T)^5 \end{aligned}$$

we conclude that the number of zeros ρ *with* $I(\gamma) \geq V$ *is* $\ll T(log\, T)^5 V^{-4}$.

STEP V. *Hence apart from* R_1 *zeros we have* $O\left(VV_1Y^{\frac{1}{2}-\sigma}\right)$ *for the O-term in* Step II. *Here*

$$R_1 \ll (XV_1^{-2} loglog\, T + TXV_1^{-6}(log\, T)^{\frac{1}{2}}(loglog\, T)^5)(log\, T)(loglog\, T) + T(log\, T)^5 V^{-4}.$$

STEP VI. *We fix* $Y = \varepsilon(VV_1)^{\frac{2}{2\sigma-1}}$, *where* $\varepsilon > 0$ *is a small constant. Thus we have to estimate* R_2, *the number of zeros under question which satisfy*

$$\tfrac{3}{4} \leq \left| \sum_{n \geq X} a_n n^{-\rho} \Delta(\tfrac{Y}{n}) \right|.$$

Let $M = \max(X, Y^2)$. Then it is easily seen that the contribution to the infinite sum from $n \geq M$ is $\ll \sum_{n \geq M} \frac{d(n)}{n^\beta} \cdot \frac{Y^2}{n^2} \ll \frac{Y^{2+\epsilon}}{M^{1+\beta}} = o(1)$. Thus we have

$$\tfrac{1}{2} \leq \left| \textstyle\sum_1 \right| + \left| \textstyle\sum_2 \right|$$

where $\sum_1$ and $\sum_2$ are the portions $X \leq n \leq M_1$ and $M_1 \leq n \leq M, M_1$ being $\max(X, Y)$. Thus $R_2 \leq R_3 + R_4$ where R_3 is the number of zeros with $| \sum_1 | \geq \frac{1}{4}$ and R_4 those with $| \sum_2 | \geq \frac{1}{4}$. We split up $\sum_2$ into $\sum_2^{(U)}$ and find that at least one U should have $| \sum_2^{(U)} | \gg (\log T)^{-1}$. By second part of Lemma 12 with $j = 1$, we now find a bound for the number of zeros associated to one such U and take the sum over all U. Thus

$$\begin{aligned} R_4 &\ll \log T \sum_{M_1 \leq U \leq M} \left\{ (\log T)^5 \cdot \frac{Y^4}{U^{2\sigma+2}} + (\log T)^{15\frac{1}{2}} \frac{TY^{12}}{U^{6\sigma+8}} \right\} \\ &\ll (\log T) \left\{ (\log T)^5 Y^{2-2\sigma} + (\log T)^{15\frac{1}{2}} \frac{T}{Y^{6\sigma-4}} \right\} \end{aligned}$$

Similarly (applying Lemma 12 with $j = 0$) we get

$$R_3 \ll (\log T) \left\{ (\log T)^5 M_1^{2-2\sigma} + (\log T)^{15\frac{1}{2}} \frac{T}{X^{6\sigma-4}} \right\}$$

and here we can replace M_1 by X since $Y^{2-2\sigma}$ is already there.

STEP VII. *Thus (since $R_2 \ll R_3 + R_4$) denoting the number of zeros in question by H, we have*

$$\begin{aligned} \frac{H}{\log T} &\ll \frac{R_1 + R_3 + R_4}{\log T} \\ &\ll \frac{X}{V_1^2} (\log\log T)^2 + \frac{TX}{V_1^6} (\log T)^{\frac{1}{2}} (\log\log T)^6 + \frac{T}{V^4} (\log T)^4 \\ &\quad + Y^{2-2\sigma} (\log T)^5 + \frac{T}{Y^{6\sigma-4}} (\log T)^{15\frac{1}{2}} + X^{2-2\sigma} (\log T)^5 + \frac{T}{X^{6\sigma-4}} (\log T)^{15\frac{1}{2}} \end{aligned}$$

We recall that $Y = \varepsilon (VV_1)^{\frac{2}{2\sigma-1}}$ where $\varepsilon (> 0)$ is a small constant.

