Hardy-Ramanujan Journal 38 (2014), 1-28 submitted 31/10/2014, accepted 11/06/2015, revised 27/04/2015

On congruences for certain sums of E. Lehmer's type

Shigeru Kanemitsu, Takako Kuzumaki and Jerzy Urbanowicz*

Abstract. Let $n > 1$ be an odd natural number and let r $(1 < r < n)$ be a natural number relatively prime to n. Denote by χ_n the principal character modulo n. In Section 3 we prove some new congruences for the sums $T_{r,k}(n) = \sum_{i=1}^{[\frac{n}{r}]}(\frac{\chi_n(i)}{i^k}) \pmod{n^{s+1}}$ for $s \in \{0,1,2\}$, for all divisors r of 24 and for some natural numbers k.We obtain 82 new congruences for $T_{r,k}(n)$, which generalize those obtained in [Ler05], [Leh38] and [Sun08] if $n = p$ is an odd prime.

Section 4 is an appendix by the second and third named authors. It contains some new congruences for the sums $U_r(n) = \sum_{i=1}^{[\frac{n}{r}]}\frac{\chi_n(i)}{n-ri} \pmod{n^{s+1}}$ for $s \in \{0,1,2\}$ and $r \mid 24$. The congruences obtained for the sums $U_r(n)$ extend those proved in [Leh38],[CFZ07] and [CP09] for $r \in \{2,3,4,6\}$ and $s = 1$. The sums are rational linear combinations of Euler's quotients and in the cases when $r \in \{8,12,24\}$[1] , also of the numbers $\frac{1}{n\varphi(n)}B_{n\varphi(n),\chi}\prod_{p|n}(1-p^{n\varphi(n)-1})$, where the generalized Bernoulli numbers $B_{n\varphi(n),\chi}$ are attached to even quadratic characters χ of conductors dividing 24[2].

Keywords. Congruence, generalized Bernoulli number, special value of L-function, ordinary Bernoulli number, Bernoulli polynomial, Euler number.

2010 Mathematics Subject Classification. primary 11B68; secondary 11R42, 11A07.

1. Notation and introduction

Let $n > 1$ be odd and let $\chi_{0,n}$ (sometimes abbreviated as χ_n) be the principal Dirichlet character modulo n (with $\chi_{0.1}$ designating the constant function $\chi_{0.1}(x) = 1$ for all integers x). For $r > 1$ prime to n denote by $q_r(n)$ the Euler quotient, i.e.,

$$q_r(n) = \frac{r^{\varphi(n)} - 1}{n}.$$

Here and throughout the paper φ is the Euler φ-function and $B_{n,\chi}$ denotes the n-th generalized Bernoulli number attached to the Dirichlet character χ modulo n defined by the generating function

$$\sum_{a=1}^{n}\frac{\chi(a)te^{at}}{e^{nt}-1} = \sum_{m=0}^{\infty}B_{m,\chi}\frac{t^m}{m!}.$$

Given the discriminant d of a quadratic field, let χ_d denote its quadratic character (Kronecker symbol). We shall denote by $\chi_{d,n}$ the character χ_d modulo n.

It was proved in [Car59] that the numbers $B_{i,\chi_d}/i$ are rational integers unless $d = -4$ or $d = \pm p$, where p is an odd prime of a special form. If $d = -4$ and i is odd, then the numbers $E_{i-1} = -2B_{i,\chi_{-4}}/i$ are odd integers, called the Euler numbers. If $d = \pm p$, then the numbers B_{i,χ_d} have p in their denominators and $pB_{i,\chi_d} \equiv p - 1 (\bmod\, p^{\mathrm{ord}_p(i)+1})$.

We consider the ordinary Bernoulli numbers B_i (i.e., generalized Bernoulli numbers attached to the trivial primitive character $\chi_{0,1}$, except when $i = 1$ for which $B_{1,\chi_{0,1}} = \frac{1}{2} = -B_1$) and the so-called D-numbers defined in [Kle55] and [Ern79] by $D_{i-1} = -3B_{i,\chi_{-3}}/i$ for i odd, having powers of 3 in

*Passed away on September 6, 2012.

We thank episciences.org for providing open access hosting of the electronic journal *Hardy-Ramanujan Journal*
The first author was supported by the JSPS, Grant-in-aid for scientific research No.25400032

[1] Which were omitted in [Leh38], [CFZ07] and [CP09].

[2] E. Lehmer proved her congruences in the case when $n = p$ is an odd prime. The congruences proved in [CFZ07] and [CP09] are for n odd and not divisible by 3. See also [Ler05].

their denominators. We also consider the rational integers $A_{i-1} = B_{i,\chi_8}/i$, $F_{i-1} = B_{i,\chi_{-3}\chi_{-4}}/i$ and $G_{i-1} = B_{i,\chi_{-3}\chi_{-8}}/i$, if $i \geq 2$ even, and $C_{i-1} = -B_{i,\chi_{-8}}/i$ and $H_{i-1} = -B_{i,\chi_{-3}\chi_8}/i$ if $i \geq 1$ odd.

In this paper we shall consider congruences for the character sums with negative weight

$$T_{r,k}(n) = \sum_{0<i<\frac{n}{r}} \frac{\chi_n(i)}{i^k}$$

modulo powers n^{s+1} for $n > 1$ odd and $s \in \{0,1,2\}$ where $\chi_n = \chi_{0,n}$ and r ($r \mid 24$ and $1 < r < n$) is coprime to n, and $k \geq 1$ is subject to the condition $k \leq n^s\varphi(n)$. Note that since $\chi_n(i) = 0$ for $(i,n) > 1$, the sum is over $(i,n) = 1$.

The central role in this paper is played by an identity proved in [SUZ95, p.276,(6)]. Let χ be a Dirichlet character modulo M, N a positive integral multiple of M, and $r(> 1)$ a positive integer prime to N. Then for any integer $m \geq 0$ we have

$$(m+1)r^m \sum_{0<i<\frac{N}{r}} \chi(i)i^m = -B_{m+1,\chi}r^m + \frac{\overline{\chi}(r)}{\varphi(r)} \sum_{\psi \in G(r)} \overline{\psi}(-N)B_{m+1,\chi\psi}(N), \tag{1.1}$$

where the sum on the right hand side is taken over all Dirichlet characters ψ modulo r. We denote by $G(r)$ the group of all such characters; then $\#G(r) = \varphi(r)$. Here $B_{n,\chi}(X) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i,\chi}X^i$ denotes the n-th generalized Bernoulli polynomial attached to χ. Since $r \mid 24$, the group $G(r)$ has exponent 2 and all characters modulo r are quadratic.

If the character χ modulo M is induced from a character $\widetilde{\chi}$ modulo some divisor of M then

$$B_{n,\chi} = B_{n,\widetilde{\chi}} \prod_{p \mid M} (1 - \widetilde{\chi}(p)p^{n-1}), \tag{1.2}$$

where the product is taken over all primes p dividing M.

If $(i,n) = 1$, then by Euler's theorem we have $i^{\varphi(n)} \equiv 1 (\mathrm{mod}\ n)$, and more generally, $\varphi(n^{s+1}) = n^s\varphi(n)$ and

$$i^{\varphi(n)n^s} \equiv 1 \pmod{n^{s+1}}$$

for $s \geq 0$.

Given r prime to n and integers $s \geq 0$, $k \geq 1$ we denote

$$S_{r,k,s}(n) = \sum_{0<i<\frac{n}{r}} \chi_n(i) i^{n^s\varphi(n)-k}.$$

Then we have the congruence

$$T_{r,k}(n) \equiv S_{r,k,s}(n) \pmod{n^{s+1}}, \tag{1.3}$$

which allows us to study $T_{r,k}(n)$ through $S_{r,k,s}(n)$.

In this paper we specialize to the case that r is a divisor of 24. Then the group $G(r) \cong (\mathbb{Z}/r\mathbb{Z})^*$ has exponent 2, so all the elements ψ are quadratic.

The main results of the paper are congruences for the sums $T_{r,k}(n)$ modulo n^{s+1} for $s \in \{0,1,2\}$ proved in Section 3. The congruences will be obtained by applying identity (1.1) to the sums $S_{r,k,s}(n)$[3]. They extend those proved by M. Lerch [Ler05], E. Lehmer [Leh38] and Z.-H. Sun [Sun08] in the case when $n = p$ is an odd prime. In principle, the congruences in this particular case have a different form from those obtained for any natural odd n. Sometimes it is not easy to derive the former congruences from the latter. We shall do it in the second part of the paper.

[3] This identity was earlier successfully exploited in [SUZ95], [SUV99] and [FUW97] to solve some other problems. See also the book [UW00] devoted to the identity and related problems.

