On the zeros of a class of generalised Dirichlet series-XVIII (a few remarks on littlewood’s theorem and Totchmarsh points)

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ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES-XVIII

(A FEW REMARKS ON LITTLEWOOD'S THEOREM AND TITCHMARSH POINTS)

BY

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(DEDICATED TO PROFESSOR K. CHANDRASEKHARAN
ON HIS SEVENTY-FIFTH BIRTHDAY)

§ 1. INTRODUCTION AND NOTATION. This paper was necessitated because we observed that the proofs of the results of the paper XVI[7] can be simplified and that the results therein can at the same time be generalised. (In § 1, § 2 and § 3 we prove six theorems in all. For an attractive application of these see Theorem 11 of § 4). We write \( s = \sigma + it \) as usual. We begin by stating a generalisation of Theorem 9.15 (A) (on page 230 of [9]). We need some definitions. (We fix two positive constants \( a \) and \( b \) with \( a < b \) throughout). The parameter \( T \) will be assumed to exceed a large positive constant.

GENERALISED DIRICHLET SERIES (GDS). Let \( \{\lambda_n\} \) be a sequence of real numbers with \( a < \lambda_1 < \lambda_2 < \cdots, \lambda_1 < b \) and \( a \leq \lambda_{n+1} - \lambda_n \leq b \) for \( n \geq 1 \). Let \( \{A_n\} \) be any sequence of complex numbers such that \( A_1 \neq 0 \)
and
\[ Z(s) = \sum_{n=1}^{\infty} A_n \lambda_n^{-s}. \] (1)

converges for some complex \( s = s_0 \). Then \( Z(s) \) is called a generalised Dirichlet series (GDS). We remark that if \( Z(s) \) converges at \( s = s_0 \), then it is absolutely convergent at \( s = s_0 + 2 \). Note that a GDS is different from zero if real part of \( s(Re s) \) exceeds a certain constant. In fact as \( Re s \to \infty, | Z(s) | \) tends to a non-zero constant. A GDS is said to be a normalised generalised Dirichlet series (NGDS) if \( \sum_{n \leq x} | A_n |^2 \ll x^{1+\epsilon} \) for every \( \epsilon > 0 \). A GDS is said to be a Dirichlet series if \( \{ \lambda_n \} \) is a subsequence of the sequence of natural numbers.

\{\alpha_n\} TRANSFORMATION OF AN NGDS. Let \( Z(s) \) be an NGDS. We consider only such sequences \( \{\alpha_n\} \) of real numbers which satisfy
\[ \sum_{n \leq x} | A_n \alpha_n |^2 \ll x^{1+\epsilon} \] for every \( \epsilon > 0 \). \( F(s) = \sum_{n=1}^{\infty} A_n (\lambda_n + \alpha_n)^{-s} \) is said to be an \( \{\alpha_n\} \) transformation of \( Z(s) \) if \( F(s) \) is a GDS. Note that
\[ F(s) = D(s) + Z(s) \] where \( D(s) = \sum_{n=1}^{\infty} A_n ((\lambda_n + \alpha_n)^{-s} - \lambda_n^{-s}) \). (2)

and that \( D(s) \) is analytic in \( \sigma > 0 \). Moreover we have

LEMMA 1. For \( \sigma > 0 \) and every \( \epsilon > 0 \) we have \( D(s) = O_\sigma((| t | + 2)^2) \) and also
\[ \frac{1}{T} \int_T^{2T} \left| D(\sigma + it) \right|^2 dt \ll \max \left( T^{2(\frac{1}{2} - \sigma) + \epsilon}, T^\epsilon \right). \] (3)

PROOF. See Theorems 7 and 7' of XV[8].

THEOREM 1 (J.E. LITTLEWOOD). Let \( Z(s) \) be a GDS which can be continued analytically in \( (\sigma \geq \frac{1}{2} - \delta_0, T - \log T \leq t \leq 2T + \log T) \), where \( \delta_0(> 0) \) is a constant, and there \( \log(\max \left| Z(s) \right| + 100) \) is \( \ll \log T \). Let \( Z(s) \to 1 \) as \( Re s \to \infty \). For \( \alpha \geq \frac{1}{2} \) let \( N(\alpha, T) \) denote the number of zeros of \( Z(s) \) in \( (\sigma \geq \alpha, T \leq t \leq 2T) \). Then (for \( \sigma_0 > \frac{1}{2} \)) we have
\[ 2\pi \int_{\sigma_0}^{\infty} N(\sigma, T) d\sigma = \int_T^{2T} \log | Z(\sigma_0 + it) | dt + O(\log T). \] (4)
and hence

$$N\left(\frac{1}{2} + 2\delta, T\right) \leq (g\delta)^{-1} T \log \left(\frac{1}{T} \int_T^{2T} \left| Z\left(\frac{1}{2} + \delta + it\right) \right|^g \, dt\right) + O(\log T) \quad (5)$$

holds uniformly for all real positive constants $g$ and $\delta$. If $\delta$ is any fixed constant we may take $\delta_0 = 0$ and then replace $O(\log T)$ by $O_\delta(\log T)$.

