On the zeros of a class of generalised Dirichlet series-XVIII (a few remarks on littlewood’s theorem and Totchmarsh points)
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DIRICHLET SERIES-XVIII

(A FEW REMARKS ON LITTLEWOOD'S THEOREM AND
TITCHMARSH POINTS)

BY

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(DEDICATED TO PROFESSOR K. CHANDRASEKHARAN
ON HIS SEVENTY-FIFTH BIRTHDAY)

§ 1. INTRODUCTION AND NOTATION. This paper was necessi-
tated because we observed that the proofs of the results of the paper XVI[7]
can be simplified and that the results therein can at the same time be gen-
eralised. (In § 1, § 2 and § 3 we prove six theorems in all. For an attractive
application of these see Theorem 11 of § 4). We write s = σ + it as usual.
We begin by stating a generalisation of Theorem 9.15 (A) (on page 230 of
[9]). We need some definitions. (We fix two positive constants a and b with
a < b throughout). The parameter T will be assumed to exceed a large
positive constant.

GENERALISED DIRICHLET SERIES (GDS). Let \{λ_n\} be a se-
quence of real numbers with \(a < \lambda_1 < \lambda_2 < \cdots, \lambda_1 < b\) and \(a \leq \lambda_{n+1} - \lambda_n \leq b\) for \(n \geq 1\). Let \{A_n\} be any sequence of complex numbers such that \(A_1 \neq 0\)
and

\[ Z(s) = \sum_{n=1}^{\infty} A_n \lambda_n^{-s} \]  \hspace{1cm} (1)

converges for some complex \( s = s_0 \). Then \( Z(s) \) is called a generalised Dirichlet series (GDS). We remark that if \( Z(s) \) converges at \( s = s_0 \), then it is absolutely convergent at \( s = s_0 + 2 \). Note that a GDS is different from zero if real part of \( s(Re\ s) \) exceeds a certain constant. In fact as \( Re\ s \to \infty \), \( |Z(s)| \) tends to a non-zero constant. A GDS is said to be a normalised generalised Dirichlet series (NGDS) if \( \sum_{n \leq x} |A_n|^{2} \ll_{\varepsilon} x^{1+\varepsilon} \) for every \( \varepsilon > 0 \). A GDS is said to be a Dirichlet series if \( \{\alpha_n\} \) is a subsequence of the sequence of natural numbers.

\( \{\alpha_n\} \) TRANSFORMATION OF AN NGDS. Let \( Z(s) \) be an NGDS. We consider only such sequences \( \{\alpha_n\} \) of real numbers which satisfy

\[ \sum_{n \leq x} |A_n\alpha_n| \ll_{\varepsilon} x^{1+\varepsilon} \]  \hspace{1cm} (2)

for every \( \varepsilon > 0 \). \( F(s) = \sum_{n=1}^{\infty} A_n (\lambda_n + \alpha_n)^{-s} \) is said to be an \( \{\alpha_n\} \) transformation of \( Z(s) \) if \( F(s) \) is a GDS. Note that

\[ F(s) = D(s) + Z(s) \]  \hspace{1cm} (3)

where \( D(s) = \sum_{n=1}^{\infty} A_n ((\lambda_n + \alpha_n)^{-s} - \lambda_n^{-s}) \) and that \( D(s) \) is analytic in \( \sigma > 0 \). Moreover we have

LEMMA 1. For \( \sigma > 0 \) and every \( \varepsilon > 0 \) we have \( D(s) = O_{\sigma} ((|t| + 2)^2) \) and also

\[ \frac{1}{T} \int_{T}^{2T} |D(\sigma + it)|^2 dt \ll_{\varepsilon} \max \left( T^{-2(1/2-\sigma)+\varepsilon}, T^{\varepsilon} \right) \]  \hspace{1cm} (4)

PROOF. See Theorems 7 and 7' of XV[6].

THEOREM 1 (J.E. LITTLEWOOD). Let \( Z(s) \) be a GDS which can be continued analytically in \( (\sigma \geq \frac{1}{2} - \delta_0, T - \log T \leq t \leq 2T + \log T) \), where \( \delta_0(>0) \) is a constant, and there \( \log(\max |Z(s)| + 100) \) is \( \ll \log T \). Let \( Z(s) \to 1 \) as \( Re\ s \to \infty \). For \( \alpha \geq \frac{1}{2} \) let \( N(\alpha, T) \) denote the number of zeros of \( Z(s) \) in \( (\sigma \geq \alpha, T \leq t \leq 2T) \). Then (for \( \sigma_0 > \frac{1}{2} \)) we have

\[ 2\pi \int_{\sigma_0}^{\infty} N(\sigma, T) d\sigma = \int_{T}^{2T} \log |Z(\sigma_0 + it)| \ dt + O(\log T) \]  \hspace{1cm} (5)
and hence

\[ N\left(\frac{1}{2} + 2\delta, T\right) \leq (g\delta)^{-1} T \log \left( \frac{1}{T} \int_{T}^{2T} \left| Z\left(\frac{1}{2} + \delta + it\right) \right|^2 \, dt \right) + O(\log T) \]  

holds uniformly for all real positive constants \( g \) and \( \delta \). If \( \delta \) is any fixed constant we may take \( \delta_0 = 0 \) and then replace \( O(\log T) \) by \( O_\delta(\log T) \).