We have to minimize this upper bound for $H(log\ T)^{-1}$ subject to the restriction that X, V, V_1 all lie between T^{δ} and T^{A} where the constants $A(>0)$ and $(\delta>0)$ can be chosen to suit our greatest convenience. (A should not be confused with earlier notation). It will be helpful to minimize first with respect to X, then with respect to V and finally with respect to V_1. The following minimization lemma is very helpful. We state it in the notation of B.R. Srinivasan.

LEMMA 13. *Let M and N be positive integers, $u_m(>0)$ and $v_n(>0)(1\leq m\leq M, 1\leq n\leq N)$ denote constants. Let $A_m(>0), B_n(>0)$. Then there exists a q with the properties (Q_1, Q_2 are non-negative numbers) $Q_1\leq q\leq Q_2$ and*

$$\sum_{m=1}^{M} A_m q^{u_m} + \sum_{n=1}^{N} B_n q^{-v_n}$$
$$\ll \sum_{m=1}^{M}\sum_{n=1}^{N}(A_m^{v_n} B_n^{u_m})^{(u_m+v_n)^{-1}} + \sum_{m=1}^{M} A_m Q_1^{u_m} + \sum_{n=1}^{N} B_n Q_2^{-v_n}.$$

PROOF. See [B.R.S].

By applying this lemma, we are led to a bound for H in the form

$$H \ll T^{\left(\frac{5\sigma-3}{\sigma^2+\sigma-1}\right)(1-\sigma)}(log\ T)^{A(\sigma)+1}(loglog\ T)^{B(\sigma)},$$

this leads as in § A.2 to

$$N(\sigma, T) \ll T^{\left(\frac{5\sigma-3}{\sigma^2+\sigma-1}\right)(1-\sigma)}(log\ T)^{A(\sigma)+1}(loglog\ T)^{B(\sigma)}.$$

It can be checked that for $\mid \sigma-\frac{3}{4}\mid \leq \frac{D\ loglog\ T}{log\ T}$ ($D>0$ any large constant), we have $1+A(\sigma)=8$ and $B(\sigma)=\frac{4}{5}$. apart from an error $O\left(\frac{loglog\ T}{log\ T}\right)$. This proves the theorem which we stated in the beginning of this section.

STEP VIII. *We now give some steps in the minimization.* We estimate the quantity H under question and as shown in § A.2, the same which will hold for $N(\sigma,T)$ for $1 \geq \sigma \geq \frac{3}{4} - \delta$. Since $Y = \varepsilon(VV_1)^{\frac{2}{2\sigma-1}}$, from Step VII, we have

$$\frac{H}{\log T} \ll \frac{X}{V_1^2}(\log\log T)^2 + \frac{TX}{V_1^6}(\log T)^{\frac{1}{2}}(\log\log T)^6 + \frac{T(\log T)^{15\frac{1}{2}}}{X^{6\sigma-4}}$$

$$+X^{2-2\sigma}(\log T)^5 + \frac{T(\log T)^4}{V^4} + \frac{T(\log T)^{15\frac{1}{2}}}{(VV_1)^{\frac{12\sigma-8}{2\sigma-1}}} + (\log T)^5(VV_1)^{\frac{4-4\sigma}{2\sigma-1}}.$$

We now minimize with respect to X subject to $T^\delta \leq X \leq T^A$ (A-a large constant) and obtain (Hereafter we omit terms which are $O\left(T^{A_2(\sigma)-\eta}\right)$ ref. Theorem 3.3) for $\sigma \geq \frac{3}{4} - \delta$