**REMARK.** This theorem is essentially due to J.E. Littlewood, since the special case $g = 2$ and $Z(s) = \zeta(s)$ (due to J.E. Littlewood) is dealt with on pages 229 and 230 of [9]. The general case stated as Theorem 1 above follows by a trivial generalisation of Littlewood’s method. If we do not assume $Z(s) \to 1$ as $\Re s \to \infty$, we have to replace $O(\log T)$ by $O(T)$ in (4) and (5). This does not matter for our purposes.

§ 2. A COROLLARY TO THEOREM 1.

**THEOREM 2.** Let $r \geq 1$ be any integer constant and let $\varphi_1(s), \varphi_2(s), \ldots, \varphi_r(s)$ be $r$ Dirichlet series each of which is continuuable analytically in $(\sigma \geq \frac{1}{2} - \delta_0, T - \log T \leq t \leq 2T + \log T)$ and there $\log(\max_j |\varphi_j(s)| + 100) \ll \log T$. Suppose further that

$$\max_j \left(\frac{1}{T} \int_{T - \log T}^{2T + \log T} \left| \varphi_j\left(\frac{1}{2} + it\right) \right|^2 \, dt\right) \ll T^\varepsilon \quad (6)$$

holds for every $\varepsilon > 0$. Let $P(X_1, \ldots, X_r)$ be any fixed polynomial (with complex coefficients) such that when we put $X_j = \varphi_j(s)(i = 1, 2, \ldots, r), P = P(s) = P(X_1, \ldots, X_r)$ is a normalised Dirichlet series. Let $F(s)$ be any $\{\alpha_n\}$ transformation of $P(s)$. Then the function $N(\sigma, T)$ defined (as before) for $F(s)$ satisfies

$$N(\sigma, T) \ll_{\sigma} T \quad (\sigma > \frac{1}{2}). \quad (7)$$

**REMARK.** We define the degree of a monomial $X_1^{d_1} \cdots X_r^{d_r}$ to be $d_1 + \cdots + d_r$ and the degree of $P(X_1, \ldots, X_r)$ to be the maximum of $d_1 + \cdots + d_r$ taken over all monomials occurring in $P(X_1, \ldots, X_r)$. If the degree of $P$ is 1 then we can allow each $\varphi_j(s)$ to be a GDS. Then $P$ has to be an NGDS.
PROOF. The proof follows from the fact that (6) implies

$$\max_j \left( \frac{1}{T} \int_T^{2T} | \varphi_j(\sigma + it) |^2 \, dt \right) \ll_{\sigma} 1 \quad (\sigma > \frac{1}{2}),$$

and that for a suitable small constant \(g > 0\) we have

$$| F(s) |^2 \ll | D(s) |^2 + | \varphi_1(s) |^2 + \cdots + | \varphi_r(s) |^2 + 1.$$  

Note that in view of Lemma 1 it is not hard to deduce that

$$\frac{1}{T} \int_T^{2T} | D(\sigma + it) |^2 \, dt \ll_{\sigma} 1 \quad (\sigma > \frac{1}{2}).$$

From these facts Theorem 2 follows from Theorem 1.

§ 3. TITCHMARSH POINTS. Let \(F(s)\) be a GDS continuable analytically in \((\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)\) and there \(\log(\max | F(s) | +100) \ll \log T\). A point \(s_0 = \sigma_0 + it_0\) in \((\sigma \geq \beta + \delta_1, T \leq t \leq 2T)\), where \(\delta_1 > 0\) is a constant, is said to be a Titchmarsh point with the lower bound \(T\ \ell^\ell\) for \(|F(s)|\) if \(\ell(>0)\) is bounded below independent of \(T\) and \(t_0\).