**REMARK.** This theorem is essentially due to J.E. Littlewood, since the special case \( g = 2 \) and \( Z(s) = \zeta(s) \) (due to J.E. Littlewood) is dealt with on pages 229 and 230 of [9]. The general case stated as Theorem 1 above follows by a trivial generalisation of Littlewood's method. If we do not assume \( Z(s) \to 1 \) as \( \text{Re } s \to \infty \), we have to replace \( O(\log T) \) by \( O(T) \) in (4) and (5). This does not matter for our purposes.

§ 2. A COROLLARY TO THEOREM 1.

**THEOREM 2.** Let \( r \geq 1 \) be any integer constant and let \( \varphi_1(s), \varphi_2(s), \ldots, \varphi_r(s) \) be \( r \) Dirichlet series each of which is continuable analytically in \( (\sigma \geq \frac{1}{2} - \delta_0, T - \log T \leq t \leq 2T + \log T) \) and there \( \log(\max_j |\varphi_j(s)| + 100) \ll \log T \). Suppose further that

\[ \max_j \left( \frac{1}{T} \int_{T - \log T}^{2T + \log T} |\varphi_j\left(\frac{1}{2} + it\right)|^2 \, dt \right) \ll \varepsilon T^\varepsilon \]  

holds for every \( \varepsilon > 0 \). Let \( P(X_1, \ldots, X_r) \) be any fixed polynomial (with complex coefficients) such that when we put \( X_j = \varphi_j(s) (i = 1, 2, \ldots, r) \), \( P = P(s) = P(X_1, \ldots, X_r) \) is a normalised Dirichlet series. Let \( F(s) \) be any \( \{\alpha_n\} \) transformation of \( P(s) \). Then the function \( N(\sigma, T) \) defined (as before) for \( F(s) \) satisfies

\[ N(\sigma, T) \ll_\varepsilon T^\varepsilon \quad (\sigma > \frac{1}{2}). \]  

**REMARK.** We define the degree of a monomial \( X_1^{d_1} \cdots X_r^{d_r} \) to be \( d_1 + \cdots + d_r \) and the degree of \( P(X_1, \ldots, X_r) \) to be the maximum of \( d_1 + \cdots + d_r \) taken over all monomials occurring in \( P(X_1, \ldots, X_r) \). If the degree of \( P \) is 1 then we can allow each \( \varphi_j(s) \) to be a GDS. Then \( P \) has to be an NGDS.
PROOF. The proof follows from the fact that (6) implies
\[ \max_j \left( \frac{1}{T} \int_T^{2T} | \varphi_j(\sigma + it) |^2 \, dt \right) \ll \sigma \quad (\sigma > \frac{1}{2}), \] (8)
and that for a suitable small constant \( g > 0 \) we have
\[ | F(s) |^2 \ll | D(s) |^2 + | \varphi_1(s) |^2 + \cdots + | \varphi_r(s) |^2 + 1. \] (9)
Note that in view of Lemma 1 it is not hard to deduce that
\[ \frac{1}{T} \int_T^{2T} | D(\sigma + it) |^2 \, dt \ll \sigma \quad (\sigma > \frac{1}{2}). \] (10)
From these facts Theorem 2 follows from Theorem 1.

§ 3. TITCHMARSH POINTS. Let \( F(s) \) be a GDS continuable analytically in \( (\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T) \) and there \( \log(\max | F(s) | + 100) \ll \log T \). A point \( s_0 = \sigma_0 + it_0 \) in \( (\sigma \geq \beta + \delta_1, T \leq t \leq 2T) \), where \( \delta_1 > 0 \) is a constant, is said to be a Titchmarsh point with the lower bound \( T \ell \) for \( | F(s) | \) if \( \ell(>0) \) is bounded below independent of \( T \) and \( t_0 \).

THEOREM 3. If \( s_0 = \sigma_0 + it_0 \) (with \( F(s) \) as above) is a Titchmarsh point of \( F(s) \), then the region \( (\sigma \geq \beta, | t - t_0 | \leq \delta_2) \) where \( \delta_2(>0) \) is any small constant, contains \( \gg \log T \) zeros of \( F(s) \).

