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ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES-XIX

BY

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§ 1. **INTRODUCTION AND PRELIMINARIES.** The object of this paper is to give a simpler approach to the following theorem (this theorem namely Theorem 10 of XVIII[5] constitutes the main theorem on Balasubramanian-Ramachandra functions arising out of the works III[1], IV[2], V[6], VI[3] due to R. Balasubramanian and K. Ramachandra).

THEOREM 1. (R. BALASUBRAMANIAN AND K. RAMACHANDRA)
(i) *Let* $\lambda_n (n = 1, 2, 3, \cdots)$ *be an increasing sequence of positive real numbers such that* $\lambda_{n+1} - \lambda_n$ *is bounded both above and below. This sequence will be further restricted by the condition* (vii) *or* (viii) *as the case may be.* θ *will denote a real constant.*

Let $f(x)$ *and* $g(x)$ *be positive real valued functions defined in* $x \geq 0$, *satisfying*

(ii) $f(x)x^{\eta}$ *is monotonic increasing and* $f(x)x^{-\eta}$ *is monotonic decreasing for every fixed* $\eta > 0$ *and all* $x \geq x_0(\eta)$.

(iii) $\lim_{x\to\infty} (g(x)x^{-1}) = 1$.

(iv) $g(x)$ *is differentiable once for* $x \geq 0$ *and* $g'(x)$ *lies between two positive constants. Also* $g(x)$ *is twice differentiable for* $x \geq x_0$ *and* $(g'(x))^2 - g(x)g''(x)$ *lies between two positive constants for* $x \geq x_0$.

Let $\{a_n\}$ and $\{b_n\}(n = 1, 2, 3, \cdots)$ be two sequences of complex numbers having the following properties

(v) $| b_n | (f(n))^{-1}$ *lies between two positive constants (for all integers $n \geq n_0$) and* $(\sum_{n\leq x} | a_n |^2)x^{-1}$ *does not exceed a positive constant for all $x \geq 1$.*

(vi) *For all $x \geq 1$,* $\sum_{x\leq n\leq 2x} | b_{n+1} - b_n | \ll f(x)$.

We next assume that $\{a_n\}$ and $\{b_n\}$ satisfy at least one of the following conditions. We set $\lambda_n = g(n)$.

(vii) **MONOTONICITY CONDITION.** *There exists an arithmetic progression $\mathcal{A}$ (of natural numbers) such that*

$$\lim_{x\to\infty}\left(x^{-1}{\sum_{n\leq x}}' a_n\right) = h \quad (0 < | h | < \infty),$$

where the accent denotes the restriction of n to $\mathcal{A}$. Also for every positive constant ν we have $| b_n | \lambda_n^{-\nu}$ is monotonic decreasing for all $n(\geq n_0)$ in $\mathcal{A}$.

(viii) **REAL PART CONDITION.** *There exists an arithmetic progression $\mathcal{A}$ (of natural numbers) such that*

$$\lim\inf_{x\to\infty}\left(\frac{1}{x}\ {\sum_{x\leq\lambda_n\leq 2x, Re\ a_n>0}}' Re\ a_n\right) > 0$$

and

$$\lim_{x\to\infty}\left(x^{-1}\ {\sum_{x\leq\lambda_n\leq 2x, Re\ a_n<0}}' Re\ a_n\right) = 0$$

where the accent denotes the restriction of n to $\mathcal{A}$.

(ix) *Finally let $\{\alpha_n\}(n = 1, 2, 3, \cdots)$ be a sequence of real numbers such that $| \alpha_n |$ does not exceed a small positive constant (depending on other constants). We suppose that the series*

$$F(s) = \sum_{n=1}^{\infty} a_n\, b_n e^{2\pi in\theta}(\lambda_n + \alpha_n)^{-s} \quad (Re\ s \geq 2)$$

can be continued analytically in $(\sigma \geq \frac{1}{2} - \delta, T - \log T \leq t \leq 2T + \log T)$ *(where* δ *is a positive constant) and there* $\log \max(|F(s)| + 100) \ll \log T$. *As usual we have written* $s = \sigma + it$.