THEOREM 3. If \(s_0 = \sigma_0 + it_0\) (with \(F(s)\) as above) is a Titchmarsh point of \(F(s)\), then the region \((\sigma \geq \beta, | t - t_0 | \leq \delta_2)\) where \(\delta_2(>0)\) is any small constant, contains \(\gg \log T\) zeros of \(F(s)\).

PROOF. For the proof of this theorem due to R. Balasubramanian and K. Ramachandra see Theorem 3 of III[1]. It should be mentioned that this theorem is not too-trivial a generalisation of Theorem 9.14 (on page 227 of [9]) due to E.C. Titchmarsh.

WELL-SPACED POINTS. The points \(s^{(q)} = \sigma_q + qt_q\) \((q = 1, 2, \cdots)\) in the complex plane are said to be well-spaced if \(| s^{(q)} - s^{(q')} |\) is bounded below for all pairs \((q,q')\) with \(q \neq q'\).

THEOREM 4. If there are \(N_0\) well-spaced Titchmarsh points for \(F(s)\) \((F(s)\ as \ in \ Theorem \ 3)\), then \(F(s)\) has \(\gg N_0 \log T\) zeros in \((\sigma \geq \beta, T \leq t \leq 2T)\).

PROOF. The proof follows from the fact that \(|F(s)|\) tends to a non-zero
THEOREM 5. Let $\beta < \frac{1}{2}$ be a constant and $r \geq 1$ any integer constant and $\varphi_1(s), \cdots, \varphi_r(s)$ be $r$ Dirichlet series each of which is continu-able analytically in $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$ and there $\log \max(|\varphi_j(s)| + 100)$ is $\ll \log T$. Suppose further that for $j = 1, 2, \cdots, r$ and $\sigma \geq \beta$, we have

$$\frac{1}{T} \int_{T-\log T}^{2T+\log T} |\varphi_j(\sigma + it)|^2 \, dt \ll_{\varepsilon} \max \left( T^{2m_j(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon \right)$$

where $m_j > 0$ are constants. Let $\mu > 0$ be a constant. Put $X_0 = T^{\mu(\frac{1}{2}-\sigma)-\varepsilon}$. Let $X_0^{d_0} X_1^{d_1} \cdots X_r^{d_r}$ ($d_j \geq 0$ integers, $j = 0, 1, 2, \cdots, r$) be any fixed monomial in $X_0, X_1, \cdots, X_r$. Let the weighted $\mu$-degree $d(\mu)$ of the monomial be defined as $\mu d_0 + m_1 d_1 + \cdots + m_r d_r$. Put $Q_0(s) = X_0^{\mu_0(\varphi_1(s))^{d_1} \cdots (\varphi_r(s))^{d_r}}$. Then given any well-spaced set of points $\{s_q\}$ with $s_q = \sigma + it_q (q = 1, 2, \cdots; \sigma \geq \beta + \delta_3, T \leq t_q \leq 2T)$ where $\delta_3 > 0$ is a small constant we have

$$|Q_0(\sigma + it_q)| \ll \max \left( T^{d(\mu)(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon \right)$$

except for $O(T^{1-\varepsilon})$ values of $q$.

REMARK. If $\sum_{j=1}^{r} d_j = 1$, then we can allow $\varphi_j(s)(j = 1, 2, \cdots, r)$ to be GDS.

PROOF. We use the fact that the value $|\varphi_j(s_q)|$ of $\varphi_j(s)$ is majorised by the mean value over a disc (with $s_q$ as centre and $\varepsilon$ as radius) of $|\varphi_j(s)|$. We choose a small radius and sum over all the discs taking $s_q$ to be $\sigma + it_q$. We obtain

$$\frac{1}{T} \sum_{q} |\varphi_j(s_q)|^2 \ll \max \left( T^{2m_j(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon \right).$$

Hence $|\varphi_j(s_q)| > \max(T^{m_j(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon)$ is possible for at most $O(T^{1-\varepsilon})$ values of $q$. We next sum over all $j$ and obtain the result.

