Hardy-Ramanujan Journal
Vol.20 (1997) 12-28

ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES-XVIII

(A FEW REMARKS ON LITTLEWOOD'S THEOREM AND TITCHMARSH POINTS)

BY

R. BALASUBRAMANIAN, K. RAMACHANDRA and A. SANKARANARAYANAN

(DEDICATED TO PROFESSOR K. CHANDRASEKHARAN ON HIS SEVENTY-FIFTH BIRTHDAY)

§ 1. **INTRODUCTION AND NOTATION.** This paper was necessitated because we observed that the proofs of the results of the paper XVI[7] can be simplified and that the results therein can at the same time be generalised. (In § 1, § 2 and § 3 we prove six theorems in all. For an attractive application of these see Theorem 11 of § 4). We write $s = \sigma + it$ as usual. We begin by stating a generalisation of Theorem 9.15 (A) (on page 230 of [9]). We need some definitions. (We fix two positive constants a and b with $a < b$ throughout). The parameter T will be assumed to exceed a large positive constant.

GENERALISED DIRICHLET SERIES (GDS). Let $\{\lambda_n\}$ be a sequence of real numbers with $a < \lambda_1 < \lambda_2 < \cdots, \lambda_1 < b$ and $a \leq \lambda_{n+1} - \lambda_n \leq b$ for $n \geq 1$. Let $\{A_n\}$ be any sequence of complex numbers such that $A_1 \neq 0$

and

$$Z(s) = \sum_{n=1}^{\infty} A_n \lambda_n^{-s} \tag{1}$$

converges for some complex $s = s_0$. Then $Z(s)$ is called a ***generalised Dirichlet series*** (GDS). We remark that if $Z(s)$ converges at $s = s_0$, then it is absolutely convergent at $s = s_0 + 2$. Note that a GDS is different from zero if real part of $s(Re\ s)$ exceeds a certain constant. In fact as $Re\ s \to \infty, \mid Z(s) \mid$ tends to a non-zero constant. A GDS is said to be a ***normalised generalised Dirichlet series*** (NGDS) if $\sum_{n \leq x} \mid A_n \mid^2 \ll_\epsilon x^{1+\epsilon}$ for every $\varepsilon > 0$. A GDS is said to be a *Dirichlet series* if $\{\lambda_n\}$ is a subsequence of the sequence of natural numbers.

$\{\alpha_n\}$ **TRANSFORMATION OF AN NGDS.** Let $Z(s)$ be an NGDS. We consider only such sequences $\{\alpha_n\}$ of real numbers which satisfy $\sum_{n \leq x} \mid A_n \alpha_n \mid^2 \ll_\epsilon x^{1+\varepsilon}$ for every $\varepsilon > 0$. $F(s) = \sum_{n=1}^{\infty} A_n (\lambda_n + \alpha_n)^{-s}$ is said to be an $\{\alpha_n\}$ transformation of $Z(s)$ if $F(s)$ is a GDS. Note that

$$F(s) = D(s) + Z(s) \text{ where } D(s) = \sum_{n=1}^{\infty} A_n((\lambda_n + \alpha_n)^{-s} - \lambda_n^{-s}) \tag{2}$$

and that $D(s)$ is analytic in $\sigma > 0$. Moreover we have

LEMMA 1. *For $\sigma > 0$ and every $\varepsilon > 0$ we have $D(s) = O_\sigma((\mid t \mid +2)^2)$ and also*

$$\frac{1}{T} \int_T^{2T} \mid D(\sigma + it) \mid^2 dt \ll_\varepsilon max \left(T^{\,2(\frac{1}{2}-\sigma)+\epsilon}, T^{\,\epsilon}\right). \tag{3}$$

PROOF. See Theorems 7 and 7′ of XV[5].

THEOREM 1 (J.E. LITTLEWOOD). *Let $Z(s)$ be a* GDS *which can be continued analytically in $(\sigma \geq \frac{1}{2} - \delta_0, T - log\ T \leq t \leq 2T + log\ T)$, where $\delta_0 (> 0)$ is a constant, and there $log(max \mid Z(s) \mid +100)$ is $\ll log\ T$. Let $Z(s) \to 1$ as $Re\ s \to \infty$. For $\alpha \geq \frac{1}{2}$ let $N(\alpha, T)$ denote the number of zeros of $Z(s)$ in $(\sigma \geq \alpha, T \leq t \leq 2T)$. Then (for $\sigma_0 > \frac{1}{2}$) we have*

$$2\pi \int_{\sigma_0}^{\infty} N(\sigma, T) d\sigma = \int_T^{2T} log \mid Z(\sigma_0 + it) \mid dt + O(log\ T) \tag{4}$$

and hence

$$N(\frac{1}{2}+2\delta,T) \leq (g\delta)^{-1}T\,log\left(\frac{1}{T}\int_{T}^{2T} \mid Z(\frac{1}{2}+\delta+it) \mid^{g} dt\right) + O(log\,T) \quad (5)$$

holds uniformly for all real positive constants g and δ. If δ is any fixed constant we may take $\delta_0 = 0$ and then replace $O(log\,T)$ by $O_\delta(log\,T)$.