Then on every line segment $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$, *(*$\delta_4$ *being any constant with* $0 < \delta_4 < \delta$*) there are* $\gg T(\log\log T)^{-1}$ *well-spaced Titchmarsh points with the lower bound* $\gg T^{\delta_4} f(T)$ *for* $|F(s)|$. *If further*

$$\frac{1}{T}\int_{T-\sqrt{\log T}}^{2T+\sqrt{\log T}} |F(\frac{1}{2} - \delta_4 + it)|^2\, dt \ll T^{2\delta_4}(f(T))^2$$

for every constant δ_4 *(with* $0 < \delta_4 \leq \delta$*), then there are* $\gg T$ *well-spaced Titchmarsh points on every line segment* $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$ *with the lower bound* $\gg T^{\delta_4} f(T)$ *for* $|F(s)|$.

In other words there exist real numbers $t_1, t_2, \cdots, t_r$ *(with* $r \gg T(\log\log T)^{-1}$ *and* $r \gg T$ *respectively in the two cases) such that* $T \leq t_j \leq 2T (j = 1, 2, \cdots, r)$, *the minimum of* $|t_j - t_{j'}|$ *taken over all pairs* (j, j') *with* $j \neq j'$ *is bounded below and further*

$$|F(\frac{1}{2} - \delta_4 + it_j)| \gg T^{\delta_4} f(T).$$

The proof of this theorem depends on the following two lemmas.

LEMMA 1 (van-der-CORPUT). *If* $f_1(x)$ *is real and twice differentiable and* $0 < \mu_2 \leq f_1''(x) \leq h'\mu_2$ *(or* $\mu_2 \leq -f_1''(x) \leq h'\mu_2$*) throughout the interval* $[a, b]$, *and* $b \geq a + 1$, *then*

$$\sum_{a<n\leq b} Exp(2\pi i f_1(n)) = O(h'(b-a)\mu_2^{\frac{1}{2}}) + O(\mu_2^{-\frac{1}{2}}).$$

REMARK. This result is Theorem 5.9 on page 104 of [8], with a slight change of notation.

LEMMA 2 (H.L. MONTGOMERY AND R.C. VAUGHAN). *If* $\{\lambda_n\}$ *is any increasing sequence of real numbers and* $\{A_n\}$ *and* $\{B_n\}$ *are any two sequences of complex numbers, then*

$$\left|\sum\sum_{m\neq n} \frac{A_m \overline{B}_n}{\log(\lambda_m \lambda_n^{-1})}\right| \leq K\left(\sum \delta_n^{-1} |A_n|^2\right)^{\frac{1}{2}}\left(\sum \delta_n^{-1} |B_n|^2\right)^{\frac{1}{2}},$$

where $\delta_n = \min_{m \neq n} | \lambda_n - \lambda_m |$ *and* K *is a numerical constant.*

REMARK. We need only a special case of this result where $\lambda_{n+1} - \lambda_n$ lies between two positive constants (and so the same is true of $n\delta_n$). For the proof in this special case and also for a reference to the paper of Montgomery and Vaughan see [7].

§ 2. SOME MORE LEMMAS.

LEMMA 3. *Let* $y > 0, w = u + iv, R(w) = Exp((Sin \frac{w}{100})^2)$, *and*

$$\Delta(y) = \frac{1}{2\pi i} \int_{u=2} y^w R(w) \frac{dw}{w}.$$

Then for $| u | \leq 3$ *we have* $| R(w) | \ll (Exp\ Exp | \frac{v}{100} |)^{-1}$. *Consequently* $\Delta(y) = 1 + O(y^{-2})$ *and also* $\Delta(y) = O(y^2)$.

PROOF. By trivial computation (and moving the line of integration to $u = -2$ and $u = 2$ respectively).