PROOF. For the proof of this theorem due to R. Balasubramanian and K. Ramachandra see Theorem 3 of III[1]. It should be mentioned that this theorem is not too-trivial a generalisation of Theorem 9.14 (on page 227 of [9]) due to E.C. Titchmarsh.

WELL-SPACED POINTS. The points \( s^{(q)} = \sigma_q + it_q \) \((q = 1, 2, \cdots)\) in the complex plane are said to be well-spaced if \( | s^{(q)} - s^{(q')} | \) is bounded below for all pairs \((q, q')\) with \( q \neq q' \).

THEOREM 4. If there are \( N_0 \) well-spaced Titchmarsh points for \( F(s) \) \((F(s) \text{ as in Theorem 3})\), then \( F(s) \) has \( \gg N_0 \log T \) zeros in \((\sigma \geq \beta, T \leq t \leq 2T)\).

PROOF. The proof follows from the fact that \( | F(s) | \) tends to a non-zero
THEOREM 5. Let $\beta(<\frac{1}{2})$ be a constant and $r \geq 1$ any integer constant and $\varphi_1(s), \cdots, \varphi_r(s)$ be $r$ Dirichlet series each of which is continu-able analytically in $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$ and there
\[ \log \max(|\varphi_j(s)| + 100) \leq \log T. \] Suppose further that for $j = 1, 2, \cdots, r$ and $\sigma \geq \beta$, we have
\[ \frac{1}{T} \int_{T - \log T}^{2T + \log T} |\varphi_j(\sigma + it)|^2 \, dt \ll_{\varepsilon} \max \left( T^{2m_j(1/2 - \sigma) + \varepsilon}, T^\varepsilon \right) \] (11)
where $m_j > 0$ are constants. Let $\mu > 0$ be a constant. Put $X_0 = T^{\mu(1/2 - \sigma) - \varepsilon}$. Let $X_0^{d_0} X_1^{d_1} \cdots X_r^{d_r}$ ($d_j \geq 0$ integers, $j = 0, 1, 2, \cdots, r$) be any fixed monomial in $X_0, X_1, \cdots, X_r$. Let the weighted $\mu$-degree $d(\mu)$ of the monomial be defined as $\mu d_0 + m_1 d_1 + \cdots + m_r d_r$. Put $Q_0(s) = X_0^{d_0} (\varphi_1(s))^{d_1} \cdots (\varphi_r(s))^{d_r}$. Then given any well-spaced set of points $\{s_q\}$ with $s_q = \sigma + it_q (q = 1, 2, \cdots; \sigma \geq \beta + \delta_3, T \leq t_q \leq 2T)$ where $\delta_3 > 0$ is a small constant we have
\[ |Q_0(\sigma + it_q)| \ll \max \left( T^{d(\mu)(1/2 - \sigma) + \varepsilon}, T^\varepsilon \right), \] (12)
except for $O(T^{1-\varepsilon})$ values of $q$.

REMARK. If $\sum_{j=1}^{r} d_j = 1$, then we can allow $\varphi_j(s)(j = 1, 2, \cdots, r)$ to be
GDS.

PROOF. We use the fact that the value $|\varphi_j(s_q)|$ of $\varphi_j(s)$ is majorised by the mean value over a disc (with $s_q$ as centre and $\varepsilon$ as radius) of $|\varphi_j(s)|$.

We choose a small radius and sum over all the discs taking $s_q$ to be $\sigma + it_q$. We obtain
\[ \frac{1}{T} \sum_{q} |\varphi_j(s_q)|^2 \ll \max \left( T^{2m_j(1/2 - \sigma) + \varepsilon}, T^\varepsilon \right). \]

Hence $|\varphi_j(s_q)| > \max(T^{m_j(1/2 - \sigma) + \varepsilon}, T^\varepsilon)$ is possible for at most $O(T^{1-\varepsilon})$ values of $q$. We next sum over all $j$ and obtain the result.