REMARK. This theorem is essentially due to J.E. Littlewood, since the special case $g = 2$ and $Z(s) = \zeta(s)$ (due to J.E. Littlewood) is dealt with on pages 229 and 230 of [9]. The general case stated as Theorem 1 above follows by a trivial generalisation of Littlewood's method. If we do not assume $Z(s) \to 1$ as $Re\ s \to \infty$, we have to replace $O(log\,T)$ by $O(T)$ in (4) and (5). This does not matter for our purposes.

§ 2. A COROLLARY TO THEOREM 1.

THEOREM 2. *Let $r \geq 1$ be any integer constant and let $\varphi_1(s), \varphi_2(s), \cdots, \varphi_r(s)$ be r Dirichlet series each of which is continuable analytically in $(\sigma \geq \frac{1}{2} - \delta_0, T - log\,T \leq t \leq 2T + log\,T)$ and there $log(\max_j \mid \varphi_j(s) \mid + 100) \ll log\,T$. Suppose further that*

$$\max_j \left(\frac{1}{T}\int_{T-log\,T}^{2T+log\,T} \mid \varphi_j(\frac{1}{2}+it) \mid^2 dt\right) \ll_\epsilon T^\epsilon \quad (6)$$

holds for every $\varepsilon > 0$. Let $P(X_1, \cdots, X_r)$ be any fixed polynomial (with complex coefficients) such that when we put $X_j = \varphi_j(s)(i = 1, 2, \cdots, r), P = P(s) = P(X_1, \cdots, X_r)$ is a normalised Dirichlet series. Let $F(s)$ be any $\{\alpha_n\}$ transformation of $P(s)$. Then the function $N(\sigma, T)$ defined (as before) for $F(s)$ satisfies

$$N(\sigma,T) \ll_\sigma T \qquad (\sigma > \frac{1}{2}). \quad (7)$$

REMARK. We define the degree of a monomial $X_1^{d_1} \cdots X_r^{d_r}$ to be $d_1 + \cdots + d_r$ and the degree of $P(X_1, \cdots, X_r)$ to be the maximum of $d_1 + \cdots + d_r$ taken over all monomials occuring in $P(X_1, \cdots, X_r)$. If the degree of P is 1 then we can allow each $\varphi_j(s)$ to be a GDS. Then P has to be an NGDS.

PROOF. The proof follows from the fact that (6) implies

$$\max_j \left(\frac{1}{T} \int_T^{2T} | \varphi_j(\sigma + it) |^2 \, dt \right) \ll_\sigma 1 \quad (\sigma > \frac{1}{2}), \tag{8}$$

and that for a suitable small constant $g > 0$ we have

$$| F(s) |^g \ll | D(s) |^2 + | \varphi_1(s) |^2 + \cdots + | \varphi_r(s) |^2 + 1. \tag{9}$$

Note that in view of Lemma 1 it is not hard to deduce that

$$\frac{1}{T} \int_T^{2T} | D(\sigma + it) |^2 \, dt \ll_\sigma 1 \quad (\sigma > \frac{1}{2}). \tag{10}$$

From these facts Theorem 2 follows from Theorem 1.

§ 3. **TITCHMARSH POINTS.** Let $F(s)$ be a GDS continuable analytically in $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$ and there $\log(\max | F(s) | + 100) \ll \log T$. A point $s_0 = \sigma_0 + it_0$ in $(\sigma \geq \beta + \delta_1, T \leq t \leq 2T)$, where $\delta_1 > 0$ is a constant, is said to be a *Titchmarsh point with the lower bound* T^{ℓ} for $| F(s) |$ if $\ell (> 0)$ is bounded below independent of T and t_0.

THEOREM 3. *If $s_0 = \sigma_0 + it_0$ (with $F(s)$ as above) is a Titchmarsh point of $F(s)$, then the region $(\sigma \geq \beta, | t - t_0 | \leq \delta_2)$ where $\delta_2 (> 0)$ is any small constant, contains $\gg \log T$ zeros of $F(s)$.*

PROOF. For the proof of this theorem due to R. Balasubramanian and K. Ramachandra see Theorem 3 of III[1]. It should be mentioned that this theorem is not too-trivial a generalisation of Theorem 9.14 (on page 227 of [9]) due to E.C. Titchmarsh.

WELL-SPACED POINTS. The points $s^{(q)} = \sigma_q + it_q$ $(q = 1, 2, \cdots)$ in the complex plane are said to be *well-spaced* if $| s^{(q)} - s^{(q')} |$ is bounded below for all pairs (q, q') with $q \neq q'$.