$$\begin{aligned}
\frac{H}{\log T} \ll\ & \frac{T^\delta}{V_1^2}(\log\log T)^2 + \frac{T^{1+\delta}}{V_1^6}(\log T)^{\frac{1}{2}}(\log\log T)^6 + \frac{T(\log T)^{15\frac{1}{2}}}{T^{A(6\sigma-4)}} + \\
& +(\log T)^5 T^{2\delta(1-\sigma)} + \left\{\left(\frac{(\log\log T)^2}{V_1^2}\right)^{6\sigma-4}\left(T(\log T)^{15\frac{1}{2}}\right)\right\}^{\frac{1}{6\sigma-3}} \\
+\ & T(\log T)^{\frac{1}{2}}\left\{\left(\frac{(\log\log T)^6}{V_1^6}\right)^{6\sigma-4}(\log T)^{15}\right\}^{\frac{1}{6\sigma-3}} \\
+\ & \left\{\left(T(\log T)^{10\frac{1}{2}}\right)^{2-2\sigma}\right\}^{\frac{1}{4\sigma-2}}(\log T)^5 + \frac{T(\log T)^4}{V^4} \\
+\ & \frac{T(\log T)^{15\frac{1}{2}}}{(VV_1)^{\frac{12\sigma-8}{2\sigma-1}}} + (\log T)^5(VV_1)^{\frac{4-4\sigma}{2\sigma-1}} \\
\ll\ & \frac{T^\delta}{V_1^2}(\log\log T)^2 + \frac{T^{1+\delta}}{V_1^6}(\log T)^{\frac{1}{2}}(\log\log T)^6 \\
+\ & \frac{T^{\frac{1}{6\sigma-3}}}{V_1^{\frac{12\sigma-8}{6\sigma-3}}}(\log T)^{\frac{15\frac{1}{2}}{6\sigma-3}}(\log\log T)^{\frac{12\sigma-8}{6\sigma-3}} + \frac{T(\log T)^{\frac{5}{2\sigma-1}+\frac{1}{2}}}{V_1^{\frac{12\sigma-8}{2\sigma-1}}}(\log\log T)^{\frac{12\sigma-8}{2\sigma-1}} \\
+\ & T^{\frac{1-\sigma}{2\sigma-1}}(\log T)^{5+(10\frac{1}{2})\left(\frac{1-\sigma}{2\sigma-1}\right)} + \frac{T(\log T)^4}{V^4} + \frac{T(\log T)^{15\frac{1}{2}}}{(VV_1)^{\frac{12\sigma-8}{2\sigma-1}}} \\
+\ & (\log T)^5(VV_1)^{\frac{4-4\sigma}{2\sigma-1}}
\end{aligned}$$

Now, we make the change $V_1 \to V_1^{2\sigma-1}, V \to V^{2\sigma-1}$. Then we have to minimize the RHS of

$$\begin{aligned}
\frac{H}{\log T} \ll\ & \frac{T^{\delta}}{V_1^{4\sigma-2}}(\log\log T)^2 + \frac{T^{1+\delta}}{V_1^{12\sigma-6}}(\log\log T)^6(\log T)^{\frac{1}{2}} \\
+\ & \frac{T^{\frac{1}{6\sigma-3}}}{V_1^{\frac{12\sigma-8}{3}}}(\log T)^{\frac{15\frac{1}{2}}{6\sigma-3}}(\log\log T)^{\frac{12\sigma-8}{6\sigma-3}} + \frac{T(\log T)^{\frac{5}{2\sigma-1}+\frac{1}{2}}}{V_1^{12\sigma-8}}(\log\log T)^{\frac{12\sigma-8}{2\sigma-1}} \\
+\ & \frac{T(\log T)^4}{V^{8\sigma-4}} + \frac{T(\log T)^{15\frac{1}{2}}}{(VV_1)^{12\sigma-8}} + (\log T)^5(VV_1)^{4-4\sigma}.
\end{aligned}$$

Now, we minimize this with respect to V subject to $T^{\delta} \le V \le T^{A}$. The last three terms have the minimum

$$\ll \frac{T(\log T)^4}{T^{A(8\sigma-4)}} + \frac{T(\log T)^{15\frac{1}{2}}}{T^{A(12\sigma-8)}V_1^{12\sigma-8}} + (\log T)^5 T^{\delta(4-4\sigma)}V_1^{4-4\sigma}$$

$$+ \left\{(T(\log T)^4)^{1-\sigma}((\log T)^5 V_1^{4-4\sigma})^{2\sigma-1}\right\}^{\frac{1}{\sigma}}$$