THEOREM 6. Let $\beta(<\frac{1}{2})$ be a constant and let $\varphi_0(s)$ be a Dirichlet series continu-able analytically in $(\sigma > \beta, T - \log T \leq t \leq 2T + \log T)$ and there
\log \max (|\varphi_0(s)| + 100) \ll \log T. Suppose that it has \( T \) (resp. \( T (\log \log T)^{-1} \)) well-spaced Titchmarsh points \( \{\sigma + it_0\} \) (where \( \sigma \) is any constant with \( \beta < \sigma < \frac{1}{2} \)) with the lower bound \( T^{\mu(\frac{1}{2} - \varepsilon)} \) where \( \mu(\varepsilon > 0) \) is a constant. Let \( Q(X_0, X_1, \ldots, X_r) \) be a fixed polynomial (with complex coefficients) such that for some positive integer \( M \), the maximum of \( d(\mu) \) (defined in Theorem 5) taken over all the monomials \( X_0^{d_0} X_1^{d_1} \cdots X_r^{d_r} \) occurring in \( Q(X_0, X_1, \ldots, X_r) \) is less than \( M \mu \). Put \( Q(s) = Q(\varphi_0(s), \varphi_1(s), \ldots, \varphi_r(s)) \) (where \( \varphi_j(s), j = 1, 2, \ldots \), are as in Theorem 5). Assume that \( (\varphi_0(s))^M - Q(s) \) is an NGDS, and let \( F(s) \) be its \( \{a_n\} \) transformation. Then \( F(s) \) has \( T \log T \) (resp. \( T (\log T)(\log \log T)^{-1} \)) zeros in \( (\sigma - \varepsilon, T) \).

**PROOF.** Follows from \( |(\varphi_0(s))^M - Q(s)| \geq |\varphi_0(s)|^M (1 - |Q(s)| |\varphi_0(s)|^{-M}) \).

**REMARK.** Note that none of the functions \( \varphi_0(s), (\varphi_0(s))^M \) and \( Q(s) \) need be normalised Dirichlet series. If \( M = 1 \), then \( Q(s) \) does not involve \( \varphi_0(s) \) (which can now be taken to be a GDS). If \( M = 1 \) and \( Q(X_0, X_1, \ldots, X_r) \) (now independent of \( X_0 \)) is linear in \( X_1, \ldots, X_r \) (i.e. \( \sum d_j \leq 1 \) for every monomial \( X_1^{d_1} \cdots X_r^{d_r} \) occurring in \( Q(X_0, X_1, \ldots, X_r) \) and equality holds for at least one monomial) then all of \( \varphi_1(s), \ldots, \varphi_r(s) \) can be taken to be GDS. Also \( Q(X_0, X_1, \ldots, X_r) \) can be a constant.

§ 4. **SOME APPLICATIONS OF THEOREMS 2 TO 6.** Theorems 2 to 6 are only easy formalisms. These would be completely uninteresting without examples. Finding examples is a difficult task. For example we do not know how to prove the expected result \( N(\sigma, T) \ll T(\sigma > \frac{1}{2}) \) for the abelian \( L \)-series of an algebraic number field. However we have a somewhat general theorem namely.

**THEOREM 7.** Let \( \{\lambda_n\} \) (\( n = 1, 2, \ldots \)) be a sequence of real numbers as in the definition of GDS. Let \( \sum n \leq x |a_n| \leq B(x), \sum n \leq x |a_n|^2 \leq xB(x) \) and \( \sum m \leq x \sum n \leq m |a_n|^2 \leq xB(x) \), where \( B(x) \) depends on \( x \). If \( B(x) \ll x^\varepsilon \) (for every
\( \varepsilon > 0 \) then \( Z(s) = \sum_{n=1}^{\infty} \alpha_n \lambda_n^{-s} \) converges uniformly over compact subsets of \( \sigma > 0 \) and hence is analytic there. We have

\[
N\left( \frac{1}{2} + \delta, T \right) \ll \delta T \quad (\delta > 0).
\]

If further \( \log B(z) \ll \log \log x \) then we have

\[
N\left( \frac{1}{2} + \delta, T \right) \ll \delta^{-1} T \log(\delta^{-1})
\]

uniformly for \( 0 < \delta \leq \frac{1}{2} \).

**Remark.** Results like

\[
1 \sim 2T + \log T \\
T \log T \sim T \int_{T-\log T}^{2T+\log T} |Z\left( \frac{1}{2} + it \right)|^2 \, dt \ll T^\varepsilon
\]

(15)

for every \( \varepsilon > 0 \) and more general and powerful results have been proved in paper VI\(^6\). Results like (15) imply (13) and (14). If \( \{Z(s)\} \) is any finite set of Dirichlet series each subject to (15) we can apply Theorem 2.

We now turn to series of the type

\[
\sum_{n=1}^{\infty} \alpha_n b_n e^{2\pi i n \theta} \lambda_n^{-s} \quad (\theta \text{ is a real constant}),
\]

(16)

their analytic continuations and their Titchmarsh points. Investigations dealing with such series were carried out in a series of papers by R. Balasubramanian and K. Ramachandra (see III\(^1\), IV\(^1\), V\(^6\), VI\(^3\), XIV\(^4\) and also the paper [8] by K. Ramachandra and A. Sankaranarayanan). The paper XIV\(^4\) is nearly final. In paper XIV\(^4\) the condition \( \alpha_n = O(1) \) is assumed. This can be relaxed to \( \sum_{n \leq x} |\alpha_n|^2 = O(x) \). This last mentioned condition on \( \alpha_n \) will be assumed in the rest of this paper.