THEOREM 4. *If there are N_0 well-spaced Titchmarsh points for $F(s)$ ($F(s)$ as in Theorem 3), then $F(s)$ has $\gg N_0 \log T$ zeros in $(\sigma \geq \beta, T \leq t \leq 2T)$.*

PROOF. The proof follows from the fact that $| F(s) |$ tends to a non-zero limit uniformly in t as $\sigma \to \infty$.

THEOREM 5. *Let $\beta(<\frac{1}{2})$ be a constant and $r \geq 1$ any integer constant and $\varphi_1(s), \cdots, \varphi_r(s)$ be r Dirichlet series each of which is continuable analytically in $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$ and there $\log \max_j(|\varphi_j(s)| + 100)$ is $\ll \log T$. Suppose further that for $j = 1, 2, \cdots, r$ and $\sigma \geq \beta$, we have*

$$\frac{1}{T}\int_{T-\log T}^{2T+\log T} |\varphi_j(\sigma+it)|^2 \, dt \ll_\varepsilon \max\left(T^{2m_j(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon\right) \qquad (11)$$

where $m_j > 0$ are constants. Let $\mu > 0$ be a constant. Put $X_0 = T^{\mu(\frac{1}{2}-\sigma)-\varepsilon}$. Let $X_0^{d_0} X_1^{d_1} \cdots X_r^{d_r}$ ($d_j \geq 0$ integers, $j = 0, 1, 2, \cdots, r$) be any fixed monomial in $X_0, X_1, \cdots, X_r$. Let the weighted μ-degree $d(\mu)$ of the monomial be defined as $\mu d_0 + m_1 d_1 + \cdots + m_r d_r$. Put $Q_0(s) = X_0^{d_0}(\varphi_1(s))^{d_1} \cdots (\varphi_r(s))^{d_r}$. Then given any well-spaced set of points $\{s_q\}$ with $s_q = \sigma + it_q (q = 1, 2, \cdots; \sigma \geq \beta + \delta_3, T \leq t_q \leq 2T)$ where $\delta_3 > 0$ is a small constant we have

$$|Q_0(\sigma + it_q)| \ll \max\left(T^{d(\mu)(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon\right), \qquad (12)$$

except for $O(T^{1-\varepsilon})$ values of q.

REMARK. If $\sum_{j=1}^{r} d_j = 1$, then we can allow $\varphi_j(s) (j = 1, 2, \cdots, r)$ to be GDS.

PROOF. We use the fact that the value $|\varphi_j(s_q)|$ of $\varphi_j(s)$ is majorised by the mean value over a disc (with s_q as centre and ε as radius) of $|\varphi_j(s)|$. We choose a small radius and sum over all the discs taking s_q to be $\sigma + it_q$. We obtain

$$\frac{1}{T}\sum_q |\varphi_j(s_q)|^2 \ll \max\left(T^{2m_j(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon\right).$$

Hence $|\varphi_j(s_q)| > \max(T^{m_j(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon)$ is possible for at most $O(T^{1-\varepsilon})$ values of q. We next sum over all j and obtain the result.

THEOREM 6. *Let $\beta(<\frac{1}{2})$ be a constant and let $\varphi_0(s)$ be a Dirichlet series continuable analytically in $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$ and there*

$\log \max(|\varphi_0(s)| + 100) \ll \log T$. Suppose that it has $\gg T$(resp.$T(\log\log T)^{-1}$) well-spaced Titchmarsh points $\{\sigma + it_q\}$ (where σ is any constant with $\beta < \sigma < \frac{1}{2}$) with the lower bound $T^{\mu(\frac{1}{2}-\sigma)-\varepsilon}$ where $\mu(> 0)$ is a constant. Let $Q(X_0, X_1, \cdots, X_r)$ be a fixed polynomial (with complex coefficients) such that for some positive integer M, the maximum of $d(\mu)$ (defined in Theorem 5) taken over all the monomials $X_0^{d_0} X_1^{d_1} \cdots X_r^{d_r}$ occuring in $Q(X_0, X_1, \cdots, X_r)$ is less than $M\mu$. Put $Q(s) = Q(\varphi_0(s), \varphi_1(s), \cdots, \varphi_r(s))$ (where $\varphi_j(s)$ $j = 1, 2, \cdots$, are as in Theorem 5). Assume that $(\varphi_0(s))^M - Q(s)$ is an NGDS, and let $F(s)$ be its $\{\alpha_n\}$ transformation. Then $F(s)$ has $\gg T \log T$(resp. $T(\log T)(\log\log T)^{-1}$) zeros in $(\sigma \geq \beta, T \leq t \leq 2T)$.

PROOF. Follows from $|(\varphi_0(s))^M - Q(s)| \geq |\varphi_0(s)|^M (1 - |Q(s)| |\varphi_0(s)|^{-M})$.