$$+ \left\{\left(\frac{T(\log T)^{15\frac{1}{2}}}{V_1^{12\sigma-8}}\right)^{1-\sigma}((\log T)^5 V_1^{4-4\sigma})^{3\sigma-2}\right\}^{\frac{1}{2\sigma-1}}$$

$$\ll \frac{T(\log T)^{15\frac{1}{2}}}{T^{A(12\sigma-8)}V_1^{12\sigma-8}} + (\log T)^5 T^{\delta(4-4\sigma)}V_1^{4-4\sigma}$$

$$+T^{\frac{1-\sigma}{\sigma}}(\log T)^{4+\frac{2\sigma-1}{\sigma}}V_1^{\frac{(4-4\sigma)(2\sigma-1)}{\sigma}} + T^{\frac{1-\sigma}{2\sigma-1}}(\log T)^{5+(10\frac{1}{2})(\frac{1-\sigma}{2\sigma-1})}.$$

Thus we have finally to minimize with respect to V_1 subject to $T^{\delta} \le V_1 \le T^{A}$

the RHS of

$$
\begin{aligned}
\frac{H}{\log T} \quad & \ll \quad \frac{T^{\delta}}{V_1^{4\sigma-2}}(\log\log T)^2 + \frac{T^{1+\delta}}{V_1^{12\sigma-6}}(\log T)^{\frac{1}{2}}(\log\log T)^6 \\
& + \quad \frac{T^{\frac{1}{6\sigma-3}}}{V_1^{\frac{12\sigma-8}{3}}}(\log T)^{(15\frac{1}{2})(6\sigma-3)^{-1}}(\log\log T)^{\frac{12\sigma-8}{6\sigma-3}} \\
& + \quad \frac{T(\log T)^{\frac{5}{2\sigma-1}+\frac{1}{2}}}{V_1^{12\sigma-8}}(\log\log T)^{\frac{12\sigma-8}{2\sigma-1}} + \frac{T(\log T)^{15\frac{1}{2}}}{T^{A(12\sigma-8)}V_1^{12\sigma-8}} \\
& + \quad (\log T)^5 T^{4\delta(1-\sigma)}V_1^{4-4\sigma} + T^{\frac{1-\sigma}{\sigma}}(\log T)^{4+\frac{2\sigma-1}{\sigma}}V_1^{\frac{(4-4\sigma)(2\sigma-1)}{\sigma}} \\
& = \quad T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 \text{ (say)} \\
& \ll_\varepsilon \quad \frac{T^{\delta+\varepsilon}}{T^{A(4\sigma-2)}} + \frac{T^{1+\delta+\varepsilon}}{T^{A(12\sigma-6)}} + \frac{T^{\frac{1}{6\sigma-3}+\varepsilon}}{T^{A(\frac{12\sigma-8}{3})}} + \frac{T^{1+\varepsilon}}{T^{A(12\sigma-8)}} \\
& + \quad \frac{T^{1+\varepsilon}}{T^{2A(12\sigma-8)}} + T^{8\delta(1-\sigma)+\varepsilon} + T^{\frac{1-\sigma}{\sigma}+\delta\left(\frac{(4-4\sigma)(2\sigma-1)}{\sigma}\right)+\varepsilon} \\
& + \quad T_1 * T_6 + T_2 * T_6 + T_3 * T_6 + T_4 * T_6 + T_5 * T_6 + T_1 * T_7 + T_2 * T_7 + \\
& \qquad T_3 * T_7 + T_4 * T_7 + T_5 * T_7
\end{aligned}
$$