Lest we get lost in generalities we state two special cases first.

**Theorem 8.** Let \( \theta_0 (0 < \theta_0 < \frac{1}{2}) \) be a constant and let \( \{a_n\} \) be a sequence of complex numbers satisfying the inequality \( |\sum_{m=1}^{N} a_m - N| \leq (\frac{1}{2} - \theta_0)^{-1} \) for
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N = 1, 2, 3, · · · . Also for n = 1, 2, 3, · · · , let \( \alpha_n \) be real and \( |\alpha_n| \leq C(\theta_0) \) where \( C(\theta_0) \) is a certain (small) constant depending only on \( \theta_0 \). Then the number of zeros of the function

\[
\sum_{n=1}^{\infty} a_n (n + \alpha_n)^{-s} = \zeta(s) + \sum_{n=1}^{\infty} (a_n (n + \alpha_n)^{-s} - n^{-s})
\]

in the rectangle \( |\sigma - \frac{1}{2}| \leq \delta, T \leq t \leq 2T \) is \( \geq C(\theta_0, \delta)T \log T \) where \( C(\theta_0, \delta) \) is a positive constant depending only on \( \theta_0 \) and \( \delta \), and \( T \geq T_0(\theta_0, \delta) \) a large positive constant.

**PROOF.** Theorem 10 (below) gives \( \gg T \) well spaced Titchmarsh points on every line segment \( (\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T) \) with the lower bound \( \gg T^\delta \), while actually (14) gives

\[
N \left( \frac{1}{2} + \frac{C \log \log T}{\log T}, T \right) \ll C^{-1}T \log T
\]

for every fixed \( C(> 0) \). (It is not hard to prove the required mean-square upper bound for the function).

**THEOREM 9.** In the above theorem we can relax the condition on \( a_n \) to

\[
|\sum_{m=1}^{N} a_m - N | \leq (\frac{1}{2} - \theta_0)^{-1}N^{\theta_0} \quad \text{and} \quad \sum_{n \leq x} |a_n| \leq (\frac{1}{2} - \theta_0)^{-1}x.
\]

Then the lower bound for the number of zeros in \( (\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T) \) (\( \delta \) being any constant with \( \frac{1}{2} - \delta > \theta_0 \)) is \( \geq C(\theta_0, \delta)T(\log T)(\loglog T)^{-1} \). But only when \( \sum_{n \leq x} a_n = x + O_\varepsilon(x^\varepsilon) \) we can prove that \( N(\frac{1}{2} + \delta, T) \ll \delta T \). Also if

\[
\sum_{n \leq x} a_n = x + O((\log x)^{C_1}) \quad (C_1 > 0 \text{ being a constant we can prove}
\]

\[
N \left( \frac{1}{2} + \frac{C((\log T)^2}{\log T}, T \right) \ll C^{-1}T(\log T)(\loglog T)^{-1}
\]

for every fixed \( C > 0 \).

**PROOF.** Theorem 10 (below) gives \( \gg T (\loglog T)^{-1} \) well-spaced Titchmarsh points on every line segment \( (\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T) \) with the lower
THEOREM 10. (i) Let \( \{\lambda_n\} \) be as in the definition of GDS. This sequence will be further restricted by the condition (vii) or (viii) as the case may be. \( \theta \) will denote a real constant.

Let \( f(x) \) and \( g(x) \) be positive real valued functions defined in \( x \geq 0 \) satisfying

(ii) \( f(x)x^n \) is monotonic increasing and \( f(x)x^{-n} \) is monotonic decreasing for every fixed \( \eta > 0 \) and all \( x \geq x_0(\eta) \).

(iii) \( \lim_{x \to \infty} (g(x)x^{-1}) = 1. \)

(iv) For all \( x \geq 0 \), \( g'(x) \) lies between two positive constants and \( (g'(x))^2 - g(x)g''(x) \) lies between two positive constants (it being assumed that \( g(x) \) is twice continuously differentiable for \( x \geq x_0 \)).

Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of complex numbers having the following properties.

(v) \( |b_n| (f(n))^{-1} \) lies between two positive constants (for all integers \( n \geq n_0 \)) and \( (\sum_{n \leq x} |a_n|^2)x^{-1} \) does not exceed a positive constant for all \( x \geq 1 \).