REMARK. Note that none of the functions $\varphi_0(s)$, $(\varphi_0(s))^M$ and $Q(s)$ need be normalised Dirichlet series. If $M = 1$, then $Q(s)$ does not involve $\varphi_0(s)$ (which can now be taken to be a GDS). If $M = 1$ and $Q(X_0, X_1, \cdots, X_r)$ (now independent of X_0) is linear in $X_1, \cdots, X_r$ (i.e. $\sum_{j=1}^{r} d_j \leq 1$ for every monomial $X_1^{d_1} \cdots X_r^{d_r}$ occuring in $Q(X_0, X_1, \cdots, X_r)$ and equality holds for at least one monomial) then all of $\varphi_1(s), \cdots, \varphi_r(s)$ can be taken to be GDS. Also $Q(X_0, X_1, \cdots, X_r)$ can be a constant.

§ 4. **SOME APPLICATIONS OF THEOREMS 2 TO 6.** Theorems 2 to 6 are only easy formalisms. These would be completely uninteresting without examples. Finding examples is a difficult task. For example we do not know how to prove the expected result $N(\sigma, T) \ll_\sigma T (\sigma > \frac{1}{2})$ for the abelian L-series of an algebraic number field. However we have a somewhat general theorem namely.

THEOREM 7. *Let $\{\lambda_n\}(n = 1, 2, \cdots)$ be a sequence of real numbers as in the definition of* GDS. *Let $|\sum_{n\leq x} a_n| \leq B(x)$, $\sum_{n\leq x} |a_n|^2 \leq xB(x)$ and $\sum_{m\leq x} |\sum_{n\leq m} a_n|^2 \leq xB(x)$, where $B(x)$ depends on x. If $B(x) \ll_\varepsilon x^\varepsilon$ (for every*

$\varepsilon > 0$) then $Z(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ converges uniformly over compact subsets of $\sigma > 0$ and hence is analytic there. We have

$$N(\frac{1}{2}+\delta, T) \ll_\delta T \qquad (\delta > 0). \tag{13}$$

If further $\log B(x) \ll \log\log x$ then we have

$$N(\frac{1}{2}+\delta, T) \ll \delta^{-1} T \log(\delta^{-1}) \tag{14}$$

uniformly for $0 < \delta \le \frac{1}{2}$.

REMARK. Results like

$$\frac{1}{T}\int_{T-\log T}^{2T+\log T} | Z(\frac{1}{2}+it) |^2 \, dt \ll_\varepsilon T^\varepsilon \tag{15}$$

for every $\varepsilon > 0$ and more general and powerful results have been proved in paper V[6]. Results like (15) imply (13) and (14). If $\{Z(s)\}$ is any finite set of Dirichlet series each subject to (15) we can apply Theorem 2.

We now turn to series of the type

$$\sum_{n=1}^{\infty} a_n b_n e^{2\pi i n\theta} \lambda_n^{-s} \quad (\theta \text{ is a real constant}), \tag{16}$$

their analytic continuations and their Titchmarsh points. Investigations dealing with such series were carried out in a series of papers by R. Balasubramanian and K. Ramachandra (see III[1], IV[2], V[6], VI[3], XIV[4] and also the paper [8] by K. Ramachandra and A. Sankaranarayanan). The paper XIV[4] is nearly final. In paper XIV[4] the condition $a_n = O(1)$ is assumed. This can be relaxed to $\sum_{n\le x} | a_n |^2 = O(x)$. *This last mentioned condition on a_n will be assumed in the rest of this paper.*

Lest we get lost in generalities we state two special cases first.

THEOREM 8. *Let $\theta_0 (0 < \theta_0 < \frac{1}{2})$ be a constant and let $\{a_n\}$ be a sequence of complex numbers satisfying the inequality $| \sum_{m=1}^{N} a_m - N | \le (\frac{1}{2} - \theta_0)^{-1}$ for*

$N = 1, 2, 3, \cdots$. Also for $n = 1, 2, 3, \cdots$, let α_n be real and $|\alpha_n| \leq C(\theta_0)$ where $C(\theta_0)$ is a certain (small) constant depending only on θ_0. Then the number of zeros of the function

$$\sum_{n=1}^{\infty} a_n (n + \alpha_n)^{-s} = \zeta(s) + \sum_{n=1}^{\infty} (a_n (n + \alpha_n)^{-s} - n^{-s})$$

in the rectangle $(|\sigma - \frac{1}{2}| \leq \delta, T \leq t \leq 2T)$ is $\geq C(\theta_0, \delta) T \log T$ where $C(\theta_0, \delta)$ is a positive constant depending only on θ_0 and δ, and $T \geq T_0(\theta_0, \delta)$ a large positive constant.