(where for $\alpha > 0, \beta > 0, E_1 > 1$ and $E_2 > 0$, we define $\left(\frac{E_1}{V_1^{\alpha}} * E_2 V_1^{\beta}\right)$ to be $(E_1^{\beta} E_2^{\alpha})^{(\alpha+\beta)^{-1}}$)

$$\ll \sum_{j=1}^{5}(T_j * T_6) + \sum_{j=1}^{5}(T_j * T_7).$$

We now make the following remarks : $T_1 * T_6$ is small,

$$T_2 * T_6 \ll \left\{(T^{1+\delta+\varepsilon})^{4-4\sigma}(T^{4\delta(1-\sigma)+\varepsilon})^{12\sigma-6}\right\}^{\frac{1}{8\sigma-2}}$$

and so small,

$$T_3 * T_6 \ll \left\{\left(T^{\frac{1}{6\sigma-3}+\varepsilon}\right)^{4-4\sigma}\left(T^{4\delta(1-\sigma)+\varepsilon}\right)^{\frac{12\sigma-8}{3}}\right\}^{\frac{3}{4}} \ll T^{\frac{1-\sigma}{2\sigma-1}+100\delta}$$

and so negligible,

$$T_4 * T_6 \ll \left\{(T^{1+\varepsilon})^{4-4\sigma}(T^{4\delta(1-\sigma)+\varepsilon})^{12\sigma-8}\right\}^{(8\sigma-4)^{-1}} \ll T^{\frac{1-\sigma}{2\sigma-1}+100\delta}$$

and so negligible and $T_5 * T_6 \ll T^{\varepsilon}$ (for large A) and so negligible. Clearly $T_5 * T_7$ is small. Now,

$$T_1 * T_7 \ll \left\{ \left(T^{\delta+\varepsilon}\right)^{\frac{4(1-\sigma)(2\sigma-1)}{\sigma}} \left(T^{\frac{1-\sigma}{\sigma}+\varepsilon}\right)^{4\sigma-2} \right\}^{\frac{1}{4\sigma-2}} \ll T^{\frac{1-\sigma}{\sigma}+100\delta}$$

and so negligible,

$$T_2 * T_7 \ll \left\{ \left(T^{1+\delta+\varepsilon}\right)^{\frac{4(1-\sigma)(2\sigma-1)}{\sigma}} \left(T^{\frac{1-\sigma}{\sigma}+\varepsilon}\right)^{12\sigma-6} \right\}^{J_1}$$

where

$$J_1 = \{4(1-\sigma)(2\sigma-1)\sigma^{-1} + 12\sigma - 6\}^{-1} = \left\{(2\sigma-1)\left(\frac{4-4\sigma}{\sigma}+6\right)\right\}^{-1}$$

$$= \frac{\sigma}{(2\sigma-1)(4+2\sigma)}.$$

Thus the exponent of T in $T_2 * T_7$ is $\leq \frac{4(1-\sigma)}{4+2\sigma} + \frac{6(1-\sigma)}{4+2\sigma} \leq 2(1-\sigma)$ (for $\sigma \geq \frac{1}{2}$) and so negligible.

In $T_3 * T_7$, the exponent of T is

$$\left\{ \frac{(4-4\sigma)(2\sigma-1)}{\sigma(6\sigma-3)} + \frac{(1-\sigma)(12\sigma-8)}{3\sigma} \right\} \left\{ \frac{(4-4\sigma)(2\sigma-1)}{\sigma} + \frac{12\sigma-8}{3} \right\}^{-1}$$

$$= \left\{ \frac{1-\sigma}{3\sigma}(1+3\sigma-2) \right\} \left\{ \frac{3(1-\sigma)(2\sigma-1)+\sigma(3\sigma-2)}{3\sigma} \right\}^{-1}$$

$$= (1-\sigma)\left\{ \frac{(3\sigma-1)}{3(1-\sigma)(2\sigma-1)+\sigma(3\sigma-2)} \right\} = (1-\sigma)J_2 \quad \text{(say)}.$$

where

$$J_2 = J_2(\sigma) = \frac{3\sigma-1}{3(1-\sigma)(2\sigma-1)+\sigma(3\sigma-2)} = \frac{3\sigma-1}{-3\sigma^2+7\sigma-3}.$$