THEOREM 6. Let $\beta < \frac{1}{2}$ be a constant and let $\varphi_0(s)$ be a Dirichlet series continu-able analytically in $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$ and there
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log max(|φ₀(s)| + 100) ≪ log T. Suppose that it has ≥ T(resp. T(loglog T)⁻¹)
well-spaced Titchmarsh points {σ + itₙ} (where σ is any constant with β <
σ < ½) with the lower bound Tμ(½−σ)−ε where μ(> 0) is a constant. Let
Q(X₀, X₁, · · · , Xᵣ) be a fixed polynomial (with complex coefficients) such that
for some positive integer M, the maximum of d(μ) (defined in Theorem 5)
taken over all the monomials X₀ᵈ₀ X₁ᵈ₁ · · · Xᵣᵈᵣ occurring in Q(X₀, X₁, · · · , Xᵣ)
is less than Mμ. Put Q(s) = Q(φ₀(s), φ₁(s), · · · , φᵣ(s))) (where φᵢ(s) j =
1, 2, · · · , are as in Theorem 5). Assume that (φ₀(s))M − Q(s) is an NGDS,
and let F(s) be its {αₙ} transformation. Then F(s) has ≥ T log T(resp.
T(log T)(loglog T)−¹) zeros in (σ ≥ β, T − t ≤ 2T).

PROOF. Follows from |(φ₀(s))M − Q(s)| ≥ |φ₀(s)|^M (1 − |Q(s)|
| |φ₀(s)|^−M).

REMARK. Note that none of the functions φ₀(s), (φ₀(s))^M and Q(s) need
be normalised Dirichlet series. If M = 1, then Q(s) does not involve φ₀(s)
(which can now be taken to be a GDS). If M = 1 and Q(X₀, X₁, · · · , Xᵣ)
(now independent of X₀) is linear in X₁, · · · , Xᵣ (i.e. ∑dᵢ ≤ 1 for every
monomial X₁ᵈ₁ · · · Xᵣᵈᵣ occurring in Q(X₀, X₁, · · · , Xᵣ) and equality holds for
at least one monomial) then all of φ₁(s), · · · , φᵣ(s) can be taken to be GDS.
Also Q(X₀, X₁, · · · , Xᵣ) can be a constant.

§ 4. SOME APPLICATIONS OF THEOREMS 2 TO 6. Theorems
2 to 6 are only easy formalisms. These would be completely uninteresting
without examples. Finding examples is a difficult task. For example we do
not know how to prove the expected result N(σ, T) ≪ T(σ > ½) for the
abelian L-series of an algebraic number field. However we have a somewhat
general theorem namely.

THEOREM 7. Let {λₙ}(n = 1, 2, · · · ) be a sequence of real numbers as
in the definition of GDS. Let |∑ₙ≤x aₙ| ≤ B(x), ∑ₙ≤x |aₙ|^2 ≤ xB(x) and
∑ₙ≤x |∑ₙ≤x aₙ|^2 ≤ xB(x), where B(x) depends on x. If B(x) ≪ x^ε (for every
If \( \varepsilon > 0 \), then \( Z(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \) converges uniformly over compact subsets of \( \sigma > 0 \) and hence is analytic there. We have

\[
N\left(\frac{1}{2} + \delta, T\right) \ll \delta T \quad (\delta > 0).
\]

If further \( \log B(z) \ll \log \log x \) then we have

\[
N\left(\frac{1}{2} + \delta, T\right) \ll \delta^{-1} T \log(\delta^{-1})
\]

uniformly for \( 0 < \delta \leq \frac{1}{2} \).

**REMARK.** Results like

\[
\frac{1}{T} \int_{-T}^{T} \left| Z\left(\frac{1}{2} + it\right) \right|^2 \, dt \leq \varepsilon T^{\varepsilon}
\]

for every \( \varepsilon > 0 \) and more general and powerful results have been proved in paper V[6]. Results like (15) imply (13) and (14). If \( \{Z(s)\} \) is any finite set of Dirichlet series each subject to (15) we can apply Theorem 2.

We now turn to series of the type

\[
\sum_{n=1}^{\infty} a_n b_n e^{2\pi i n \theta} \lambda_n^{-s} \quad (\theta \text{ a real constant}),
\]

their analytic continuations and their Titchmarsh points. Investigations dealing with such series were carried out in a series of papers by R. Balasubramanian and K. Ramachandra (see III[1], IV[2], V[6], VI[3], XIV[4] and also the paper [8] by K. Ramachandra and A. Sankaranarayanan). The paper XIV[4] is nearly final. In paper XIV[4] the condition \( a_n = O(1) \) is assumed. This can be relaxed to \( \sum_{n \leq x} |a_n|^2 = O(x) \). This last mentioned condition on \( a_n \) will be assumed in the rest of this paper.

Lest we get lost in generalities we state two special cases first.