(vi) For all \( X \geq 1 \), \( \sum_{X \leq n \leq 2X} |b_{n+1} - b_n| \leq f(X). \)

We next assume that \( \{a_n\} \) and \( \{b_n\} \) satisfy at least one of the two following conditions.

(vii) MONOTONICITY CONDITION. There exists an arithmetic progression \( A \) (of integers) such that

\[
\lim_{x \to \infty} \left( x^{-1} \sum_{n \leq x} a_n \right) = h \quad (h \neq 0),
\]

where the accent denotes the restriction of \( n \) to \( A \). Also for every positive constant \( \nu \) we have that \( |b_n| \lambda_n^{-\nu} \) is monotonic decreasing for all \( n(\geq n_0) \) in \( A \).

(viii) REAL PART CONDITION. There exists an arithmetic pro-
gression \( \mathcal{A} \) (of integers) such that

\[
\lim_{x \to \infty} \inf \left( \frac{1}{x} \sum_{x \leq \lambda_n \leq 2x, \text{Re } a_n > 0} \text{Re } a_n \right) > 0
\]

and

\[
\lim_{x \to \infty} \left( \frac{1}{x} \sum_{x \leq \lambda_n \leq 2x, \text{Re } a_n < 0} \text{Re } a_n \right) = 0
\]

where the accent denotes the restriction of \( n \) to \( \mathcal{A} \).

(ix) Finally we set \( \lambda_n = g(n) \) and let \( \{\alpha_n\} \) be a sequence of real numbers such that \( |\alpha_n| \) does not exceed a small positive constant (depending on other constants). We suppose that the GDS

\[
F(s) = \sum_{n=1}^{\infty} a_n b_n e^{2\pi i \theta} (\lambda_n + \alpha_n)^{-s}
\]

can be continued analytically in \((\sigma \geq \frac{1}{2} - \delta, T - \log T \leq t \leq 2T + \log T)\) and there \( \log \max(|F(s)| + 100) \ll \log T \).

Then on every line segment \((\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)\) (\( \delta_4 \) being a constant with \( 0 < \delta_4 \leq \delta \)) there are \( \gg T \left(\log \log T\right)^{-1} \) well-spaced Titchmarsh points with the lower bound \( \gg T^{\delta_4} f(T) \). If further we have

\[
\frac{1}{T} \int_{T}^{2T} |F(\frac{1}{2} - \delta_4 + it)|^2 dt \ll T^{2\delta_4} (f(T))^2
\]

for every constant \( \delta_4 \) (with \( 0 < \delta_4 \leq \delta \)), then the number of well-spaced Titchmarsh points on the line segment \((\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)\) (with the lower bound \( \gg T^{\delta_4} f(T) \)) is \( \gg T \).

REMARK. This theorem is proved by R. Balasubramanian and K. Ramachandra in this form in the paper XIV[4] except that we have now to use \( \sum_{n \leq x} |a_n|^2 \ll x \) in place of \( a_n = O(1) \) and also except that we have to involve \( \theta \). Lemmas necessary (see Lemma 6 of IV[2]) for these generalities and also the method have been developed in previous papers mentioned before by R. Balasubramanian and K. Ramachandra.
Finally we would like to mention paper XV[6] of this series of papers. Here we assume a functional equation of a very general type for a GDS and prove that a large class of \( \{\alpha_n\} \) transformations of it have \( \gg T \) well-spaced Titchmarsh points on every line segment \( (\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T) \) with a lower bound of the type \( \gg \epsilon T^{m\delta - \varepsilon} \) where \( m > 0 \) is a real constant and \( \varepsilon(> 0) \) is an arbitrary constant (for example for the zeta function of a ray class in an algebraic number field of degree \( m \). If \( m \geq 2 \) we can allow \( \sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\varepsilon} \) in place of \( \sum_{n \leq x} |A_n\alpha_n|^2 \ll x^{1+\varepsilon} \), see the definition in §1 for the meaning of \( A_n \). Note that if \( A_n = O_x(n^\varepsilon) \) then the condition on \( \alpha_n \) is simply \( \sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\varepsilon} \). These results are very general. But out of these GDS only in very special cases (but still a somewhat large class of GDS) we can prove that

\[
\frac{1}{T} \int_T^{2T} |F(\frac{1}{2} + \delta + it)|^2 \, dt \ll \varepsilon, \delta T^\varepsilon
\]

for all \( \delta > 0 \) and \( \varepsilon > 0 \). Some examples (not already covered by Theorem 7) are (i) zeta function of any ray class of a quadratic field (ii) zeta function of a positive definite quadratic form \( Q(X_1, \cdots, X_\ell) \) (in \( \ell \geq 2 \) variables and with integer coefficients) namely \( \sum_{n=1}^{\infty} (a_n n^{-\frac{1}{2}+1}) n^{-s} \), where \( a_n \) is the number of \( \ell \)-tuples \( (m_1, \cdots, m_\ell) \) of integers with \( Q(m_1, \cdots, m_\ell) = n \). In this case \( m = 1 \) and the lower bound is \( \gg T^\delta \) (resp. \( \gg T^{2\delta-\varepsilon} \)) according as \( \ell > 2 \) or \( \ell = 2 \) see [8].