Two such congruences modulo n^2 were earlier obtained, by using (1.1), in [Cai02] for $r = 2, k = 1$ and in [KUW12] for $r = 4, k = 2$. In the present paper we find 82 new congruences for the sums $T_{r,k}(n) \pmod{n^{s+1}}$ for $s \in \{0, 1, 2\}$, $r \mid 24$ and $k \geq 1$, in particular for $k = 1$ or 2. Most of our congruences for $T_{r,k}(n)$ have not been known earlier even in the particular case when $n = p$ is a prime. The machinery introduced in [SUZ95] is much more efficient than the methods exploited in [Ler05], [Leh38] and [Sun08].

In Section 4, which is an appendix by the second and third named authors, we find congruences for the sums

$$U_r(n) = \sum_{0<i<\frac{n}{r}} \frac{\chi_n(i)}{n - ri}$$

modulo n^{s+1} for $s \in \{0, 1, 2\}$ and all $r \mid 24$ $(1 < r < n)$ coprime to n. To obtain such congruences it suffices to use appropriate congruences for the sums $T_{r,k}(n)$ or, by virtue of (1.3), for the sums $S_{r,k,s}(n)$ for $k \in \{1, 2, 3\}$. The congruences are consequences of those proved in Section 3 and for $s = 1$ extend those obtained in [CFZ07],[CP09] for $r \in \{2, 3, 4, 6\}$. They have the same form as those proved by E. Lehmer [Leh38] if $n = p$ is an odd prime. The sums are rational linear combinations of Euler's quotients and in the cases when $r \in \{8, 12, 24\}$, omitted in [Leh38], [CFZ07] and [CP09], also of the numbers $\frac{1}{n\varphi(n)} B_{n\varphi(n),\chi} \prod_{p|n}(1-p^{n\varphi(n)-1})$, where $B_{n\varphi(n),\chi}$ are the generalized Bernoulli numbers attached to even quadratic characters χ of conductors dividing 24. Also some new congruences for $s = 2$ with an additional summand $-\frac{n^2}{2r^3} B_{n^2\varphi(n)-2} \prod_{p|n}(1 - p^{n^2\varphi(n)-3})$ for all $r \mid 24$ are obtained.

2. Some auxiliary formulae

The idea exploited in [Cai02] and [KUW12] to use identity (1.1) to extend classical congruences for the sums $T_{r,k}(n)$ seems to be very efficient. This identity allows us to obtain almost automatically many new congruences. Usually the proofs using (1.1) are much easier, more unified and much shorter than those applying other methods.

The general scheme of reasoning is uniform. To obtain congruences for the sums $T_{r,k}(n)$ modulo n^{s+1} we first determine, using (1.1), the sums $S_{r,k,s}(n)$ modulo n^{s+1}. We substitute in (1.1) $m = n^s\varphi(n) - k$ and $N = M = n$, by the definition of $S_{r,k,s}(n)$. We assume that $r \mid 24$, $n > 1$ is odd. If $3 \not| r$ then we have $(n, r) = 1$. If $3 \mid r$, then we additionally assume that n is not divisible by 3. Note that, since $r \mid 24$, all generalized Bernoulli numbers occurring in $S_{r,k,s}(n)$ are rational.

Thus, throughout the paper, we write $m = n^s\varphi(n) - k \geq 0$. Consequently, we obtain

$$S_{r,k,s}(n) \;=\; S_1 + S_2, \tag{2.4}$$

where, by (1.2),

$$S_1 = -\frac{B_{m+1,\chi_{0,n}}}{m+1} = -\frac{B_{m+1}}{m+1}\prod_{p|n}(1 - p^m) \tag{2.5}$$

and

$$S_2 = \frac{1}{\varphi(r)(m+1)r^m} \sum_{\psi\in G(r)} \psi(-n) B_{m+1,\chi_{0,n}\psi}(n).$$

Note that $\chi_{0,n}$ is even. Thus, if $m \neq 0$ is even, then $B_{m+1} = 0$, and so $S_1 = 0$. If $m = 0$, then $1 - p^m = 0$, and so $S_1 = 0$ too. Otherwise, in view of (2.5), we have $S_1 \neq 0$. Furthermore,

$$S_2 = \frac{1}{\varphi(r)(m+1)r^m} \sum_{\psi\in G(r)} \psi(-n) \sum_{i=0}^{m+1} \binom{m+1}{i} B_{i,\chi_{0,n}\psi} n^{m+1-i} \quad \text{cf.[SUZ95, p.274,(6)]}$$

and $B_{0,\chi_{0,n}\psi} = 0$ if ψ is not trivial modulo r and

$$B_{0,\chi_{0,n}\chi_{0,r}} = \frac{\varphi(rn)}{rn}$$

otherwise, and hence (recall that $(r,n) = 1$)

$$\frac{n^{m+1}}{\varphi(r)(m+1)r^m} \sum_{\psi \in G(r)} \psi(-n) B_{0,\chi_{0,n}\psi} \;=\; \frac{n^m \varphi(n)}{(m+1)r^{m+1}}.$$

$$\begin{aligned} S_2 = {} & \frac{n^{m+1}}{\varphi(r)(m+1)r^m} \sum_{\psi \in G(r)} \psi(-n) B_{0,\chi_{0,n}\psi} \\ & + \frac{1}{\varphi(r)(m+1)r^m} \sum_{i=1}^{m+1} \binom{m+1}{i} n^{m+1-i} \sum_{\psi \in G(r)} \psi(-n) B_{i,\chi_{0,n}\psi} \\ = {} & \frac{n^m \varphi(n)}{(m+1)r^{m+1}} \\ & + \frac{1}{\varphi(r)(m+1)r^m} \sum_{i=0}^{m} \binom{m+1}{i+1} n^{m-i} \sum_{\psi \in G(r)} \psi(-n) B_{i+1,\chi_{0,n}\psi}. \end{aligned}$$

Consequently,

$$S_2 \;=\; \Theta_s + \frac{1}{\varphi(r)r^m} \sum_{i=0}^{m} \binom{m}{i} n^{m-i} U_i(r), \tag{2.6}$$

where

$$\Theta_s = \Theta_s(n,m,r) = \frac{n^m \varphi(n)}{(m+1)r^{m+1}} \tag{2.7}$$

and

$$U_i(r) = \sum_{\psi \in G(r)} \psi(-n) \frac{B_{i+1,\chi_{0,n}\psi}}{i+1}. \tag{2.8}$$

2.A. $U_i(r)$ for $r \mid 24$

Let $n > 1$ be odd and relatively prime to r. Here and subsequently, we set

$$
\begin{aligned}
\widetilde{B}_i &= B_i \prod_{p|n} (1 - p^{i-1}),\\
\widetilde{A}_i &= (-1)^{\frac{n^2-1}{8}} A_i \prod_{p|n} (1 - (-1)^{\frac{p^2-1}{8}} p^i) = (-1)^{\frac{n^2-1}{8}} \frac{B_{i+1,\chi_8}}{i+1} \prod_{p|n} (1 - (-1)^{\frac{p^2-1}{8}} p^i),\\
\widetilde{C}_i &= (-1)^{\frac{(n-1)(n+5)}{8}} C_i \prod_{p|n} (1 - (-1)^{\frac{(p-1)(p+5)}{8}} p^i)\\
&= (-1)^{\frac{(n+1)(n+3)}{8}} \frac{B_{i+1,\chi_{-8}}}{i+1} \prod_{p|n} (1 - (-1)^{\frac{(p-1)(p+5)}{8}} p^i),\\
\widetilde{D}_i &= (-1)^{\nu(n)} D_i \prod_{p|n} (1 - (-1)^{\nu(p)} p^i) = (-1)^{\nu(n)+1} \frac{3B_{i+1,\chi_{-3}}}{i+1} \prod_{p|n} (1 - (-1)^{\nu(p)} p^i),\\
\widetilde{E}_i &= (-1)^{\frac{n-1}{2}} E_i \prod_{p|n} (1 - (-1)^{\frac{p-1}{2}} p^i) = (-1)^{\frac{n+1}{2}} \frac{2B_{i+1,\chi_{-4}}}{i+1} \prod_{p|n} (1 - (-1)^{\frac{p-1}{2}} p^i),\\
\widetilde{F}_i &= (-1)^{\frac{n-1}{2}+\nu(n)} F_i \prod_{p|n} (1 - (-1)^{\frac{p-1}{2}+\nu(p)} p^i)\\
&= (-1)^{\frac{n-1}{2}+\nu(n)} \frac{B_{i+1,\chi_{-3}\chi_{-4}}}{i+1} \prod_{p|n} (1 - (-1)^{\frac{p-1}{2}+\nu(p)} p^i),\\
\widetilde{G}_i &= (-1)^{\frac{(n-1)(n+5)}{8}+\nu(n)} G_i \prod_{p|n} (1 - (-1)^{\frac{(p-1)(p+5)}{8}+\nu(p)} p^i)\\
&= (-1)^{\frac{(n-1)(n+5)}{8}+\nu(n)} \frac{B_{i+1,\chi_{-3}\chi_{-8}}}{i+1} \prod_{p|n} (1 - (-1)^{\frac{(p-1)(p+5)}{8}+\nu(p)} p^i),\\
\widetilde{H}_i &= (-1)^{\frac{n^2-1}{8}+\nu(n)} H_i \prod_{p|n} (1 - (-1)^{\frac{p^2-1}{8}+\nu(p)} p^i)\\
&= (-1)^{\frac{n^2+7}{8}+\nu(n)} \frac{B_{i+1,\chi_{-3}\chi_8}}{i+1} \prod_{p|n} (1 - (-1)^{\frac{p^2-1}{8}+\nu(p)} p^i),
\end{aligned}
$$

where $\chi_{-3}(n) = (-1)^{\nu(n)}, \nu(n) = 0$, resp. 1 if $n \equiv 1$, resp. $-1 (\mathrm{mod}\ 3)$.