PROOF. Theorem 10 (below) gives $\gg T$ well spaced Titchmarsh points on every line segment $(\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)$ with the lower bound $\gg T^{\delta}$, while actually (14) gives

$$N\left(\frac{1}{2} + \frac{C \log\log T}{\log T}, T\right) \ll C^{-1} T \log T$$

for every fixed $C (> 0)$. (It is not hard to prove the required mean-square upper bound for the function).

THEOREM 9. *In the above theorem we can relax the condition on a_n to*

$$\left| \sum_{m=1}^{N} a_m - N \right| \leq \left(\frac{1}{2} - \theta_0\right)^{-1} N^{\theta_0} \text{ and } \sum_{n \leq x} |a_n|^2 \leq \left(\frac{1}{2} - \theta_0\right)^{-1} x.$$

Then the lower bound for the number of zeros in $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)$ (δ being any constant with $\frac{1}{2} - \delta > \theta_0$) is $\geq C(\theta_0, \delta) T (\log T)(\log\log T)^{-1}$. But only when $\sum_{n \leq x} a_n = x + O_{\epsilon}(x^{\epsilon})$ we can prove that $N(\frac{1}{2} + \delta, T) \ll_{\delta} T$. Also if $\sum_{n \leq x} a_n = x + O((\log x)^{C_1})$ ($C_1 > 0$ being a constant we can prove

$$N\left(\frac{1}{2} + \frac{C(\log\log T)^2}{\log T}, T\right) \ll C^{-1} T (\log T)(\log\log T)^{-1}$$

for every fixed $C > 0$.

PROOF. Theorem 10 (below) gives $\gg T (\log\log T)^{-1}$ well-spaced Titchmarsh points on every line segment $(\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)$ with the lower bound $\gg T^{\delta}$ (δ being a constant subject to $\frac{1}{2} - \delta > \theta_0$).

THEOREM 10. (i) *Let $\{\lambda_n\}$ be as in the definition of* GDS. *This sequence will be further restricted by the condition* (vii) *or* (viii) *as the case may be. θ will denote a real constant.*

Let $f(x)$ and $g(x)$ be positive real valued functions defined in $x \geq 0$ satisfying

(ii) *$f(x)x^{\eta}$ is monotonic increasing and $f(x)x^{-\eta}$ is monotonic decreasing for every fixed $\eta > 0$ and all $x \geq x_0(\eta)$.*

(iii) $\lim_{x\to\infty}(g(x)x^{-1}) = 1$.

(iv) *For all $x \geq 0$, $g'(x)$ lies between two positive constants and $(g'(x))^2 - g(x)g''(x)$ lies between two positive constants (it being assumed that $g(x)$ is twice continuously differentiable for $x \geq x_0$).*

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers having the following properties.

(v) *$| b_n | (f(n))^{-1}$ lies between two positive constants (for all integers $n \geq n_0$) and $(\sum_{n\leq x} | a_n |^2)x^{-1}$ does not exceed a positive constant for all $x \geq 1$.*

(vi) *For all $X \geq 1$, $\sum_{X\leq n\leq 2X} | b_{n+1} - b_n | \ll f(X)$.*

We next assume that $\{a_n\}$ and $\{b_n\}$ satisfy at least one of the two following conditions.

(vii) **MONOTONICITY CONDITION.** *There exists an arithmetic progression A (of integers) such that*

$$\lim_{x\to\infty}\left(x^{-1}{\sum_{n\leq x}}' a_n\right) = h \quad (h \neq 0),$$

where the accent denotes the restriction of n to A. Also for every positive constant ν we have that $| b_n | \lambda_n^{-\nu}$ is monotonic decreasing for all $n(\geq n_0)$ in A.

(viii) **REAL PART CONDITION.** *There exists an arithmetic pro-*

gression $\mathcal{A}$ (of integers) such that

$$\liminf_{x\to\infty} \left(\frac{1}{x} \sum_{x\le\lambda_n\le 2x, Re\ a_n>0}{}' Re\ a_n \right) > 0$$

and

$$\lim_{x\to\infty} \left(\frac{1}{x} \sum_{x\le\lambda_n\le 2x, Re\ a_n<0}{}' Re\ a_n \right) = 0$$

where the accent denotes the restriction of n to $\mathcal{A}$.

(ix) *Finally we set $\lambda_n = g(n)$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $|\alpha_n|$ does not exceed a small positive constant (depending on other constants). We suppose that the* GDS

$$F(s) = \sum_{n=1}^{\infty} a_n b_n e^{2\pi i n\theta} (\lambda_n + \alpha_n)^{-s}$$

can be continued analytically in $(\sigma \ge \frac{1}{2} - \delta, T - \log T \le t \le 2T + \log T)$ and there $\log \max(|F(s)| + 100) \ll \log T$.