**THEOREM 8.** Let \( \theta_0 (0 < \theta_0 < \frac{1}{2}) \) be a constant and let \( \{a_n\} \) be a sequence of complex numbers satisfying the inequality

\[
|\sum_{m=1}^{N} a_m - N| \leq (\frac{1}{2} - \theta_0)^{-1}
\]

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$N = 1, 2, 3, \ldots$. Also for $n = 1, 2, 3, \ldots$, let $\alpha_n$ be real and $|\alpha_n| \leq C(\theta_0)$ where $C(\theta_0)$ is a certain (small) constant depending only on $\theta_0$. Then the number of zeros of the function

$$\sum_{n=1}^{\infty} a_n (n + \alpha_n)^{-s} = \zeta(s) + \sum_{n=1}^{\infty} (a_n (n + \alpha_n)^{-s} - n^{-s})$$

in the rectangle ($\lfloor \sigma - \frac{1}{2} \rfloor \leq \delta, T \leq t \leq 2T$) is $\geq C(\theta_0, \delta)T \log T$ where $C(\theta_0, \delta)$ is a positive constant depending only on $\theta_0$ and $\delta$, and $T \geq T_0(\theta_0, \delta)$ a large positive constant.

**PROOF.** Theorem 10 (below) gives $\gg T$ well spaced Titchmarsh points on every line segment ($\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T$) with the lower bound $\gg T^5$, while actually (14) gives

$$N \left( \frac{1}{2} + \frac{C \log \log T \log T}{\log T} \right) \ll C^{-1} T \log T$$

for every fixed $C(> 0)$. (It is not hard to prove the required mean-square upper bound for the function).

**THEOREM 9.** In the above theorem we can relax the condition on $a_n$ to

$$\left| \sum_{m=1}^{N} a_m - N \right| \leq \left( \frac{1}{2} - \theta_0 \right)^{-1} N^{\theta_0} \quad \text{and} \quad \sum_{n \leq x} \left| a_n \right|^2 \leq \left( \frac{1}{2} - \theta_0 \right)^{-1} x.$$

Then the lower bound for the number of zeros in $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)$ ($\delta$ being any constant with $\frac{1}{2} - \delta > \theta_0$) is $\geq C(\theta_0, \delta)T(\log T)(\log \log T)^{-1}$. But only when $\sum_{n \leq x} a_n = x + O(x^e)$ we can prove that $N(\frac{1}{2} + \delta, T) \ll \delta T$. Also if

$$\sum_{n \leq x} a_n = x + O((\log x)^{C_1}) \quad (C_1 > 0 \text{ being a constant})$$

we can prove

$$N \left( \frac{1}{2} + \frac{C(\log \log T)^{2}}{\log T} \right) \ll C^{-1} T(\log T)(\log \log T)^{-1}$$

for every fixed $C > 0$.

**PROOF.** Theorem 10 (below) gives $\gg T (\log \log T)^{-1}$ well-spaced Titchmarsh points on every line segment ($\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T$) with the lower
THEOREM 10. (i) Let \{\lambda_n\} be as in the definition of GDS. This sequence will be further restricted by the condition (vii) or (viii) as the case may be. \theta will denote a real constant.

Let \( f(x) \) and \( g(x) \) be positive real valued functions defined in \( x \geq 0 \) satisfying

(ii) \( f(x)x^n \) is monotonic increasing and \( f(x)x^{-\eta} \) is monotonic decreasing for every fixed \( \eta > 0 \) and all \( x \geq x_0(\eta) \).

(iii) \( \lim_{x \to \infty} (g(x)x^{-1}) = 1. \)

(iv) For all \( x \geq 0, g'(x) \) lies between two positive constants and \( (g'(x))^2 - g(x)g''(x) \) lies between two positive constants (it being assumed that \( g(x) \) is twice continuously differentiable for \( x \geq x_0 \)).

Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of complex numbers having the following properties.

(v) \( |b_n| (f(n))^{-1} \) lies between two positive constants (for all integers \( n \geq n_0 \)) and \( \left( \sum_{n \leq x} |a_n|^2 \right) x^{-1} \) does not exceed a positive constant for all \( x \geq 1. \)

(vi) For all \( X \geq 1, \sum_{X \leq n \leq 2X} |b_{n+1} - b_n| \ll f(X). \)

We next assume that \( \{a_n\} \) and \( \{b_n\} \) satisfy at least one of the two following conditions.

(vii) **MONOTONICITY CONDITION.** There exists an arithmetic progression \( A \) (of integers) such that

\[
\lim_{x \to \infty} \left( x^{-1} \sum_{n \leq x} \hat{a}_n \right) = h \quad (h \neq 0),
\]

where the accent denotes the restriction of \( n \) to \( A \). Also for every positive constant \( \nu \) we have that \( |b_n| \lambda_n^{-\nu} \) is monotonic decreasing for all \( n(\geq n_0) \) in \( A \).