$\mathcal{A}$ will denote the arithmetic progression consisting of an infinite subset of natural numbers. Let $\lambda (0 < \lambda < 1)$ be a constant. We put $X = T\lambda$ (later we will choose λ to be a small constant). S will denote the set $\mathcal{A} \cap [\frac{1}{2}X, X]$. All our O-constants and the constants implied by the Vinogradov symbols $\gg$ and $\ll$ will be independent of λ.

LEMMA 4. *For* $T \leq t \leq 2T$, *we have,*

$$| \sum_{n \in S} Exp(-2\pi i n\theta + it \log g(n)) | \ll T^{\frac{1}{2}}.$$

PROOF. Noting that the second derivative of $-2\pi x\theta + t \log g(x)$ is $t((g'(x))^2 - g(x)g''(x))(g(x))^{-2}$ the lemma follows by Lemma 1.

LEMMA 5. *For* $T \leq t \leq 2T$, *we have*

$$| \sum_{n \in S} \bar{b}_n Exp(-2\pi i n\theta + it \log g(n)) | \ll T^{\frac{1}{2}} f(X).$$

PROOF. The proof follows by partial summation (from Lemma 4) on using $\sum_{x \leq n \leq 2x} | b_{n+1} - b_n | \ll f(x)$ for all $x \geq 1$.

Next we put $g(n) = \lambda_n$. We consider the case $\alpha_n \equiv 0$ first. Our object is to obtain a good lower bound for the LHS of (2) below.

LEMMA 6. *For* $s = \frac{1}{2} - \delta + it, T \leq t \leq 2T$, *put*

$$F_X(s) = \sum_{n=1}^{\infty} a_n b_n Exp(2\pi i n\theta)\lambda_n^{-s}\Delta(\frac{X}{\lambda_n}). \tag{1}$$

Then

$$\frac{1}{T}\int_T^{2T} | F_X(s) | dt \gg (T^{\frac{3}{2}} f(X))^{-1} | I |, \tag{2}$$

where

$$I = \int_T^{2T} F_X(s) \sum_{n\in S} \bar{b}_n Exp(-2\pi i n\theta + it \log \lambda_n) dt. \tag{3}$$

PROOF. Follows from Lemma 5.

LEMMA 7. *We have,*

$$I = T\sum_{n\in S} a_n | b_n |^2 \lambda_n^{-\frac{1}{2}+\delta} + O(J) \tag{4}$$

where

$$J = (J_1 J_2)^{\frac{1}{2}}, J_1 = \sum_{n=1}^{\infty} | a_n b_n |^2 n^{2\delta}(\Delta(\frac{X}{\lambda_n}))^2 \tag{5}$$

and

$$J_2 = \sum_{n\in[\frac{1}{2}X, X]} n | b_n |^2 . \tag{6}$$

PROOF. Follows from Lemma 2.

LEMMA 8. *We have*

$$J_1 = O(X^{1+2\delta}(f(X))^2) \tag{7}$$

and

$$J_2 = O(X^2(f(X))^2). \tag{8}$$

PROOF. Follows from $\Delta(y) = 1 + O(y^{-2}) = O(y^2)$ and also from (ii) of Theorem 1.

LEMMA 9. *Let*

$$\sum\nolimits_0 = \sum_{n \in S} a_n \mid b_n \mid^2 \lambda_n^{-\frac{1}{2}+\delta} \Delta(\frac{X}{\lambda_n}). \tag{9}$$

Then

$$I = T\sum\nolimits_0 + O(X^{\frac{3}{2}+\delta}(f(X))^2) \tag{10}$$

PROOF. Follows from Lemmas 7 and 8.

LEMMA 10. *Under monotonicity condition, we have,*

$$\mid \sum\nolimits_0 \mid \gg \mid h \mid X^{\frac{1}{2}+\delta}(f(X))^2. \tag{11}$$

PROOF. We write

$$\frac{1}{x} \sum_{n \in A, n \leq x} a_n = h + \varepsilon_x$$

where $\varepsilon_x \to 0$ as $x \to \infty$. We obtain the result by the monotonicity of $\mid b_n \mid^2 \lambda_n^{-\frac{1}{4}+\frac{1}{2}\delta}$.