Then on every line segment $(\sigma = \frac{1}{2} - \delta_4, T \le t \le 2T)$ (δ_4 being a constant with $0 < \delta_4 \le \delta$) there are $\gg T(\log\log T)^{-1}$ well-spaced Titchmarsh points with the lower bound $\gg T^{\delta_4} f(T)$. If further we have

$$\frac{1}{T} \int_T^{2T} |F(\tfrac{1}{2} - \delta_4 + it)|^2\, dt \ll T^{2\delta_4} (f(T))^2$$

for every constant δ_4 (with $0 < \delta_4 \le \delta$), then the number of well-spaced Titchmarsh points on the line segment $(\sigma = \frac{1}{2} - \delta_4, T \le t \le 2T)$ (with the lower bound $\gg T^{\delta_4} f(T)$) is $\gg T$.

REMARK. This theorem is proved by R. Balasubramanian and K. Ramachandra in this form in the paper XIV[4] except that we have now to use $\sum_{n\le x} |a_n|^2 \ll x$ in place of $a_n = O(1)$ and also except that we have to involve θ. Lemmas necessary (see Lemma 6 of IV[2]) for these generalities and also the method have been developed in previous papers mentioned before by R. Balasubramanian and K. Ramachandra.

Finally we would like to mention paper XV[5] of this series of papers. Here we assume a functional equation of a very general type for a GDS and prove that a large class of $\{\alpha_n\}$ transformations of it have $\gg T$ well-spaced Titchmarsh points on every line segment $(\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)$ with a lower bound of the type $\gg_\varepsilon T^{m\delta-\varepsilon}$ where $m > 0$ is a real constant and $\varepsilon(> 0)$ is an arbitrary constant (for example for the zeta function of a ray class in an algebraic number field of degree m. If $m \geq 2$ we can allow $\sum_{n\leq x} |\alpha_n|^2 \ll x^{1+\varepsilon}$ in place of $\sum_{n\leq x} |A_n\alpha_n|^2 \ll x^{1+\varepsilon}$, see the definition in § 1 for the meaning of A_n). Note that if $A_n = O_\varepsilon(n^\varepsilon)$ then the condition on α_n is simply $\sum_{n\leq x} |\alpha_n|^2 \ll x^{1+\varepsilon}$. These results are very general. But out of these GDS only in very special cases (but still a somewhat large class of GDS) we can prove that

$$\frac{1}{T}\int_T^{2T} |F(\frac{1}{2}+\delta+it)|^2\, dt \ll_{\varepsilon,\delta} T^\varepsilon$$

for all $\delta > 0$ and $\varepsilon > 0$. Some examples (not already covered by Theorem 7) are (i) zeta function of any ray class of a quadratic field (ii) zeta function of a positive definite quadratic form $Q(X_1, \cdots X_\ell)$ (in $\ell \geq 2$ variables and with integer coefficients) namely $\sum_{n=1}^{\infty}\left(a_n n^{-\frac{\ell}{2}+1}\right) n^{-s}$, where a_n is the number of ℓ-tuples $(m_1, \cdots, m_\ell)$ of integers with $Q(m_1, \cdots, m_\ell) = n$. In this case $m = 1$ and the lower bound is $\gg T^\delta$(resp. $\gg T^{2\delta-\varepsilon}$) according as $\ell > 2$ or $\ell = 2$ see [8].

Instead of enumerating all the applications of this theory we state a beautiful theorem (namely Theorem 11 below). Many other theorems can be deduced in a similar manner by the interested readers from the results of papers mentioned above and the results of § 1, § 2 and § 3, (see also the post-script at the end of this paper).

THEOREM 11. *Let $\mathcal{F}$ denote the class of Dirichlet series of the form $\zeta(s) + \sum_{n=1}^{\infty} a_n n^{-s}$ with complex number sequence $\{a_n\}$ satisfying $\sum_{n\leq x} a_n = O(1)$. Let $\varphi_j = \varphi_j(s)(j = 0, 1, 2, \cdots, r)$ be any $r+1$ Dirichlet series (may* *not be distinct) of the class* $\mathcal{F}$. *Let* $P(X_0, X_1, \cdots, X_r)$ *be any fixed polynomial (with complex coefficients) of degree* d *(being the maximum of* $d_0+d_1+\cdots+d_r$ *taken over all monomials* $X_0^{d_0} X_1^{d_1} \cdots X_r^{d_r}$ *occuring in* $P(X_0, \cdots, X_r)$*). Let* Q *be defined by*

$$Q = (\varphi_0(s))^{d+1} - P(\varphi_0, \varphi_1, \cdots, \varphi_r) = \sum_{n=1}^{\infty} B_n n^{-s}, (\sigma > 1).$$

Then first we have $B_n \neq 0$ *for at least one* n *(also* Q *is analytic in* $\sigma > 0$, $t \geq 1$*). Next put*

$$F(s) = \sum B_n((n+\alpha_n)^{-s} - n^{-s}) + Q$$

where $\{\alpha_n\}$ *is any sequence of real numbers with* $|\alpha_n| \leq \frac{1}{3}$. *Then in* $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)$, $F(s)$ *has* $\gg T \log T$ *zeros and in* $\left(\sigma \geq \frac{1}{2} + \frac{C \log\log T}{\log T}, T \leq t \leq 2T\right)$ *only* $\ll C^{-1} T \log T$ *zeros (*$C \geq 1$ *being any constant).*

REMARK 1. If $d \geq 1$ we can allow $\sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\epsilon}$ in place of $|\alpha_n| \leq \frac{1}{3}$. But then we have to stipulate that $F(s)$ should be a GDS.