(viii) **REAL PART CONDITION.** There exists an arithmetic pro-
gession $\mathcal{A}$ (of integers) such that
\[
\liminf_{x \to \infty} \left( \frac{1}{x} \sum_{x \leq \lambda_n \leq 2x, \Re a_n > 0} \Re a_n \right) > 0
\]
and
\[
\lim_{x \to \infty} \left( \frac{1}{x} \sum_{x \leq \lambda_n \leq 2x, \Re a_n < 0} \Re a_n \right) = 0
\]
where the accent denotes the restriction of $n$ to $\mathcal{A}$.

(ix) Finally we set $\lambda_n = g(n)$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $|\alpha_n|$ does not exceed a small positive constant (depending on other constants). We suppose that the GDS
\[
F(s) = \sum_{n=1}^{\infty} a_n b_n e^{2\pi i n\theta} (\lambda_n + \alpha_n)^{-s}
\]
can be continued analytically in $(\sigma \geq \frac{1}{2} - \delta, T - \log T \leq t \leq 2T + \log T)$ and there $\log \max(|F(s)| + 100) \ll \log T$.

Then on every line segment $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$ ($\delta_4$ being a constant with $0 < \delta_4 \leq \delta$) there are $\gg T((\log \log T)^{-1}$ well-spaced Titchmarsh points with the lower bound $\gg T^{\delta_4} f(T)$. If further we have
\[
\frac{1}{T} \int_{T}^{2T} |F(\frac{1}{2} - \delta_4 + it)|^2 \, dt \ll T^{2\delta_4} (f(T))^2
\]
for every constant $\delta_4$ (with $0 < \delta_4 \leq \delta$), then the number of well-spaced Titchmarsh points on the line segment $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$ (with the lower bound $\gg T^{\delta_4} f(T)$) is $\gg T$.

REMARK. This theorem is proved by R. Balasubramanian and K. Ramachandra in this form in the paper XIV[4] except that we have now to use $\sum_{n \leq x} |a_n|^2 \ll x$ in place of $a_n = O(1)$ and also except that we have to involve $\theta$. Lemmas necessary (see Lemma 6 of IV[2]) for these generalities and also the method have been developed in previous papers mentioned before by R. Balasubramanian and K. Ramachandra.
Finally we would like to mention paper XV[6] of this series of papers. Here we assume a functional equation of a very general type for a GDS and prove that a large class of \( \{\alpha_n\} \) transformations of it have \( \gg T \) well-spaced Titchmarsh points on every line segment \( (\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T) \) with a lower bound of the type \( \gg \varepsilon T^{m\delta - \varepsilon} \) where \( m > 0 \) is a real constant and \( \varepsilon(> 0) \) is an arbitrary constant (for example for the zeta function of a ray class in an algebraic number field of degree \( m \)). If \( m \geq 2 \) we can allow \( \sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\varepsilon} \) in place of \( \sum_{n \leq x} |A_n\alpha_n|^2 \ll x^{1+\varepsilon} \), see the definition in § 1 for the meaning of \( A_n \). Note that if \( A_n = O_x(n^\delta) \) then the condition on \( \alpha_n \) is simply \( \sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\varepsilon} \). These results are very general. But out of these GDS only in very special cases (but still a somewhat large class of GDS) we can prove that

\[
\frac{1}{T} \int_T^{2T} |F(\frac{1}{2} + \delta + it)|^2 dt \ll \varepsilon, \delta T^\varepsilon
\]

for all \( \delta > 0 \) and \( \varepsilon > 0 \). Some examples (not already covered by Theorem 7) are (i) zeta function of any ray class of a quadratic field (ii) zeta function of a positive definite quadratic form \( Q(X_1, \cdots X_\ell) \) (in \( \ell \geq 2 \) variables and with integer coefficients) namely \( \sum_{n=1}^\infty \left( a_n n^{-\frac{1}{2} + 1} \right) n^{-\sigma} \), where \( a_n \) is the number of \( \ell \)-tuples \( (m_1, \cdots, m_\ell) \) of integers with \( Q(m_1, \cdots, m_\ell) = n \). In this case \( m = 1 \) and the lower bound is \( \gg T^{\delta} \) (resp. \( \gg T^{2\delta - \varepsilon} \)) according as \( \ell > 2 \) or \( \ell = 2 \) see [8].