In the following, we compute $U_i(r)$ for $r = 2, 3, 4, 6, 8, 12$ or 24.

1. Case $r = 2$

   Then $\#G(2) = 1$ and $G(2) = \{\chi_{0,2}\}$. Then, by definition and identity (1.2),

$$
U_i(2) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}(1 - 2^i), & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even.} \end{cases} \tag{2.9}
$$

2. Case $r = 3$

   Then $\#G(3) = 2$ and $G(3) = \{\chi_{0,3}, \chi_{-3}\}$. Then, by definition and identity (1.2),

$$
U_i(3) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}(1 - 3^i), & \text{if } i \text{ is odd;} \\ \frac{1}{3}\widetilde{D}_i, & \text{if } i \text{ is even.} \end{cases} \tag{2.10}
$$

3. Case $r = 4$

   Then $\#G(4) = 2$ and $G(4) = \{\chi_{0,4}, \chi_{-4}\}$. Thus, by definition and the same arguments as in the case $r = 3$ (note that both characters χ_{-3} and χ_{-4} are odd), in view of (1.2) we obtain

$$U_i(4) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}(1-2^i), & \text{if } i \text{ is odd;} \\ \frac{1}{2}\widetilde{E}_i, & \text{if } i \text{ is even.} \end{cases} \tag{2.11}$$

4. Case $r = 6$

   Then $\#G(6) = 2$ and $G(6) = \{\chi_{0,6}, \chi_{-3,6}\}$. Consequently, by (1.2) and the same arguments as in the previous case we obtain

$$U_i(6) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}(1-2^i)(1-3^i), & \text{if } i \text{ is odd;} \\ \frac{1}{3}\widetilde{D}_i(1+2^i), & \text{if } i \text{ is even.} \end{cases} \tag{2.12}$$

5. Case $r = 8$

   Then $\#G(8) = 4$ and $G(8) = \{\chi_{0,8}, \chi_{-4,8}, \chi_{-8}, \chi_8\}$. Therefore, in view of (1.2),

$$U_i(8) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}(1-2^i) + \widetilde{A}_i, & \text{if } i \text{ is odd;} \\ \frac{1}{2}\widetilde{E}_i + \widetilde{C}_i, & \text{if } i \text{ is even.} \end{cases} \tag{2.13}$$

6. Case $r = 12$

   Then $\#G(12) = 4$ and $G(12) = \{\chi_{0,12}, \chi_{-3,12}, \chi_{-4,12}, \chi_{(-3)(-4)}\}$. Consequently, by definition and (1.2),

$$U_i(12) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}(1-2^i)(1-3^i) + \widetilde{F}_i, & \text{if } i \text{ is odd;} \\ \frac{1}{3}\widetilde{D}_i(1+2^i) + \frac{1}{2}\widetilde{E}_i(1+3^i), & \text{if } i \text{ is even.} \end{cases} \tag{2.14}$$

7. Case $r = 24$

   Then $\#G(24) = 8$ and
   $G(24) = \{\chi_{0,24}, \chi_{-3,24}, \chi_{-4,24}, \chi_{(-3)(-4),24}, \chi_{(-3)(-8)}, \chi_{(-3)8}, \chi_{-8,24}, \chi_{8,24}\}$. Consequently, in view of (1.2),

$$U_i(24) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}(1-2^i)(1-3^i) + \widetilde{F}_i + \widetilde{G}_i + \widetilde{A}_i(1+3^i), & \text{if } i \text{ is odd;} \\ \frac{1}{3}\widetilde{D}_i(1+2^i) + \frac{1}{2}\widetilde{E}(1+3^i) + \widetilde{H}_i + \widetilde{C}_i(1-3^i), & \text{if } i \text{ is even.} \end{cases} \tag{2.15}$$

2.B. The sums $S_{r,k,s}(n) \pmod{n^{s+1}}$ for $m > s,\ r \mid 24,\ s \le 2$

The generalized Bernoulli numbers attached to Dirichlet characters modulo r, with $r \mid 24$, are rational numbers. In what follows we consider congruences for $S_{r,k,s}(n)$ modulo n^{s+1} for $n > 1$ odd and $s \in \{0, 1, 2\}$. We assume that n is not divisible by 3 if $3 \mid r$; then r and $\varphi(r)$ are coprime to n.

It is shown in the previous section that the numbers $U_i(r)$ are linear combinations of the numbers $\widetilde{A}_i, \widetilde{C}_i, \widetilde{D}_i, \widetilde{E}_i, \widetilde{F}_i, \widetilde{G}_i, \widetilde{H}_i$ and the quotients $\dfrac{B_{i+1}}{i+1}$. Denote by $U_i^{odd}(r)$, resp. $U_i^{even}(r)$ the sum $U_i(r)$ taken over odd, resp. even characters ψ modulo r. Note that $U_i(r) = U_i^{odd}(r) + U_i^{even}(r)$ and $U_i^{odd}(r) = 0,\ U_i^{even}(r) = 0$ if i is odd or even, respectively.

First we recall some divisibility properties of the quotients $\dfrac{B_{i+1,\chi}}{i+1}$ for primitive Dirichlet characters χ of conductors $f_\chi \mid nr$. These quotients, multiplied by some Euler factors, are summands of U_i . We start with some elementary lemmas on the quotients $\dfrac{B_{i+1}}{i+1}$ of the ordinary Bernoulli numbers. Lemma 2.1 is called the von Staudt and Clausen theorem. Lemma 2.2 due to L. Carlitz is its generalization.

Lemma 2.1. (See [Wash97, Theorem 5.10] or [IR90, Corollary to Theorem 3, p. 233]). *Let k be an even natural number and let p be a prime number. Then B_k contains p in its denominator if and only if $p-1 \mid k$ and $pB_k \equiv -1 (\mathrm{mod}\ p)$.*

Lemma 2.2. (See [Car2]) *If $p^\nu(p-1) \mid k,\ \nu \geq 0$ then $pB_k \equiv p-1 \pmod{p^{\nu+1}}$.*

Lemma 2.3. (See [Ern79, Proposition 15.2.4, p. 238].) *If $p-1 \not| k$ then the quotients $\dfrac{B_k}{k}$ are p-integral.*

Since conductors of non-trivial characters occurring in $U_i(r)$ are coprime to n, they are not powers of a prime divisor of n. In such cases we have a useful lemma:

Lemma 2.4. (See [Ern79, Theorem 1.5].) *Let χ be a primitive Dirichlet character with conductor f_χ . If f_χ is not a power of a given prime number p, then the quotients $\dfrac{B_{n,\chi}}{n}(n \geq 1)$ are p-integral.*

We set $NTU_i^{even}(r) = U_i^{even}(r) - \frac{\widetilde{B}_{i+1}}{i+1}\prod_{p|r}(1-p^i)$. By Lemma 2.4 we obtain:

Lemma 2.5. *Let r be coprime to p for a given prime number $p \mid n$. Then the numbers $U_i^{odd}(r)$ for i even and the numbers $NTU_i^{even}(r)$ for i odd are p-integral.*

Assume that $m = n^s\varphi(n) - k > s$ for $s \in \{0,1,2\}$. Since for odd $n > 1$, $\varphi(n)$ is even, m and k are of the same parity. We divide each of the cases $s = 0, 1$ or 2 .

Our purpose is to obtain some congruences for the sums $S_{r,k,s}(n)$ modulo n for $s \in \{0,1,2\}$, and next using congruence (1.3) to obtain congruences for the sums $T_{r,k}(n)$. We prove that the latter sums are congruent modulo n^{s+1} to linear combinations of the quotients $\widetilde{B}_m/m$ and some of the numbers $\widetilde{A}_{m-1}, \widetilde{C}_m, \widetilde{C}_{m-2}, \widetilde{D}_m, \widetilde{D}_{m-2}, \widetilde{E}_m, \widetilde{E}_{m-2}, \widetilde{F}_{m-1}, \widetilde{G}_{m-1}, \widetilde{H}_m, \widetilde{H}_{m-2}$ if k is even, and of the quotients $\widetilde{B}_{m-1}/(m-1), \widetilde{B}_{m+1}/(m+1)$ and some of the numbers $\widetilde{A}_m, \widetilde{A}_{m-2}, \widetilde{C}_{m-1}, \widetilde{D}_{m-1}, \widetilde{E}_{m-1}, \widetilde{F}_m, \widetilde{F}_{m-2}, \widetilde{G}_m, \widetilde{G}_{m-2}, \widetilde{H}_{m-1}$ if k is odd[4].