LEMMA 11. *Under the real part condition, we have,*

$$Re \sum\nolimits_0 \gg X^{\frac{1}{2}+\delta}(f(X))^2 \tag{12}$$

PROOF. Follows since the contribution from those a_n with $Re\, a_n < 0$ is of a smaller order.

LEMMA 12. *We have*

$$\mid I \mid > C_1 T(f(X))^2 X^{\frac{1}{2}+\delta} - C_2(f(X))^2 X^{\frac{3}{2}+\delta} \tag{13}$$

where C_1 and C_2 are positive constants independent of λ.

PROOF. Follows from Lemma 7 to 11.

LEMMA 13. *We have, with $s = \frac{1}{2} - \delta + it, X = T\lambda$, where $\lambda(> 0)$ is some fixed small constant, the inequality*

$$\frac{1}{T} \int_T^{2T} \mid F_X(s) \mid dt \gg T^{\delta} f(T). \tag{14}$$

PROOF. RHS of (13) is

$$\left(C_1 T^{\frac{3}{2}+\delta}\lambda^{\frac{1}{2}+\delta} - C_2 T^{\frac{3}{2}}\lambda^{\frac{3}{2}+\delta}\right)(f(X))^2.$$

Using (ii) it follows that $f(X)X^{-1} \geq f(T)T^{-1}$ and so $f(X) \geq \lambda f(T)$. Lemma 13 follows on fixing λ to be a small positive constant.

LEMMA 14. *Let now $Y = T\lambda'$ where $\lambda'(0 < \lambda' < \lambda)$ is a small constant. We have*

$$\frac{1}{T}\int_T^{2T} \mid F_Y(s) \mid dt \leq \eta_0 T^\delta f(T), \tag{15}$$

where η_0 depends on λ' and is small enough if λ' is small.

PROOF. Note that

$$\frac{1}{T}\int_T^{2T} \mid F_Y(s) \mid^2 dt \ll Y^{2\delta}(f(Y))^2$$

and that here RHS is $\leq Y^\delta (Y^{\frac{1}{2}\delta} f(Y))^2 \leq (\lambda')^\delta\ T^{2\delta}(f(T))^2$. Lemma 14 follows from this on using Hölder's inequality.

LEMMA 15. *We have, with $X = T\lambda, Y = T\lambda'$ where λ is as before and $\lambda'(0 < \lambda' < \lambda)$ is fixed to be a sufficiently small constant, the inequality*

$$\frac{1}{T}\int_T^{2T} \mid F_X(s) - F_Y(s) \mid dt \gg T^\delta f(T). \tag{16}$$

PROOF. Follows from Lemma 13 and 14.

From now on we fix the positive constants λ and λ' so that (16) is satisfied.

LEMMA 16. *Now let α_n be real and let $\mid \alpha_n \mid$ be bounded above by a small positive constant. Then with $s = \frac{1}{2} - \delta + it, X = T\lambda, Y = T\lambda'$ we have*

$$\frac{1}{T}\int_T^{2T} \mid \sum_{n=1}^{\infty} a_n b_n Exp(2\pi i n\theta)(\lambda_n + \alpha_n)^{-s}$$

$$\left(\Delta\left(\frac{X}{\lambda_n + \alpha_n}\right) - \Delta\left(\frac{Y}{\lambda_n + \alpha_n}\right)\right) \mid dt \gg T^\delta f(T). \tag{17}$$