Instead of enumerating all the applications of this theory we state a beautiful theorem (namely Theorem 11 below). Many other theorems can be deduced in a similar manner by the interested readers from the results of papers mentioned above and the results of §1, §2 and §3, (see also the post-script at the end of this paper).

**Theorem 11.** Let \( F \) denote the class of Dirichlet series of the form \( \zeta(s) + \sum_{n=1}^{\infty} a_n n^{-s} \) with complex number sequence \( \{a_n\} \) satisfying \( \sum_{n \leq x} a_n = O(1) \). Let \( \varphi_j = \varphi_j(s)(j = 0, 1, 2, \cdots, r) \) be any \( r + 1 \) Dirichlet series (may
not be distinct) of the class $\mathcal{F}$. Let $P(X_0, X_1, \ldots, X_r)$ be any fixed polynomial (with complex coefficients) of degree $d$ (being the maximum of $d_0 + d_1 + \cdots + d_r$ taken over all monomials $X_0^{d_0} X_1^{d_1} \cdots X_r^{d_r}$ occurring in $P(X_0, \ldots, X_r)$). Let $Q$ be defined by

$$Q = (\varphi_0(s))^{d+1} - P(\varphi_0, \varphi_1, \ldots, \varphi_r) = \sum_{n=1}^{\infty} B_n n^{-s}, (\sigma > 1).$$

Then first we have $B_n \neq 0$ for at least one $n$ (also $Q$ is analytic in $\sigma > 0$, $t \geq 1$). Next put

$$F(s) = \sum B_n ((n + \alpha_n)^{-s} - n^{-s}) + Q$$

where $\{\alpha_n\}$ is any sequence of real numbers with $|\alpha_n| \leq \frac{1}{3}$. Then in $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T), F(s) \gg T \log T$ zeros and in $(\sigma \geq \frac{1}{2} + \frac{C \log \log T}{\log T}, T \leq t \leq 2T)$ only $\ll C^{-1} T \log T$ zeros ($C \geq 1$ being any constant).

**Remark 1.** If $d \geq 1$ we can allow $\sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\varepsilon}$ in place of $|\alpha_n| \leq \frac{1}{3}$. But then we have to stipulate that $F(s)$ should be a GDS.

**Remark 2.** That $B_n \neq 0$ for at least one $n$ of course follows since $Q$ has a pole of order $(d + 1)$ at $s = 1$. But then we mention that the conclusion of Theorem 11 are valid for $\varphi_0(s) = (1 - 2^{1-s})\zeta(s)$ and $\varphi_j(s) = \sum_{n=1}^{\infty} a_n^{(j)} n^{-s} (j = 1, 2, \ldots, r)$ where $\max \sum_{n \leq x} |a_n^{(j)}| = O(1)$. 


REFERENCES


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1. In view of Theorem 2, it is important to find Dirichlet series which satisfy (6). This will enable us to prove \( N(\frac{1}{2} + \delta, T) \ll \delta T(\delta > 0) \) for larger and larger class of GDS.

2. In view of Theorem 4, it is important to find \( N_0 \) (as large as possible) well-spaced Titchmarsh points with the lower bound \( T^{k\delta - \varepsilon} \) (for some \( k > 0 \) and every \( \varepsilon > 0 \)) on the line segment \( (\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T) \) for a large class of Dirichlet series. In this direction we have Balasubramanian-Ramachandra functions given by Theorem 10 (Theorems 8 and 9 are special cases of these functions). Also we have the \( \ell \)th derivative \( (\ell \geq 0 \) integer) of a class of GDS which satisfy a very general functional equation (see equation (5) of XV[5]). The case \( \ell = 0 \) is treated in XV[6] and it is proved that \( N_0 \gg T \). We can cover all integers \( \ell \) as follows. We make use of the following lemma.