We start with the study of the case $s = 2$. Next, similarly, we derive the remaining congruences modulo n^2 and modulo n. First we show when the numbers Θ_s (defined in (2.7)) are congruent to 0 modulo n^{s+1}.

Lemma 2.6. *Let $n > 1$ be odd and let $1 < r \leq n$ be coprime to n. Assume that $m > s$ and $p \mid n$ is a prime. Then the numbers Θ_s in (2.7) are p-integral and*

$$\Theta_s \quad = \quad \frac{n^m\varphi(n)}{(m+1)r^{m+1}} \equiv 0 \pmod{n^{s+1}}$$

except when $s = 1,\ 3 \parallel n,\ 3 \not| \varphi(n)$ and $m = 2$ [5].

Proof. First we prove that the numbers Θ_s are p-integral for $m \geq s+1$. It suffices to show that $m\mathrm{ord}_p(n) - \mathrm{ord}_p(m+1) \geq 0$. Let us define the function $g(x) = x - \log_p(x+1)$, which is increasing for $x \geq 1$. Since $\log_p(m+1) \geq \mathrm{ord}_p(m+1)$ and $\mathrm{ord}_p(n) \geq 1$ we obtain that

$$m\mathrm{ord}_p(n) - \mathrm{ord}_p(m+1) \geq m - \log_p(m+1) = g(m) \geq g(s+1) > 0$$

because $g(3) = 3 - \log_p(5) > 0,\ g(2) = 2 - \log_p(4) > 0$ and $g(1) = 1 - \log_p(2) > 0$ for any prime p.

[4] As well as of Euler's quotients $q_2(n)$ or $q_3(n)$ if $k = 1$.

[5] Then $\Theta_1 = n^2\varphi(n)/3r^3$ and the exceptional n's have the form $n = 3\prod_{i=1}^u p_i^{e_i}$ where $p_i \equiv 2(\mathrm{mod}\ 3)$ for $i = 1, \ldots, u$. Moreover $k = n\varphi(n) - 2$ is even. Obviously, if $k \geq 2$ and $(k-1, n) = 1$, then the congruence $\Theta_1 \equiv 0(\mathrm{mod}\ n^{s+1})$ is true because $m+1$ and n are coprime. We leave it to the reader to verify that the congruence holds if $k = 1$.

Let us consider the functions $f_s(x) = x - s - \log_p(x+1)$ for $x \geq 1$, which are increasing for $x \geq 1$[6]. Note that the congruence $\Theta_s \equiv 0 (\bmod\ n^{s+1})$ for $m > s$ holds if and only if

$$(m-s)\mathrm{ord}_p(n) + \mathrm{ord}_p(\varphi(n)) - \mathrm{ord}_p(m+1) > 0$$

for every $p \mid n$.

In view of $\log_p(m+1) \geq \mathrm{ord}_p(m+1)$ and $\mathrm{ord}_p(n) \geq 1$ the above follows from the inequality $f_s(m) > 0$ for $m \geq 3$ if $s = 1, 2$, and for $m \geq 1$ if $s = 0$ because

$$(m-s)\mathrm{ord}_p(n) - \mathrm{ord}_p(m+1) \geq (m-s) - \log_p(m+1) = f_s(m)$$

and $f_s(m) \geq f_2(3) = 1 - \log_p(4) > 0$ if $s = 2$, $f_s(m) \geq f_1(3) = 2 - \log_p(4) > 0$ if $s = 1$ and $f_s(m) \geq f_0(1) = 1 - \log_p(2) > 0$ if $s = 0$ for every $p \mid n$. This gives the congruence $\Theta_s \equiv 0(\bmod\ n^{s+1})$ for $s = 0, 2$ and $m > s$ and $s = 1$ and $m \geq 3$.

In the case when $s = 1$ and $m = 2$ we have $f_1(2) = 1 - \log_p(3) > 0$ if $p \geq 5$, and so the congruence holds for $3 \nmid n$. We are left with the task of checking when the congruence holds for $s = 1$, $m = 2$ and $3 \mid n$. Then it is easily seen that the congruence $\Theta_1 = \dfrac{n^2\varphi(n)}{3r^3} \equiv 0(\bmod\ n^2)$ holds if and only if $\mathrm{ord}_3(\varphi(n)) \geq 1$. This does not hold if and only if $s = 1,\ 3\|n,\ 3 \nmid \varphi(n),\quad m = 2$, as claimed. □

2.B.a. The case when $s = 2$

Assume that $m = n^2\varphi(n) - k$ and $1 \leq k < n^2\varphi(n) - 2$ $(m > 2)$. Then, by Lemma 2.6, $\Theta_2 \equiv 0 \ (\bmod\ n^3)$.

Case (i):

If $k \geq 2$ is even, then $m + 1 = n^2\varphi(n) - k + 1$ is odd. Consequently $S_1 = 0$ in (2.4). Thus, combining (2.4) and (2.6) gives $S_{r,k,2}(n) = \Theta_2 + S_2 \equiv S_2(\bmod\ n^3)$, and

$$\begin{aligned} S_{r,k,2}(n) \equiv S_2 \equiv & \frac{1}{\varphi(r)r^m}\Bigg(U_m^{odd}(r) + mnU_{m-1}^{even}(r) \\ & + \binom{m}{2}n^2U_{m-2}^{odd}(r) + \binom{m}{3}n^3U_{m-3}^{even}(r)\Bigg) \ (\bmod\ n^3) \end{aligned} \tag{2.16}$$

because for every prime number $p \mid n$ by Lemma 2.5, the summands $U_m^{odd}(r)$, $\binom{m}{2}n^2U_{m-2}^{odd}(r)$, $mnU_{m-1}^{even}(r)$ and $\binom{n}{3}n^3U_{m-3}^{even}(r)$[7] are p-integral.

Case (ii):

If $k \geq 1$ is odd, then $m+1$ is even and $S_1 \neq 0$. Moreover, by Lemma 2.6, $\Theta_2 \equiv 0(\bmod\ n^3)$. Thus, by (2.4), (2.5), (2.6), we obtain

$$\begin{aligned} S_{r,k,2}(n) \equiv -\frac{\widetilde{B}_{m+1}}{m+1} + & \frac{1}{\varphi(r)r^m}\Bigg(U_m^{even}(r) \\ & + mnU_{m-1}^{odd}(r) + \binom{m}{2}n^2U_{m-2}^{even}(r)\Bigg) \ (\bmod\ n^3) \end{aligned}$$

since, by Lemmas 2.4 or 2.5, $\binom{m}{3}n^3U_{m-3}^{odd}(r)$[8] is p-integral for any $p \mid n$ and divisible by n^3.

[6] The functions $g(x)$ and $f_s(x)$ are increasing since $g'(x) = f_s'(x) = 1 - \frac{1}{(x+1)\log p} > 0$ for $x \geq 1$.

[7] With $m, m-2$ even and $m-1, m-3$ odd.

[8] With $m-3$ even.

Consequently, if k is odd and $r, \varphi(r)$ are relatively prime to n, we find, by Lemma 2.4, that

$$S_{r,k,2}(n) \equiv \frac{\widetilde{B}_{m+1}}{m+1}\left(-1+\frac{1}{\varphi(r)r^m}\prod_{q|r}(1-q^m)\right) + \frac{1}{\varphi(r)r^m}\left(NTU_m^{even}(r)+mnU_{m-1}^{odd}(r)+\binom{m}{2}n^2U_{m-2}^{even}(r)\right) \pmod{n^3}. \tag{2.17}$$

Note that for $p \mid n$, by Lemma 2.5, the summands $NTU_m^{even}(r)$, $\binom{m}{2}n^2U_{m-2}^{even}(r)$ and $mnU_{m-1}^{odd}(r)$[9] are p-integral.

Moreover, if $p \mid n$ and $p-1 \mid m+1$, i.e., p is in the denominator of B_{m+1}, then by the little Fermat theorem, we have $q^m \equiv q^{-1} (\bmod\, p^{ord_p(m+1)+1})$ and $r^m \equiv r^{-1} (\bmod\, p^{\mathrm{ord}_p(m+1)+1})$ (recall that r is coprime to n), and

$$-1+\frac{1}{\varphi(r)r^m}\prod_{q|r}(1-q^m) \equiv -1+\frac{r}{\varphi(r)}\prod_{q|r}(1-q^{-1}) = 0 \,(\bmod\, p^{\mathrm{ord}_p(m+1)+1}).$$

Hence and from Lemma 2.2, it follows that for $p \mid n$ the first summand of the right hand side of (2.17) is p-integral in the case when $p-1 \mid m+1$. If $p-1 \nmid m+1$, then the same conclusion follows from Lemma 2.3.