Instead of enumerating all the applications of this theory we state a beautiful theorem (namely, Theorem 11 below). Many other theorems can be deduced in a similar manner by the interested readers from the results of papers mentioned above and the results of § 1, § 2 and § 3, (see also the post-script at the end of this paper).

**THEOREM 11.** Let \( F \) denote the class of Dirichlet series of the form

\[
\zeta(s) + \sum_{n=1}^\infty a_n n^{-s} \text{ with complex number sequence } \{a_n\} \text{ satisfying } \sum_{n \leq x} a_n = O(1).
\]

Let \( \varphi_j = \varphi_j(s) (j = 0, 1, 2, \cdots, r) \) be any \( r + 1 \) Dirichlet series (may
not be distinct) of the class $F$. Let $P(X_0, X_1, \ldots, X_r)$ be any fixed polynomial (with complex coefficients) of degree $d$ (being the maximum of $d_0+d_1+\cdots+d_r$ taken over all monomials $X_0^{d_0}X_1^{d_1}\cdots X_r^{d_r}$ occurring in $P(X_0, \ldots, X_r)$). Let $Q$ be defined by
\[ Q = (\varphi_0(s))^{d+1} - P(\varphi_0, \varphi_1, \ldots, \varphi_r) = \sum_{n=1}^{\infty} B_n n^{-z}, (\sigma > 1). \]

Then first we have $B_n \neq 0$ for at least one $n$ (also $Q$ is analytic in $\sigma > 0$, $t \geq 1$). Next put
\[ F(s) = \sum B_n ((n + \alpha_n)^{-z} - n^{-z}) + Q \]
where \( \{\alpha_n\} \) is any sequence of real numbers with \( |\alpha_n| \leq \frac{1}{3} \). Then in \( (\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T) \), $F(s)$ has $\gg T \log T$ zeros and in \( (\sigma \geq \frac{1}{2} + \frac{C \log \log T}{\log T}, T \leq t \leq 2T) \) only $\ll C^{-1} T \log T$ zeros ($C \geq 1$ being any constant).

**Remark 1.** If $d \geq 1$ we can allow $\sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\varepsilon}$ in place of $|\alpha_n| \leq \frac{1}{3}$. But then we have to stipulate that $F(s)$ should be a GDS.

**Remark 2.** That $B_n \neq 0$ for at least one $n$ of course follows since $Q$ has a pole of order $(d+1)$ at $s = 1$. But then we mention that the conclusion of Theorem 11 are valid for $\varphi_0(s) = (1 - 2^{1-s})\zeta(s)$ and $\varphi_j(s) = \sum_{n=1}^{\infty} a_n^{(j)} n^{-z} (j = 1, 2, \ldots, r)$ where $\max \sum_{n \leq x} n^{-z} = O(1)$. 
REFERENCES


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1. In view of Theorem 2, it is important to find Dirichlet series which satisfy (6). This will enable us to prove \( N(\frac{1}{2} + \delta, T) \ll \delta T(\delta > 0) \) for larger and larger class of GDS.

2. In view of Theorem 4, it is important to find \( N_0 \) (as large as possible) well-spaced Titchmarsh points with the lower bound \( \geq T^{k+\varepsilon} \) (for some \( k > 0 \) and every \( \varepsilon > 0 \)) on the line segment \( (\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T) \) for a large class of Dirichlet series. In this direction we have Balasubramanian-Ramachandra functions given by Theorem 10 (Theorems 8 and 9 are special cases of these functions). Also we have the \( \ell \)th derivative \((\ell \geq 0 \text{ integer})\) of a class of GDS which satisfy a very general functional equation (see equation (5) of XVII). The case \( \ell = 0 \) is treated in XVII and it is proved that \( N_0 \gg T \). We can cover all integers \( \ell \) as follows. We make use of the following lemma.

**LEMMA 2.** Let \( h(x) \) be an \( n \)-times continuously differentiable function defined in \( x_0 \leq x \leq x_0 + nd_0 \), where \( x_0 > 0, d_0 > 0 \) are constants and \( n \) is any fixed integer \( \geq 1 \). Then

\[
\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} h(a_0 + rd_0) = \int_0^{d_0} \cdots \int_0^{d_0} h(a_0 + u_1 + \cdots + u_n) du_1 du_2 \cdots du_n.
\]

**PROOF.** Follows by trivial induction.

We apply this lemma to \( h(x) = h(x,t) = \chi(x + it) \) of equation (5) of XVII and obtain \( |\chi^{(\ell)}(s_0)| \gg T^{k(\frac{1}{2} - \sigma)}(\log T)^{\ell} \) for any fixed \( t(T \leq t \leq 2T) \) and a suitable \( s_0 = \sigma_0 + it \) (with \( \sigma_0 \) at a distance of \( O((\log T)^{-1}) \) from any arbitrarily given \( \sigma \)). At the same time for all \( s \) and \( \ell \) we have (by Cauchy's theorem), \( |\chi^{(\ell)}(s)| \ll T^{k(\frac{1}{2} - \sigma)}(\log T)^{\ell} \).