**Lemma 2.** Let \( h(x) \) be an \( n \)-times continuously differentiable function defined in \( a_0 \leq x \leq a_0 + nd_0 \), where \( a_0 > 0, d_0 > 0 \) are constants and \( n \) is any fixed integer \( \geq 1 \). Then

\[
\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} h(a_0 + rd_0) = \int_{0}^{d_0} \cdots \int_{0}^{d_0} h(a_0 + u_1 + \cdots + u_n) du_1 du_2 \cdots du_n.
\]

**Proof.** Follows by trivial induction.

We apply this lemma to \( h(x) = h(x, t) = \chi(x + it) \) of equation (5) of XV[5] and obtain \( |\chi(x)(s_0)| \gg T^{k(\frac{1}{2} - \sigma)}(\log T)^\ell \) for any fixed \( t(T \leq t \leq 2T) \) and a suitable \( s_0 = \sigma_0 + it \) (with \( \sigma_0 \) at a distance of \( O((\log T)^{-1}) \) from any arbitrarily given \( \sigma \)). At the same time for all \( s \) and \( \ell \) we have (by Cauchy's theorem), \( |\chi(x)(s)| \ll T^{k(\frac{1}{2} - \sigma)}(\log T)^\ell \).

Next we apply local convexity (see for example the references [PS-1] and [PS-2] below, see especially Theorem 6-C of [PS-2] for a correction in [PS-1]) to the zeta-function like analytic function \( Z(\ell)(s)(\chi(s))^{-1}(\log T)^{-\ell} \) to prove that the integral of its absolute value taken over \( |t - t_0| \leq C(\varepsilon) \) on \( \sigma = \frac{1}{2} + \delta \) exceeds \( t_0^{-\varepsilon}(T \leq t_0 \leq 2T) \), where \( C(\varepsilon) \) depends only on \( \varepsilon \). From this it follows that for \( Z(\ell)(s) \) we have \( N_0 \gg T \) and the lower bound
is $\geq T^{k\delta - \epsilon}$.

3. Next given (arbitrarily) $N_0$ well-spaced points on $(\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)$ we can sometimes obtain a subset (of these points) of cardinality $\gg N_0$ ($= T$, sometimes $T'(\log \log T)^{-1}$) Titchmarsh points for a class of Dirichlet series or GDS. But this class of Dirichlet series is a very restricted one. Let $Z(s)$ be a Dirichlet series (see equation (5) of XV[5]) which have

(a) Euler product for $Z_1(s)$.

(b) Functional equation with $1 \leq k \leq 2$.

(c) Mean-square on the critical line (see equation (6) of the present paper) $\sigma = \frac{1}{2}$.

(We have to mention that (c) follows from (b))

(d) $|\chi^{(\ell)}(s)| \asymp t^{k(\frac{1}{2} - \sigma)}(\log t)^\ell$ for all integers $\ell \geq 0$.

From these we can deduce.

**Lemma 3.** Let $\{t_j\}(T \leq t_j \leq 2T)$ be a well-spaced set of points with cardinality $\gg T$. Then out of these points we can select a subset of points $t'_j$ (with cardinality $\gg T$) satisfying

$$|Z_1(\frac{1}{2} + \delta + it'_j)| \gg 1 \text{ and } |Z_1^{(\ell)}(\frac{1}{2} + \delta + it'_j)| \ll 1$$

$(\ell = 1, 2, \cdots, \ell_0)$ where $\ell_0 \geq 1$ is any integer.

**Proof.** This lemma is contained implicitly in the proof of Theorem 1 of [PS-3].

From these we can formulate a general principle.

**General Principle.** In Theorem 11 we can replace $(\varphi_0(s))^{d+1}$ by $Q_1 \equiv (F_1(s))^{M_1}(\varphi^{(\ell_1)}(s))^{M_2}$ with integers $\ell_1 \geq 0, M_1 \geq 0, M_2 \geq 0, M_1 + M_2 \geq 1$, where $F_1(s)$ is a power product (with non-negative integral exponents) of derivatives of functions like $Z(s)$ satisfying (a),(b),(c) and (d) above and $\varphi(s)$ is either a Balasubramanian-Ramachandra function or a function which has a functional equation such as (5) of XV[5]. In place of
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\( P(\varphi_0, \varphi_1, \cdots, \varphi_r) \) of Theorem 11, we can have a suitable modification say \( Q_2 \) such that \( Q_1 - Q_2 \) has \( \gg T \) (resp. \( T(\log \log T)^{-1} \)) well-spaced Titchmarsh points on \( \sigma = \frac{1}{2} - \delta \). Accordingly we have lower bounds for the number of zeros of \( Q_1 - Q_2 \) in \( (\sigma \geq \frac{1}{2} - 2\delta, T \leq t \leq 2T) \) (and upper bounds for \( N(\frac{1}{2} + \delta, T) \) only sometimes). We can say similar things about the \( \{a_n\} \) transformations of \( Q_1 - Q_2 \).

REFERENCES ADDED TO POST-SCRIPT


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