We note that the derivative of the denominator is > 0 for $0 < \sigma < 1$ and that $-3(\frac{3}{4})^2 + \frac{21}{4} - 3 > 0$. Also

$$\begin{aligned} J_2'(\sigma)(-3\sigma^2+7\sigma-3)^2 &= (-3\sigma^2+7\sigma-3)3-(3\sigma-1)(-6\sigma+7) \\ &= 9\sigma^2-6\sigma-2 \\ &< 0 \end{aligned}$$

if $\sigma \geq \frac{3}{4} - \eta$. Hence $J_2(\sigma)$ is decreasing in $\sigma \geq \frac{3}{4} - \eta$ for some $\eta > 0$. But $J_2(\frac{3}{4}) < \frac{12}{5}$. Thus $T_3 * T_7 \ll T^{(1-\sigma)(\frac{12}{5}-100\delta)}$ and so it is negligible. Also we have

$$T_5 * T_7 \ll \left\{ \left(T^{1+\varepsilon-A(12\sigma-8)}\right)^{\frac{4(1-\sigma)(2\sigma-1)}{\sigma}} \left(T^{\frac{1-\sigma}{\sigma}+\varepsilon}\right)^{12\sigma-8} \right\}^{J_3(\sigma)}$$

where $J_3(\sigma) = \frac{\sigma}{4(\sigma^2+\sigma-1)}$. For large A and $\sigma \geq \frac{3}{4} - \eta$, we note that $T_5 * T_7$ is also small. Finally, we are left with the only term $T_4 * T_7$. Now

$$T_4 * T_7 \ll \left\{ \left(T(\log T)^{\frac{5}{2\sigma-1}+\frac{1}{2}}(\log\log T)^{\frac{12\sigma-8}{2\sigma-1}}\right)^{\frac{4(1-\sigma)(2\sigma-1)}{\sigma}} \times \right.$$

$$\left. \times \left(T^{\frac{1-\sigma}{\sigma}}(\log T)^{4+\frac{2\sigma-1}{\sigma}}\right)^{12\sigma-8} \right\}^{\left(12\sigma-8+\frac{4(1-\sigma)(2\sigma-1)}{\sigma}\right)^{-1}}$$

Note that $3\sigma - 2 + \frac{(1-\sigma)(2\sigma-1)}{\sigma} = \frac{(\sigma^2+\sigma-1)}{\sigma}$. Thus

$$\begin{aligned} T_4 * T_7 &\ll \left\{ \left(T(\log T)^{\frac{5}{2\sigma-1}+\frac{1}{2}}(\log\log T)^{\frac{12\sigma-8}{2\sigma-1}}\right)^{(1-\sigma)(2\sigma-1)} \times \right. \\ &\quad \left. \times \left(T^{1-\sigma}(\log T)^{6\sigma-1}\right)^{3\sigma-2} \right\}^{(\sigma^2+\sigma-1)^{-1}} \\ &= T^{\frac{(5\sigma-3)(1-\sigma)}{\sigma^2+\sigma-1}}(\log T)^{A(\sigma)}(\log\log T)^{B(\sigma)} \end{aligned}$$

where

$$\begin{aligned} A(\sigma) &= \left\{\frac{(2\sigma+9)}{2(2\sigma-1)}(1-\sigma)(2\sigma-1)+(3\sigma-2)(6\sigma-1)\right\}(\sigma^2+\sigma-1)^{-1} \\ &= \frac{1}{2}(34\sigma^2-37\sigma+13)(\sigma^2+\sigma-1)^{-1} \end{aligned}$$

and

$$B(\sigma) = 4(1-\sigma)(3\sigma-2)(\sigma^2+\sigma-1)^{-1}.$$

Finally we have $N(\sigma, T) \ll H \ll T^{\left(\frac{5\sigma-3}{\sigma^2+\sigma-1}\right)(1-\sigma)}(\log T)^{1+A(\sigma)}(\log\log T)^{B(\sigma)}$ and this proves the theorem.

POST-SCRIPT

K. Ramachandra has recently deduced (see K. RAMACHANDRA, *A large value theorem for* $\zeta(s)$, Hardy-Ramanujan J., 18 (1995),1-9) the following large value theorem from Montgomery's fundamental Lemma 8.1 (see [H.L.M]). We follow the notation of Lemma 1 of § A.3 except that we write T_0 for T.