Next we apply local convexity (see for example the references [PS-1] and [PS-2] below, see especially Theorem 6-C of [PS-2] for a correction in [PS-1]) to the zeta-function like analytic function \( Z^{(\ell)}(s)(\chi(s))^{-1}(\log T)^{-\ell} \) to prove that the integral of its absolute value taken over \( |t - t_0| \leq C(\varepsilon) \) on \( \sigma = \frac{1}{2} + \delta \) exceeds \( t_0^{\varepsilon}(T \leq t_0 \leq 2T) \), where \( C(\varepsilon) \) depends only on \( \varepsilon \). From this it follows that for \( Z^{(\ell)}(s) \) we have \( N_0 \gg T \) and the lower bound
is \( \geq T^{k \delta - \varepsilon} \).

3. Next given (arbitrarily) \( N_0 \) well-spaced points on \( (\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T) \)
we can sometimes obtain a subset (of these points) of cardinality \( \gg N_0 \)
(\( = T \), sometimes \( T(\log \log T)^{-1} \)) Titchmarsh points for a class of Dirichlet
series or GDS. But this class of Dirichlet series is a very restricted one. Let
\( Z(s) \) be a Dirichlet series (see equation (5) of XV[5]) which have

(a) Euler product for \( Z_1(s) \).

(b) Functional equation with \( 1 \leq k \leq 2 \).

(c) Mean-square on the critical line (see equation (6) of the present paper)
\( \sigma = \frac{1}{2} \).

(We have to mention that (c) follows from (b))

(d) \( |\chi^{(\ell)}(s)| \leq t^{(k - 1 - \sigma)}(\log t)^{\ell} \) for all integers \( \ell \geq 0 \).

From these we can deduce.

**Lemma 3.** Let \( \{t_j\}(T \leq t_j \leq 2T) \) be a well-spaced set of points with
cardinality \( \gg T \). Then out of these points we can select a subset of points \( t'_j \)
(with cardinality \( \gg T \)) satisfying

\[
|Z_1(\frac{1}{2} + \delta + it'_j)| \gg 1 \quad \text{and} \quad |Z_1^{(\ell)}(\frac{1}{2} + \delta + it'_j)| \ll 1
\]

\( (\ell = 1, 2, \ldots, \ell_0) \) where \( \ell_0 \geq 1 \) is any integer.

**Proof.** This lemma is contained implicitly in the proof of Theorem 1 of [PS-3].

From these we can formulate a general principle.

**General Principle.** In Theorem 11 we can replace \((\varphi_0(s))^{\ell_1 + 1}\) by
\( Q_1 \equiv (F_1(s))^{M_1}(\varphi^{(\ell_1)}(s))^{M_2} \) with integers \( \ell_1 \geq 0, M_1 \geq 0, M_2 \geq 0, M_1 + M_2 \geq 1 \), where \( F_1(s) \) is a power product (with non-negative integral exponents) of derivatives of functions like \( Z(s) \) satisfying (a),(b),(c) and (d)
above and \( \varphi(s) \) is either a Balasubramanian-Ramachandra function or a
function which has a functional equation such as (5) of XV[5]. In place of
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\[ P(\varphi_0, \varphi_1, \ldots, \varphi_r) \] of Theorem 11, we can have a suitable modification say \( Q_2 \) such that \( Q_1 - Q_2 \) has \( \gg T(\text{resp. } T(\log \log T)^{-1}) \) well-spaced Titchmarsh points on \( \sigma = \frac{1}{2} - \delta \). Accordingly we have lower bounds for the number of zeros of \( Q_1 - Q_2 \) in \( (\sigma \geq \frac{1}{2} - 2\delta, T \leq t \leq 2T) \) (and upper bounds for \( N(\frac{1}{2} + \delta, T) \) only sometimes). We can say similar things about the \( \{\alpha_n\} \) transformations of \( Q_1 - Q_2 \).

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