2.B.b. The case when s = 1

Assume that $m = n\varphi(n) - k$ and $1 \le k < n\varphi(n) - 1$ $(m > 1)$. Then, by Lemma 2.6, $\Theta_1 \equiv 0 \pmod{n^2}$ if $m > 2$. If $m = 2$ and $r \mid 8$, then the congruence holds if n is not divisible by 3 or divisible by 9. If $m = 2$ and $3 \| n$, then it is true for $3 \mid \varphi(n)$.

Case (i):

If $k \ge 2$ is even, then analysis similar to that in the proof of (2.16) shows that

$$S_{r,k,1}(n) \equiv \frac{1}{\varphi(r)r^m}(U_m^{odd}(r)+mnU_{m-1}^{even}(r) \pmod{n^2} \tag{2.18}$$

if $m > 2$ or $m = 2$ and n is not exceptional in the sense of Lemma 2.6 since $\binom{m}{2}n^2U_{m-2}^{odd}(r) + \binom{m}{3}n^3U_{m-3}^{even}(r)$ is divisible by n^2. If $m = 2$ and n is exceptional, i.e. $3\|n$ and $3 \nmid \varphi(n)$, then we should add to the right hand side of (2.18) the correction $\Theta_1 = n^2\varphi(n)/3r^3$, but we prefer to exclude the case when $m = 2$, i.e., $k = n\varphi(n) - 2$.

Case (ii):

If $k \ge 1$ is odd, then by Lemma 2.6 we have $\Theta_1 \equiv 0 (\bmod\ n^2)$ and a similar argument to that in the proof of (2.17) shows that

$$S_{r,k,1}(n) \equiv \frac{\widetilde{B}_{m+1}}{m+1}\left(-1+\frac{1}{\varphi(r)r^m}\prod_{q|r}(1-q^m)\right) + \frac{1}{\varphi(r)r^m}\left(NTU_m^{even}(r)+mnU_{m-1}^{odd}(r)+\binom{m}{2}n^2U_{m-2}^{even}(r)\right) \pmod{n^2}. \tag{2.19}$$

[9] With $m, m-2$ odd and $m-1$ even.

2.B.c. The case when $s = 0$

Assume that $m = \varphi(n) - k$ and $1 \le k < \varphi(n)$. Then, by Lemma 2.6, $\Theta_0 \equiv 0 \pmod{n}$.

Case (i):

If $k \ge 2$ is even, then in the same way as in the proof of (2.18) we obtain

$$S_{r,k,0}(n) \equiv \frac{1}{\varphi(r) r^m}\left(U_m^{odd}(r) + mnU_{m-1}^{even}(r)\right) \pmod{n}. \tag{2.20}$$

Case (ii):

If $k \ge 1$ is odd, then by a similar argument to that in the proof of (2.19) we find

$$S_{r,k,0}(n) \equiv \frac{\widetilde{B}_{m+1}}{m+1}\left(-1 + \frac{1}{\varphi(r)r^m}\prod_{q|r}(1-q^m)\right) + \frac{1}{\varphi(r)r^m} NTU_m^{even}(r) \pmod{n} \tag{2.21}$$

because $mnU_{m-1}^{odd}(r) + \binom{m}{2} n^2 U_{m-2}^{even}(r)$ is divisible by n, which is an easy con sequence of Lemmas 1 and 5.

3. The main results of the paper

In this section we compute the sums $T_{r,k}(n) \pmod{n^{s+1}}$ for $s \in \{0,1,2\}$ and all $r \mid 24$, using congruence (1.3) and congruences for the sums $S_{r,k,s}(n)$, namely congruences (2.16) and (2.17) if $s = 2$, (2.18) and (2.19) if $s = 1$, and (2.20) and (2.21) if $s = 0$.

We divide each of the three cases $s = 0, 1$ or 2 into seven subcases: $r = 2, 3, 4, 6, 8, 12, 24$, obtaining congruences for $T_{r,k}(n)$ for $1 \le k < n^s\varphi(n) - s$. In the second part of the paper we shall derive from obtained congruences some congruences in the case when $n = p$ is an odd prime. Some of such congruences were proved by M. Lerch [Ler05], E. Lehmer [Leh38] and Z.-H. Sun [Sun08], but most of them were not earlier known.

We substitute formulae (2.9-15) into congruences (2.16), (2.18) and (2.20) if k is even and congruences (2.17), (2.19) and (2.21) if k is odd. Consequently, after some calculations, we obtain Theorems and Corollaries.

In the theorems below, given any $k \ge 1$ and $\rho \in \mathbb{Z}$, we write

$$I(k,\rho) = \{n > 1 : 2 \not| n \text{ and } p \not| n \text{ if } p - 1 \mid k + \rho\}^{10}$$

for example $I(1,1) = \{n > 1 : 2,3 \not| n\}, I(3,1) = I(2,2) = \{n > 1 : 2,3,5 \not| n\}$ or $I(5,1) = I(4,2) = \{n > 1 : 2,3,7 \not| n\}$ and

$$Q_2(n) = -2q_2(n) + nq_2^2(n) - \frac{2}{3}n^2q_2^3(n),$$
$$Q_3(n) = -\frac{3}{2}q_3(n) + \frac{3}{4}nq_3^2(n) - \frac{1}{2}n^2q_3^3(n).$$

The sums $T_{r,1}(n)$ presented in Corollaries below are congruent to linear combinations of Euler's quotients $\widehat{EQ}_r(n)$ plus some generalized Bernoulli numbers where $\widehat{EQ}_2(n) = Q_2(n)$, $\widehat{EQ}_3(n) = Q_3(n)$, $\widehat{EQ}_4(n) = \frac{3}{2}Q_2(n)$, $\widehat{EQ}_6(n) = Q_2(n) + Q_3(n)$, $\widehat{EQ}_8(n) = 2Q_2(n)$, $\widehat{EQ}_{12}(n) = \frac{3}{2}Q_2(n) + Q_3(n)$ and $\widehat{EQ}_{24}(n) = 2Q_2(n) + Q_3(n)$. For $i = 2, 3$ set $Q_i''(n) = Q_i(n) \pmod{n^2}$ and $Q_i'(n) = Q_i(n) \pmod{n}$.

1. Case $r = 2$

[10] Note that if k and ρ are of the same parity and $n \in I(k,\rho)$, then $3 \not| n$.

Theorem 3.1. *Given an odd $n > 1$ and $1 \le k < n^s\varphi(n) - s$, write $m = n^s\varphi(n) - k$. Then:*
(i) *In the case $s = 2$*

$$T_{2,k}(n) \equiv \begin{cases} \frac{1}{2}(2^{k+1}-1)n\widetilde{B}_m + \frac{1}{24}\binom{k+1}{2}(2^{k+3}-1)n^3\widetilde{B}_{m-2} & \pmod{n^3} \text{ for } k \text{ even}, \\ 2^k(1-2^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} - \frac{k}{8}(2^{k+2}-1)n^2\widetilde{B}_{m-1} & \pmod{n^3} \text{ for } k \text{ odd}, \end{cases}$$

in particular, k is even and $n \in I(k, 2)$, then

$$T_{2,k}(n) \equiv \frac{1}{2}(2^{k+1}-1)n\widetilde{B}_m \pmod{n^3}.$$

(ii) *In the case $s = 1$, (cf. [Sun08] if k is odd and $n = p$ is an odd prime number)*

$$T_{2,k}(n) \equiv \begin{cases} \frac{1}{2}(2^{k+1}-1)n\widetilde{B}_m & \pmod{n^2} \text{ for } k \text{ even}, \\ 2^k(1-2^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} - \frac{k}{8}(2^{k+2}-1)n^2\widetilde{B}_{m-1} & \pmod{n^2} \text{ for } k \text{ odd}, \end{cases}$$

in particular, k is odd and $n \in I(k, 1)$, then

$$T_{2,k}(n) \equiv 2^k(1-2^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} \pmod{n^2}.$$

(iii) *In the case $s = 0$*

$$T_{2,k}(n) \equiv \begin{cases} \frac{1}{2}(2^{k+1}-1)n\widetilde{B}_m & \pmod{n} \text{ for } k \text{ even}, \\ 2^k(1-2^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} & \pmod{n} \text{ for } k \text{ odd}, \end{cases}$$

in particular, k is even and $n \in I(k, 0)$, then $T_{2,k}(n) \equiv 0 \pmod{n}$.