THEOREM 1. *We have*

$$\sum_{s\in S^*} | S(s) |^2 \ll_\varepsilon GN + G(T_0^\varepsilon + | S^* | (\log T_0)^\varepsilon)T_0^{\frac{1}{2}}$$

where $T_0 \geq 2, G = \sum_{n=N}^{2N} | a_n |^2 n^{-2\sigma_0}$ *and* $\varepsilon > 0$ *is an arbitrary constant.*

REMARK. The proof depends on the fact that if $T \leq t_1 < t_2 < \cdots < t_R \leq 2T, | t_{j+1} - t_j | \geq 1$ $(j = 1, 2, \cdots, R-1)$ and $| \log \zeta(1+it_j) | \geq \varepsilon \log\log T$ $(j = 1, 2, \cdots, R)$ then $R \ll_\varepsilon T^{100\varepsilon}$ (where $0 < \varepsilon < \frac{1}{10}$ and $T \geq 100$). This will be proved in Ramachandra's paper referred to above.

COROLLARY. *Consider any set* S^{**} *of complex numbers with the following properties* (i) $\sigma_0 \leq Re\ s \leq \sigma_0 + \frac{1}{4}$ *for all* $s \in S^{**}$ (ii) $| Im\ s | \leq T (\geq 100)$ *for all* $s \in S^{**}$ *and* (iii) $| Im(s - s') | \geq 1$ *for any two* $s, s' \in S^{**}$ *with* $s \neq s'$. *Then for* $V > 0$ *we have*

$$| \{s \mid s \in S^{**}, | S(s) | \geq V\} | \ll_\varepsilon GNV^{-2} + TG^3NV^{-6}(\log T)^\varepsilon + T^\varepsilon(1 + TG^2V^{-4}),$$

where $\varepsilon > 0$ *is an arbitrary constant.*

REMARK. Earlier some imperfect theorems in place of Theorem 1 above were known (due to H.L. Montgomery). Corresponding corollaries were obtained by M.N. Huxley. Our method of deducing the above corollary is similar to Huxley's.

From the corollary it is not hard to deduce (from the arguments of § A.3) the following theorem.

THEOREM 2. *With* $N(\sigma,T)$ *as usual,* $\frac{3}{4} \leq \sigma \leq \frac{3}{4} + \frac{D\, loglog\, T}{log\, T}, T \geq 100$ *and* $A_2(\sigma) = \left(\frac{5\sigma-3}{\sigma^2+\sigma-1}\right)$, *we have*

$$N(\sigma,T) \ll_{D,\varepsilon} T^{A_2(\sigma)(1-\sigma)}(log\, T)^{8-\frac{1}{5}+\varepsilon}$$

where $\varepsilon > 0$ *and* $D > 0$ *are any two arbitrary constants.*

We now borrow a theorem of Barban and Vehov (ref: M.B. Barban and P.P. Vehov, *ob odnoi ekstremal' noi zudače,* Trudy Mosk. Mat. Obsč. 18 (1968), 83-90, English translation : Trans. Moscow Math. Soc. 18 (1968), 91-99) from Jutila's paper (M. Jutila, Zeros of the zeta-function near the critical line, *Studies in pure mathematics, To the memory of Paul Turán,* 385-394 (Birkhaüser, Basel-Stuttgart, (1982)). Let $1 < v_1 < v_2$, and let us define $H_d = H_d(v_1, v_2)$ for integers $d > 0$ by $H_d = 1$ if $1 \leq d \leq v_1, H_d = \left(log \frac{v_2}{d}\right)\left(log \frac{v_2}{v_1}\right)^{-1}$ if $v_1 < d < v_2$ and $H_d = 0$ if $d \geq v_2$.

Again for any z_1, z_2 with $1 < z_1 < z_2$ define

$$\lambda_d = \lambda_d(z_1, z_2) = \mu(d)H_d(z_1, z_2).$$

Then we have

THEOREM 3. (M.B. BARBAN AND P.P. VEHOV)

$$\sum_{n\le x}\left(\sum_{d/n}\lambda_d(z_1,z_2)\right)^2 \ll x\left(log\frac{z_2}{z_1}\right)^{-1}.$$

We now indicate the proof of the following theorem.