REMARK 2. That $B_n \neq 0$ for at least one n of course follows since Q has a pole of order $(d+1)$ at $s = 1$. But then we mention that the conclusion of Theorem 11 are valid for $\varphi_0(s) = (1 - 2^{1-s})\zeta(s)$ and $\varphi_j(s) = \sum_{n=1}^{\infty} a_n^{(j)} n^{-s} (j = 1, 2, \cdots, r)$ where $\max |\sum_{n \leq x} a_n^{(j)}| = O(1)$.

REFERENCES

[1] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *On the zeros of a class of generalised Dirichlet series*-III, J. Indian Math. Soc., 41 (1977), 301-315.

[2] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *On the zeros of a class of generalised Dirichlet series*-IV, J. Indian Math. Soc., 42 (1978), 135-142.

[3] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *On the zeros of a class of generalised Dirichlet series*-VI, Arkiv för Matematik, 19 (1981), 239-250.

[4] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *On the zeros of a class of generalised Dirichlet series*-XIV, Proc. Indian Acad. Sci., (Math. Sci.), 104 (1994), 167-176.

[5] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *On the zeros of a class of generalised Dirichlet series*-XV, Indag. Math., 5 (2) (1994), 129-144.

[6] K. RAMACHANDRA, *On the zeros of a class of generalised Dirichlet series*-V, J. Reine u. Angew. Math., 303/304, (1978), 295-313.

[7] K. RAMACHANDRA AND A. SANKARANARAYANAN, *On the zeros of a class of generalised Dirichlet series*-XVI, Math. Scand., 75 (1994), 178-184.

[8] K. RAMACHANDRA AND A. SANKARANARAYANAN, *Hardy's theorem for zeta-functions of quadratic forms*, (to appear) .

[9] E.C. TITCHMARSH, The theory of the Riemann zeta-function, (Revised and edited by D.R. HEATH-BROWN), Clarendon Press, Oxford (1986).

POST-SCRIPT

1. In view of Theorem 2, it is important to find Dirichlet series which satisfy (6). This will enable us to prove $N(\frac{1}{2}+\delta,T) \ll_\delta T(\delta > 0)$ for larger and larger class of GDS.

2. In view of Theorem 4, it is important to find N_0 (as large as possible) well-spaced Titchmarsh points with the lower bound $\geq T^{\,k\delta-\varepsilon}$ (for some $k>0$ and every $\varepsilon>0$) on the line segment $(\sigma=\frac{1}{2}-\delta, T\leq t\leq 2T)$ for a large class of Dirichlet series. In this direction we have Balasubramanian-Ramachandra functions given by Theorem 10 (Theorems 8 and 9 are special cases of these functions). Also we have the ℓth derivative ($\ell \geq 0$ integer) of a class of GDS which satisfy a very general functional equation (see equation (5) of XV[5]). The case $\ell=0$ is treated in XV[5] and it is proved that $N_0 \gg T$. We can cover all integers ℓ as follows. We make use of the following lemma.

LEMMA 2. *Let $h(x)$ be an n-times continuously differentiable function defined in $a_0 \leq x \leq a_0+nd_0$, where $a_0>0, d_0>0$ are constants and n is any fixed integer ≥ 1. Then*

$$\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}h(a_0+rd_0)=\int_0^{d_0}\cdots\int_0^{d_0}h(a_0+u_1+\cdots+u_n)du_1du_2\cdots du_n.$$

PROOF. Follows by trivial induction.

We apply this lemma to $h(\sigma)=h(\sigma,t)=\chi(\sigma+it)$ of equation (5) of XV[5] and obtain $|\chi^{(\ell)}(s_0)| \gg T^{\,k(\frac{1}{2}-\sigma)}(\log T)^{\ell}$ for any fixed $t(T\leq t\leq 2T)$ and a suitable $s_0=\sigma_0+it$ (with σ_0 at a distance of $O((\log T)^{-1})$ from any arbitrarily given σ). At the same time for all s and ℓ we have (by Cauchy's theorem), $|\chi^{(\ell)}(s)| \ll T^{\,k(\frac{1}{2}-\sigma)}(\log T)^{\ell}$.