Proof. If k is even, resp. odd, then it suffices to apply congruence (2.16), (2,18), (2,20) resp. (2.17), (2.19), (2.21). Substituting (2.9) into these congruences gives the theorem immediately.
□

Corollary 3.2. *Let $n > 1$ be odd. Then:*
(i) *(cf. [Sun08], [Cai02] and [Leh38] if $n = p$ is an odd prime)*

$$\begin{aligned} T_{2,1}(n) &\equiv Q_2(n) - \frac{7}{8}n^2\widetilde{B}_{n^2\varphi(n)} && \pmod{n^3}, \\ T_{2,1}(n) &\equiv Q_2''(n) && \pmod{n^2} \quad \text{if } 3 \nmid n, \\ T_{2,1}(n) &\equiv Q_2'(n) && \pmod{n}. \end{aligned}$$

(ii)

$$\begin{aligned} T_{2,2}(n) &\equiv \frac{7}{2}n\widetilde{B}_{n^2\varphi(n)-2} + \frac{31}{8}n^3\widetilde{B}_{n^2\varphi(n)-4} \pmod{n^3}, \\ T_{2,2}(n) &\equiv \frac{7}{2}n\widetilde{B}_{n^2\varphi(n)-2} && \pmod{n^3} \quad \text{if } 3, 5 \nmid n, \\ T_{2,2}(n) &\equiv \frac{7}{2}n\widetilde{B}_{n\varphi(n)-2} && \pmod{n^2}, \\ T_{2,2}(n) &\equiv 0 && \pmod{n} \quad \text{if } 3 \nmid n. \end{aligned}$$

Proof. (i)This is a particular case of Theorems 3.1 for $k = 1$. Then $m+1 = n^s\varphi(n)$ and, by $2^{\varphi(n)} = nq_2(n)+1$, we have

$$\begin{aligned}2(1-2^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} &= 2(1-(1+nq_2(n))^{n^s})\frac{\widetilde{B}_{n^s\varphi(n)}}{n^s\varphi(n)}\\ &= (Q_2(n)+\alpha n^3)\frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)} \equiv Q_2(n) \pmod{n^{s+1}}\end{aligned}$$

because $\alpha\in\mathbb{Z},\ s\le 2$ and

$$\frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)} \equiv 1 \pmod{n^{s+1}} \tag{3.22}$$

Indeed, if $p_0 \mid n$ is a prime, then $(p_0-1)p_0^{(s+1)\mathrm{ord}_{p_0}(n)-1} \mid n^s\varphi(n)$ and, by Lemma 2.2,

$$\frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)} \equiv \frac{n(p_0-1)}{p_0\varphi(n)}\prod_{p\mid n, p\ne p_0}(1-p^{-1}) = 1 \pmod{p_0^{(s+1)\mathrm{ord}_{p_0}(n)}}.$$

This completes the proof of (3.22) and of Corollary 3.2 (i).

(ii) is an immediate consequence of Theorems 3.1 for $k=2$. □

2. Case $r = 3$

Theorem 3.3. *Given an odd $n>1$ not divisible by 3 and $1\le k< n^s\varphi(n)-s$, write $m = n^s\varphi(n)-k$. Then:*
(i) *In the case $s = 2$*

$$T_{3,k}(n) \equiv \begin{cases} \frac{3^{k-1}}{2}\widetilde{D}_m + \frac{1}{6}(3^{k+1}-1)n\widetilde{B}_m + \frac{3^{k-1}}{2}\binom{k+1}{2}n^2\widetilde{D}_{m-2} \pmod{n^3} & \text{for } k \text{ even}, n\in I(k,2),\\ \frac{3^k}{2}(1-3^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} - \frac{3^{k-1}}{2}kn\widetilde{D}_{m-1} - \frac{k}{36}(3^{k+2}-1)n^2\widetilde{B}_{m-1} \pmod{n^3} & \text{for } k \text{ odd.}\end{cases}$$

(ii) *In the case $s = 1$*

$$T_{3,k}(n) \equiv \begin{cases} \frac{3^{k-1}}{2}\widetilde{D}_m + \frac{1}{6}(3^{k+1}-1)n\widetilde{B}_m \pmod{n^2} & \text{for } k \text{ even},\\ \frac{3^k}{2}(1-3^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} - \frac{3^{k-1}}{2}kn\widetilde{D}_{m-1} \pmod{n^2} & \text{for } k \text{ odd},\ n\in I(k,1).\end{cases}$$

(iii) *In the case $s = 0$, (cf. [Sun08] if $n = p$ is a prime)*

$$T_{3,k}(n) \equiv \begin{cases} \frac{3^{k-1}}{2}\widetilde{D}_m \pmod{n} & \text{for } k \text{ even}, n\in I(k,0),\\ \frac{3^k}{2}(1-3^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} \pmod{n} & \text{for } k \text{ odd.}\end{cases}$$

Proof. For k even, resp. odd we combine formula (2.10) with congruence (2.16),(2.18),(2.20), resp. (2.17),(2.19),(2.21). Hence the theorem follows at once. □

Corollary 3.4. *Let $n>1$ be odd and not divisible by 3. Then:*
(i) *(cf. [Sun08] if $n = p$ is a prime)*

$$\begin{aligned}T_{3,1}(n) \equiv& Q_3(n) - \frac{1}{2}n\widetilde{D}_{n^2\varphi(n)-2} - \frac{13}{18}n^2\widetilde{B}_{n^2\varphi(n)-2} \pmod{n^3},\\ T_{3,1}(n) \equiv& Q_3''(n) - \frac{1}{2}n\widetilde{D}_{n\varphi(n)-2} \pmod{n^2},\\ T_{3,1}(n) \equiv& Q_3'(n) \pmod{n}.\end{aligned}$$

(ii)

$$\begin{aligned}
T_{3,2}(n) &\equiv \frac{3}{2}\widetilde{D}_{n^2\varphi(n)-2} + \frac{13}{3}n\widetilde{B}_{n^2\varphi(n)-2} + \frac{9}{2}n^2\widetilde{D}_{n^2\varphi(n)-4} \pmod{n^3} \quad \textit{if } \ 5 \not| n,\\
T_{3,2}(n) &\equiv \frac{3}{2}\widetilde{D}_{n\varphi(n)-2} + \frac{13}{3}n\widetilde{B}_{n\varphi(n)-2} \pmod{n^2},\\
T_{3,2}(n) &\equiv \frac{3}{2}\widetilde{D}_{\varphi(n)-2} \pmod{n}.
\end{aligned}$$

Proof. (i) This is a particular case of Theorems 3.3 for $k = 1$. Then $m+1 = n^s\varphi(n)$ and, by $3^{\varphi(n)} = nq_3(n)+1$ and (3.22), we obtain

$$\begin{aligned}
\frac{3}{2}(1-3^{m+1})\frac{\widetilde{B}_{m+1}}{m+1} &= \frac{3}{2}(1-(1+nq_3(n))^{n^s})\frac{\widetilde{B}_{n^s\varphi(n)}}{n^s\varphi(n)}\\
&\equiv (Q_3(n)+\beta n^3)\frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)} \equiv Q_3(n) \pmod{n^{s+1}}
\end{aligned}$$

because $\beta \in \mathbb{Z}$ and $s \le 2$. The rest of the proof is straightforward.

(ii) This is a particular case of Theorems 3.3 for $k = 2$. □

3. Case $r = 4$

Theorem 3.5. *Given an odd $n > 3$ and $1 \le k < n^s\varphi(n) - s$, write $m = n^s\varphi(n) - k$. Then:*
(i) *In the case s=2*

$$T_{4,k}(n) \equiv \begin{cases} 2^{2k-2}\widetilde{E}_m + 2^{k-2}(2^{k+1}-1)n\widetilde{B}_m + 2^{2k-2}\binom{k+1}{2}n^2\widetilde{E}_{m-2} & \pmod{n^3}\\ & k \ \textit{even}, n \in I(k,2),\\ 2^{2k-1}(1-2^m-2^{2m+1})\dfrac{\widetilde{B}_{m+1}}{m+1} - 2^{2k-2}kn\widetilde{E}_{m-1} & \\ \quad -2^{k-4}k(2^{k+2}-1)n^2\widetilde{B}_{m-1} & \pmod{n^3} \ \ \textit{for } k \ \textit{odd.}\end{cases}$$

(ii) *In the case $s = 1$*[11] *(cf. [Sun08] if k is odd and $n = p$ is an odd prime)*

$$T_{4,k}(n) \equiv \begin{cases} 2^{2k-2}\widetilde{E}_m + 2^{k-2}(2^{k+1}-1)n\widetilde{B}_m & \pmod{n^2} \ \ \textit{for } k \ \textit{is even},\\ 2^{2k-1}(1-2^m-2^{2m+1})\dfrac{\widetilde{B}_{m+1}}{m+1} - 2^{2k-2}kn\widetilde{E}_{m-1} & \pmod{n^2} \ \ \textit{for } k \ \textit{odd}, n \in I(k,1).\end{cases}$$

(iii) *(cf. [Sun08] if $n = p$ is an odd prime)*

$$T_{4,k}(n) \equiv \begin{cases} 2^{2k-2}\widetilde{E}_m & \pmod{n} \ \ \textit{for } k \ \textit{even}, n \in I(k,0),\\ 2^{2k-1}(1-2^m-2^{2m+1})\dfrac{\widetilde{B}_{m+1}}{m+1} & \pmod{n} \ \ \textit{for } k \ \textit{odd.}\end{cases}$$

Proof. This is an immediate consequence of (2.16-21). We apply formula (2.11). □

Corollary 3.6. *Let $n > 3$ be odd. Then:*
(i) *(cf. [Sun08] if $n = p$ is an odd prime)*

$$\begin{aligned}
T_{4,1}(n) &\equiv \frac{3}{2}Q_2(n) - n\widetilde{E}_{n^2\varphi(n)-2} - \frac{7}{8}n^2\widetilde{B}_{n^2\varphi(n)-2} \pmod{n^3},\\
T_{4,1}(n) &\equiv \frac{3}{2}Q_2''(n) - n\widetilde{E}_{n\varphi(n)-2} \pmod{n^2} \quad \textit{if } \ 3 \not| n,\\
T_{4,1}(n) &\equiv \frac{3}{2}Q_2'(n) \pmod{n}.
\end{aligned}$$

[11] Theorem 3.12(i) is also true for $k = n\varphi(n) - 2$ if we assume that n is not exceptional in the sense of Lemma 2.6; for exceptional n we should add the correction $\Theta_1 = \frac{1}{192}n^2\varphi(n)$ to the right hand side of the congruence.