THEOREM 4. *Let $\frac{1}{2}+\delta\le\sigma\le 1-\delta$ where $0<\delta<\frac{1}{100}$. Then*

$$N(\sigma,T)\ll_\delta (T\ log\ T\ loglog\ T)^{\frac{3(1-\sigma)}{2-\sigma}}(log\ T)^2(loglog\ T)^{-1}.$$

COROLLARY. *In the neighbourhood $|\ \sigma-\frac{3}{4}\ |\le\frac{D\ loglog\ T}{log\ T}$ ($D>0$ is any arbitrary constant) of $\frac{3}{4}$ we have*

$$N(\sigma,T)\ll_D T^{\frac{3(1-\sigma)}{2-\sigma}}(log\ T)^{\frac{13}{5}}(loglog\ T)^{-\frac{2}{5}}.$$

PROOF OF THEOREM 4. Put $z_1=T, z_2=T\ log\ T$ and $M(T,s)=\sum_{n=1}^{\infty}\lambda_n n^{-s}$. By a well-known theorem of Montgomery and Vaughan we have

$$\int_T^{2T}|\ M(T,\frac{1}{2}+it)\ |^2\ dt=\sum_{n=1}^{\infty}(T+O(n))\lambda_n^2 n^{-1}\ll T\ log\ T.$$

Also by the same theorem of Montgomery and Vaughan we have (using Theorem 3),

$$\int_T^{2T}|\ \zeta(1+it)M(T,1+it)-1\ |^2\ dt\ll log\ T(loglog\ T)^{-1}.$$

From these two results Theorem 4 follows as in the appendix § A.2.

REMARK 1. Using Theorem 2 and the corollary to Theorem 4, we can in the results involving $\psi(x)$ (stated in the introduction) replace $8\frac{23}{24}, 10\frac{5}{8}$ and

$12\frac{19}{24}$ by the numbers $7\frac{7}{12}, 9\frac{1}{4}$ and $11\frac{5}{12}$ respectively.

REMARK 2. The paper by K. Ramachandra (referred to in the beginning of the post-script) has since appeared. In Theorem 1 he has improved $(\log T_0)^{\epsilon}$ to $\log\log T_0$. As a consequence we can replace (in Theorem 2) $(\log T)^{8-\frac{1}{5}+\epsilon}$ by $(\log T)^{8-\frac{1}{5}}(\log\log T)^{\frac{8}{5}}$. This results in minor improvements of results stated in the previous remark.

REMARK 3. Using the density results proved in the appendix and post-script we can prove the following result.

Let $H = X^{\frac{1}{6}}(\log X)^{11\frac{5}{12}}(\log\log X)^{3\frac{11}{12}}f(X)$ where $f(X)(\leq \log\log\log X)$ tends to ∞ as $X \to \infty$. Then there holds

$$\frac{1}{X}\int_X^{2X}(\psi(x+h)-\psi(x)-H)^2dx = o(H^2(\log X)^{-1}).$$

This result will be used in the forthcoming paper (with the title problems and results on $\alpha p - \beta q$) by us.

ACKNOWLEDGEMENTS. We are thankful to Professor M. Jutila for suggesting the use of Barban-Vehov theorem. We are also thankful to Professor A. Ivić for pointing out the paper (D. WOLKE, *On the explicit formula of Riemann von-Mongoldt*-II, (2) 28 (1983), 106-116) of D. Wolke. Here he has already proved a result which is nearly the same as the one stated in Theorem 3.1 of our paper. However our method of proof is slightly different. Finally we are thankful to Professor K. Soundararajan for some helpful remarks.

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MANUSCRIPT COMPLETED ON 7 AUGUST 1994

CLASSIFICATIONS. 11XX, 11NXX, 11N05, also 11XX, 11JXX, 11J25.

KEY WORDS. Ramanujan-Hardy-Littlewood Lattice point problem, Riemann zeta-function, Theory of prime numbers, L-series, sieve methods.

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