PROOF. We split the infinite series on the LHS of (17) to be $\sum_1$ with $n \leq T\lambda''$ (where $\lambda''(>0)$ is a small constant) and $\sum_2$ the rest. Clearly (by Lemma 2)

$$\frac{1}{T}\int_T^{2T} \mid \sum\nolimits_1 \mid dt \ll (\lambda'')^{\frac{1}{4}\delta} T^{\delta} f(T)$$

and also in $\sum_2$ using

$$(\lambda_n+\alpha_n)^{-s}\left(\Delta\left(\frac{X}{\lambda_n+\alpha_n}\right)-\Delta\left(\frac{Y}{\lambda_n+\alpha_n}\right)\right)-\lambda_n^{-s}\left(\Delta\left(\frac{X}{\lambda_n}\right)-\Delta\left(\frac{Y}{\lambda_n}\right)\right)$$

$$=\int_0^{\alpha_n}\frac{d}{dk}\left((\lambda_n+k)^{-s}\left(\Delta\left(\frac{X}{\lambda_n+k}\right)-\Delta\left(\frac{Y}{\lambda_n+k}\right)\right)\right)dk$$

and Lemma 2 we are led to Lemma 16. (For details see page 173 of XIV[4]).

LEMMA 17. *We have, with α_n as in Lemma 16,*

$$\frac{1}{T}\int_T^{2T} \mid \sum_{n=1}^{\infty} a_n b_n Exp(2\pi i n\theta)(\lambda_n+\alpha_n)^{-s}$$

$$\left(\Delta\left(\frac{X}{\lambda_n+\alpha_n}\right)-\Delta\left(\frac{Y}{\lambda_n+\alpha_n}\right)\right) \mid^2 dt \ll T^{2\delta}(f(T))^2. \qquad (18)$$

PROOF. Follows from Lemma 2.

THEOREM 2. *Denote by $G(s)$ the infinite series in the LHS of* (18). *Then there are real numbers $t_1, t_2, \cdots, t_r$ as in Theorem 1 with $r \gg T$ and*

$$\mid G(\frac{1}{2}-\delta+it_j) \mid \gg T^{\delta} f(T). \qquad (19)$$

PROOF. Divide the interval $[T, 2T]$ (of integration) on the LHS of (17) into abutting intervals of length 1, ignoring a bit at one end. Ignore the integrals over intervals of length 1 which do not exceed a small (positive) constant times $T^{\delta} f(T)$. Now apply Hölder's inequality for the rest and apply Lemma 17. We obtain (19).

THEOREM 3. *We have*

$$\frac{1}{T}\int_{T-\sqrt{\log T}}^{2T+\sqrt{\log T}} \mid F(\frac{1}{2}-\delta+it) \mid dt \gg T^{\delta} f(T). \qquad (20)$$

PROOF. Follows from Lemma 16 on writing $G(s)$ as a line integral over $u = 2$ and moving the line of integration to $u = 0$ using suitable horizontal connecting lines.

§ 3. COMPLETION OF THE PROOF OF THEOREM 1. Using the mean square upper bound for $|F(s)|$ and also Theorem 3, we can obtain (as in the proof of Theorem 2) real numbers $t_1, t_2, \cdots, t_r$ as in Theorem 1 with $r \gg T$ and

$$|F(\frac{1}{2} - \delta + it_j)| \gg T^{\delta} f(T).$$

Next we use Theorem 2. Out of the numbers $t_1, \cdots, t_r$ we can omit a minimal number of them and obtain numbers $\tau_1, \cdots, \tau_{r'}$ such that $r' \gg T(loglog\, T)^{-1}$, $|\tau_j - \tau_{j'}| \gg loglog\, T$ for all pairs (j, j') with $j \neq j'$ and

$$|G(\frac{1}{2} - \delta + i\tau_j)| \gg T^{\delta} f(T).$$

Now writing $G(s)$ as a line integral over $u = 2$ and moving the line of integration to $u = 0$ using suitable connecting horizontal lines. We thus obtain points $\tau'_1, \cdots, \tau'_{r'}$ with

$$|F(\frac{1}{2} - \delta + i\tau'_j)| \gg T^{\delta} f(T) \ (j = 1, 2, \cdots, r').$$

This proves Theorem 1 completely.

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