(ii) (*cf.* [KUW12])

$$\begin{aligned}
T_{4,2}(n) \equiv & 4\widetilde{E}_{n^2\varphi(n)-2} + 7n\widetilde{B}_{n^2\varphi(n)-2} + 12n^2\widetilde{E}_{n^2\varphi(n)-4} \pmod{n^3} \quad \textit{if } 3,5 \not| n,\\
T_{4,2}(n) \equiv & 4\widetilde{E}_{n\varphi(n)-2} + 7n\widetilde{B}_{n\varphi(n)-2} \pmod{n^2},\\
T_{4,2}(n) \equiv & 4\widetilde{E}_{\varphi(n)-2} \pmod{n}^{12} \quad \textit{if } 3 \not| n.
\end{aligned}$$

Proof. (i) This is a particular case of Theorems 3.5 for $k = 1$. Then $m + 1 = n^s\varphi(n)$ and, by $2^{\varphi(n)} = nq_2(n) + 1$ and (3.22), we have

$$\begin{aligned}
2(1-2^m-2^{2m+1})\frac{\widetilde{B}_{m+1}}{m+1} &= ((1-(1+nq_2(n))^{n^s}) + (1-(1+nq_2(n))^{2n^s}))\frac{\widetilde{B}_{n^s\varphi(n)}}{n^s\varphi(n)}\\
&\equiv \frac{1}{2}Q_2(n) + Q_2(n) + \gamma n^3)\frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)} \equiv \frac{3}{2}Q_2(n) \pmod{n^{s+1}}
\end{aligned}$$

because $\gamma \in \mathbb{Z}$ and $s \leq 2$. This gives the theorem at once since the rest of the proof is straightforward.

(ii) This is a particular case of Theorems 3.5 in case $k = 2$. □

4. Case $r = 6$

Theorem 3.7. *Given an odd $n > 5$ not divisible by 3 and $1 \leq k < n^s\varphi(n) - s$, write $m = n^s\varphi(n) - k$. Then:*

(i) *In the case $s = 2$*

$$T_{6,k}(n) \equiv \begin{cases} \frac{3^{k-1}}{2}(2^k+1)\widetilde{D}_m + \frac{1}{12}(2^{k+1}-1)(3^{k+1}-1)n\widetilde{B}_m \\ \quad + \frac{3^{k-1}}{8}\binom{k+1}{2}(2^{k+2}-1)n^2\widetilde{D}_{m-2} \pmod{n^3} & \textit{for } k \textit{ even}, n \in I(k,2),\\ 2^{k-1}3^k(1-2^m-3^m-6^m)\frac{\widetilde{B}_{m+1}}{m+1} - \frac{3^{k-1}}{4}(2^{k+1}+1)kn\widetilde{D}_{m-1} \\ \quad - \frac{k}{144}(2^{k+2}-1)(3^{k+2}-1)n^2\widetilde{B}_{m-1} \pmod{n^3} & \textit{for } k \textit{ odd.} \end{cases}$$

(ii) *In the case $s = 1$*

$$T_{6,k}(n) \equiv \begin{cases} \frac{3^{k-1}}{2}(2^k+1)\widetilde{D}_m + \frac{1}{12}(2^{k+1}-1)(3^{k+1}-1)n\widetilde{B}_m \pmod{n^2} & \textit{for } k \textit{ even},\\ 2^{k-1}3^k(1-2^m-3^m-6^m)\frac{\widetilde{B}_{m+1}}{m+1} \\ \quad - \frac{3^{k-1}}{4}(2^{k+1}+1)kn\widetilde{D}_{m-1} \pmod{n^2} & \textit{for } k \textit{ odd }, n \in I(k,1). \end{cases}$$

(iii) *In the case $s = 0$ (cf. [Sun08] if $n = p$ is an odd prime)*

$$T_{6,k}(n) \equiv \begin{cases} \frac{3^{k-1}}{2}(2^k+1)\widetilde{D}_m \pmod{n} & \textit{for } k \textit{ even}, n \in I(k,0),\\ 2^{k-1}3^k(1-2^m-3^m-6^m)\frac{\widetilde{B}_{m+1}}{m+1} \pmod{n} & \textit{for } k \textit{ odd.} \end{cases}$$

Proof. This is an immediate consequence of congruences (2.16),(2.18),(2.20) if k is even or (2.17),(2.19),(2.21) if k is odd and formula (2.12). □

Corollary 3.8. *Let $n > 5$ be odd and not divisible by 3. Then:*
(i)

$$
\begin{aligned}
T_{6,1}(n) \equiv& Q_2(n) + Q_3(n) - \frac{5}{4} n \widetilde{D}_{n^2\varphi 6(n)-2} - \frac{91}{72} n^2 \widetilde{B}_{n^2\varphi(n)-2} \pmod{n^3},\\
T_{6,1}(n) \equiv& Q''_2(n) + Q''_3(n) - \frac{5}{4} n \widetilde{D}_{n\varphi(n)-2} \pmod{n^2},\\
T_{6,1}(n) \equiv& Q'_2(n) + Q'_3(n) \pmod{n}.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
T_{6,2}(n) \equiv& \frac{15}{2} \widetilde{D}_{n^2\varphi(n)-2} + \frac{91}{6} n \widetilde{B}_{n^2\varphi(n)-2} + \frac{153}{8} n^2 \widetilde{D}_{n^2\varphi(n)-4} \pmod{n^3},\\
T_{6,2}(n) \equiv& \frac{15}{2} \widetilde{D}_{n\varphi(n)-2} + \frac{91}{6} n \widetilde{B}_{n\varphi(n)-2} \pmod{n^2},\\
T_{6,2}(n) \equiv& \frac{15}{2} \widetilde{D}_{\varphi(n)-2} \pmod{n}.
\end{aligned}
$$

Proof. (i) This is a particular case of Theorems 3.7 for $k = 1$. Then $m + 1 = n^s\varphi(n)$ and, in view of $2^{\varphi(n)} = nq_2(n) + 1, 3^{\varphi(n)} = nq_3(n) + 1$ and (3.22), we find that

$$
\begin{aligned}
3(1 - 2^m - 3^m - 6^m)\frac{\widetilde{B}_{m+1}}{m+1} =& \frac{1}{2}(3(1 - (1 + nq_2(n))^{n^s}) + 2(1 - (1 + nq_3(n))^{n^s})\\
&+ (1 - (1 + nq_2(n)^{n^s}(1 + nq_3(n))^{n^s}))\frac{\widetilde{B}_{n^s\varphi(n)}}{n^s\varphi(n)}\\
\equiv& \frac{1}{2}(\frac{3}{2}Q_2(n) + \frac{4}{3}Q_3(n) + \frac{1}{2}Q_2(n) + \frac{2}{3}Q_3(n) + \lambda n^3)\frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)}\\
\equiv& Q_2(n) + Q_3(n) \pmod{n^{s+1}}
\end{aligned}
$$

because $\lambda \in \mathbb{Z}$ and $s \le 2$. This gives the theorem.