Next we apply local convexity (see for example the references [PS-1] and [PS-2] below, see especially Theorem 6-C of [PS-2] for a correction in [PS-1]) to the zeta-function like analytic function $Z^{(\ell)}(s)(\chi(s))^{-1}(\log T)^{-\ell}$ to prove that the integral of its absolute value taken over $|t-t_0|\leq C(\varepsilon)$ on $\sigma=\frac{1}{2}+\delta$ exceeds $t_0^{-\varepsilon}(T\leq t_0\leq 2T)$, where $C(\varepsilon)$ depends only on ε. From this it follows that for $Z^{(\ell)}(s)$ we have $N_0 \gg T$ and the lower bound is $\geq T^{k\delta-\epsilon}$.

3. Next given (arbitrarily) N_0 well-spaced points on $(\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)$ we can sometimes obtain a subset (of these points) of cardinality $\gg N_0$ ($= T$, sometimes $T(loglog\ T)^{-1}$) Titchmarsh points for a class of Dirichlet series or GDS. But this class of Dirichlet series is a very restricted one. Let $Z(s)$ be a Dirichlet series (see equation (5) of XV[5]) which have

(a) Euler product for $Z_1(s)$.

(b) Functional equation with $1 \leq k \leq 2$.

(c) Mean-square on the critical line (see equation (6) of the present paper) $\sigma = \frac{1}{2}$.

(We have to mention that (c) follows from (b))

(d) $|\chi^{(\ell)}(s)| \asymp t^{k(\frac{1}{2}-\sigma)}(log\ t)^{\ell}$ for all integers $\ell \geq 0$.

From these we can deduce.

LEMMA 3. *Let $\{t_j\}(T \leq t_j \leq 2T)$ be a well-spaced set of points with cardinality $\gg T$. Then out of these points we can select a subset of points t'_j (with cardinality $\gg T$) satisfying*

$$|Z_1(\frac{1}{2} + \delta + it'_j)| \gg 1 \quad and \quad |Z_1^{(\ell)}(\frac{1}{2} + \delta + it'_j)| \ll 1$$

($\ell = 1, 2, \cdots, \ell_0$) where $\ell_0 \geq 1$ is any integer.

PROOF. This lemma is contained implicitly in the proof of Theorem 1 of [PS-3].

From these we can formulate a general principle.

GENERAL PRINCIPLE. *In Theorem 11 we can replace $(\varphi_0(s))^{d+1}$ by $Q_1 \equiv (F_1(s))^{M_1}(\varphi^{(\ell_1)}(s))^{M_2}$ with integers $\ell_1 \geq 0, M_1 \geq 0, M_2 \geq 0, M_1 + M_2 \geq 1$, where $F_1(s)$ is a power product (with non-negative integral exponents) of derivatives of functions like $Z(s)$ satisfying* (a),(b),(c) *and* (d) *above and $\varphi(s)$ is either a Balasubramanian-Ramachandra function or a function which has a functional equation such as* (5) *of* XV[5]. *In place of*

$P(\varphi_0, \varphi_1, \cdots, \varphi_r)$ of Theorem 11, we can have a suitable modification say Q_2 such that $Q_1 - Q_2$ has $\gg T$(resp. $T(loglog\ T)^{-1}$) well-spaced Titchmarsh points on $\sigma = \frac{1}{2} - \delta$. Accordingly we have lower bounds for the number of zeros of $Q_1 - Q_2$ in $(\sigma \geq \frac{1}{2} - 2\delta, T \leq t \leq 2T)$ (and upper bounds for $N(\frac{1}{2}+\delta, T)$ only sometimes). We can say similar things about the $\{\alpha_n\}$ transformations of $Q_1 - Q_2$.

REFERENCES ADDED TO POST-SCRIPT

[PS-1] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *Some local convexity theorems for the zeta-function like analytic functions*-I, Hardy-Ramanujan J., Vol. 11 (1981), 1-12.

[PS-2] R. BALASUBRAMANIAN AND K. RAMACHANDRA, *Some local convexity theorems for the zeta-function like analytic functions*-II, Hardy-Ramanujan J., Vol. 20 (1997), 2-11.

[PS-3] K. RAMACHANDRA AND A. SANKARANARAYANAN, *Notes on the Riemann zeta-function*, J. Indian Math. Soc., Vol. 57 (1991), 67-77.

ADDRESSES OF AUTHORS

1. PROFESSOR R. BALASUBRAMANIAN
MATSCIENCE
THARAMANI P.O.
MADRAS 600 0113, INDIA
e-mail : BALU@IMSC.ERNET.IN

2. PROFESSOR K. RAMACHANDRA
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
BOMBAY 400 005
e-mail : KRAM@TIFRVAX.TIFR.RES.IN

3. DR. A. SANKARANARAYANAN
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
BOMBAY 400 005
e-mail : SANK@TIFRVAX.TIFR.RES.IN

MANUSCRIPT COMPLETED ON 11th OCTOBER 1995.