(ii) The theorem follows easily from Theorems 3.7 for $k = 2$. □

5. Case $r = 8$

Theorem 3.9. *Given an odd $n > 7$ and $1 \le k < n^s\varphi(n) - s$*[13]*, write $m = n^s\varphi(n) - k$. Then:*
(i) *In the case $s = 2$*

$$
\begin{aligned}
T_{8,k}(n) \equiv& 2^{3k-3}\widetilde{E}_m + 2^{3k-2}\widetilde{C}_m + 2^{2k-3}(2^{k+1} - 1)n\widetilde{B}_m - 2^{3k-2}kn\widetilde{A}_{m-1}\\
&+ 2^{3k-3}\binom{k+1}{2}n^2\widetilde{E}_{m-2} + 2^{3k-2}\binom{k+1}{2}n^2\widetilde{C}_{m-2}\\
&\pmod{n^3} \qquad \text{for } k \text{ even}, n \in I(k, 2),\\
T_{8,k}(n) \equiv& 2^{3k-2}(1 - 2^m - 2^{3m+2})\frac{\widetilde{B}_{m+1}}{m+1} + 2^{3k-2}\widetilde{A}_m - 2^{3k-3}kn\widetilde{E}_{m-1}\\
&- 2^{3k-2}kn\widetilde{C}_{m-1} - 2^{2k-5}kn^2(2^{k+2} - 1)\widetilde{B}_{m-1} + 2^{3k-2}\binom{k+1}{2}n^2\widetilde{A}_{m-2}\\
&\pmod{n^3} \qquad \text{for } k \text{ odd}.
\end{aligned}
$$

[13] Theorem 3.9 (ii) is also true for $k = n\varphi(n) - 2$ if we assume that n is not exceptional in the sense of Lemma 2.6; for exceptional n we should add the correction $\Theta_1 = \frac{1}{1536}n^2\varphi(n)$ to the right hand side of the congruence.

(ii) *In the case* $s=1$

$$T_{8,k}(n)\equiv\begin{cases}2^{3k-3}\widetilde{E}_m+2^{3k-2}\widetilde{C}_m+2^{2k-3}(2^{k+1}-1)n\widetilde{B}_m\\ \quad -2^{3k-2}kn\widetilde{A}_{m-1} \pmod{n^2} & \text{for } k \text{ is even,}\\ 2^{3k-2}(1-2^m-2^{m+2})\dfrac{\widetilde{B}_{m+1}}{m+1}+2^{3k-2}\widetilde{A}_m\\ \quad -2^{3k-3}kn\widetilde{E}_{m-1}-2^{3k-2}kn\widetilde{C}_{m-1} \pmod{n^2} & \text{for } k \text{ odd}, n\in I(k,1).\end{cases}$$

(iii) *In the case* $s=0$

$$T_{8,k}(n)\equiv\begin{cases}2^{3k-3}\widetilde{E}_m+2^{3k-2}\widetilde{C}_m & \pmod{n} \text{ for } k \text{ even}, n\in I(k,0),\\ 2^{3k-2}(1-2^m-2^{3m+2})\dfrac{\widetilde{B}_{m+1}}{m+1}+2^{3k-2}\widetilde{A}_m & \pmod{n} \text{ for } k \text{ odd.}\end{cases}$$

Proof. This follows from congruence (2.16),(2.18),(2.20), resp. (2.17),(2.19),(2.21) for k even, resp. odd and formula (2.13). □

Corollary 3.10. *Let* $n>7$ *be odd. Then:*
(i)

$$\begin{aligned}T_{8,1}(n)\equiv&2Q_2(n)+2\widetilde{A}_{n^2\varphi(n)-1}-n\widetilde{E}_{n^2\varphi(n)-2}-2n\widetilde{C}_{n^2\varphi(n)-2}\\ &-\frac{7}{8}n^2\widetilde{B}_{n^2\varphi(n)-2}+2n^2\widetilde{A}_{n^2\varphi(n)-3} \pmod{n^3},\\ T_{8,1}(n)\equiv&2Q_2''(n)+2\widetilde{A}_{n\varphi(n)-1}-n\widetilde{E}_{n\varphi(n)-2}-2n\widetilde{C}_{n\varphi(n)-2} \pmod{n^2} \quad \text{if } 3\nmid n,\\ T_{8,1}(n)\equiv&2Q_2'(n)+2\widetilde{A}_{\varphi(n)-1} \pmod{n}.\end{aligned}$$

(ii)

$$\begin{aligned}T_{8,2}(n)\equiv&8\widetilde{E}_{n^2\varphi(n)-2}+16\widetilde{C}_{n^2\varphi(n)-2}+14n\widetilde{B}_{n^2\varphi(n)-2}-32n\widetilde{A}_{n^2\varphi(n)-3}\\ &+24n^2\widetilde{E}_{n^2\varphi(n)-4}+48n^2\widetilde{C}_{n^2\varphi(n)-4} \pmod{n^3} \quad \text{if } 3,5\nmid n,\\ T_{8,2}(n)\equiv&8\widetilde{E}_{n\varphi(n)-2}+16\widetilde{C}_{n\varphi(n)-2}+14n\widetilde{B}_{n\varphi(n)-2}-32n\widetilde{A}_{n\varphi(n)-3} \pmod{n^2},\\ T_{8,2}(n)\equiv&8\widetilde{E}_{\varphi(n)-2}+16\widetilde{C}_{\varphi(n)-2} \pmod{n} \quad \text{if } 3\nmid n.\end{aligned}$$

Proof. (i) This is a particular case of Theorems 3.9 for $k=1$. Then $m+1=n^s\varphi(n)$ and, by virtue of $2^{\varphi(n)}=nq_2(n)+1$ and (3.22), we obtain

$$\begin{aligned}2(1-2^m-2^{3m+2})\frac{\widetilde{B}_{m+1}}{m+1}&=(2-(1+nq_2(n))^{n^s}-(1+nq_2(n))^{3n^s})\frac{\widetilde{B}_{n^s\varphi(n)}}{n^s\varphi(n)}\\ &\equiv(\frac{1}{2}Q_2(n)+\frac{3}{2}Q_2(n)+\xi n^3)\frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)}\\ &\equiv 2Q_2(n) \pmod{n^{s+1}}\end{aligned}$$

because $\xi\in\mathbb{Z}$ and $s\le 2$. This gives the theorem at once. (ii) It is trivial. □

6. Case $r=12$

Theorem 3.11. *Given an odd* $n>11$ *not divisible by 3 and* $1\le k<n^s\varphi(n)-s$*, write* $m=n^s\varphi(n)-k$*. Then:*

(i) *In the case* $s = 2$

$$\begin{aligned}
T_{12,k}(n) \equiv & 2^{k-2}3^{k-1}(2^k+1)\widetilde{D}_m + 2^{2k-3}(3^k+1)\widetilde{E}_m \\
& + \frac{2^{k-3}}{3}(2^{k+1}-1)(3^{k+1}-1)n\widetilde{B}_m - 2^{2k-2}3^k kn\widetilde{F}_{m-1} \\
& + 2^{k-4}3^{k-1}\binom{k+1}{2}(2^{k+2}+1)n^2\widetilde{D}_{m-2} \\
& + \frac{2^{2k-3}}{9}\binom{k+1}{2}(3^{k+2}+1)n^2\widetilde{E}_{m-2} \pmod{n^3} \quad \textit{for } k \textit{ even}, n \in I(k,2), \\
T_{12,k}(n) \equiv & 2^{2k-2}3^k(1-2^m-3^m+6^m-4\cdot 12^m)\frac{\widetilde{B}_{m+1}}{m+1} + 2^{2k-2}3^k\widetilde{F}_m \\
& - 2^{k-3}3^{k-1}(2^{k+1}+1)kn\widetilde{D}_{m-1} - \frac{2^{2k-3}}{3}(3^{k+1}+1)kn\widetilde{E}_{m-1} \\
& - \frac{2^{k-5}}{9}(2^{k+2}-1)(3^{k+2}-1)kn^2\widetilde{B}_{m-1} \\
& + 2^{2k-2}3^k\binom{k+1}{2}n^2\widetilde{F}_{m-2} \qquad \pmod{n^3} \quad \textit{for } k \textit{ odd.}
\end{aligned}$$

(ii) *In the case* $s = 1$

$$T_{12,k}(n)$$

$$\equiv \begin{cases} 2^{k-2}3^{k-1}(2^k+1)\widetilde{D}_m + 2^{2k-3}(3^k+1)\widetilde{E}_m \\ \quad + \dfrac{2^{k-1}}{3}(2^{k+1}-1)(3^{k+1}-1)n\widetilde{B}_m - 2^{2k-2}3^k kn\widetilde{F}_{m-1} \pmod{n^2} & \textit{for } k \textit{ even}, \\ 2^{2k-2}3^k(1-2^m-3^m+6^m-4\cdot 12^m)\dfrac{\widetilde{B}_{m+1}}{m+1} + 2^{2k-2}3^k\widetilde{F}_m \\ \quad -2^{k-3}3^{k-1}(2^{k+1}+1)kn\widetilde{D}_{m-1} - \dfrac{2^{2k-3}}{3}(3^{k+1}+1)kn\widetilde{E}_{m-1} \pmod{n^2} & \textit{for } k \textit{ odd}, n \in I(k,1). \end{cases}$$

(iii) *In the case* $s = 0$

$$T_{12,k}(n) \equiv \begin{cases} 2^{k-2}3^{k-1}(2^k+1)\widetilde{D}_m + 2^{2k-3}(3^k+1)\widetilde{E}_m \pmod{n} & \textit{for } k \textit{ even}, n \in I(k,1), \\ 2^{2k-2}3^k(1-2^m-3^m+6^m-4\cdot 12^m)\dfrac{\widetilde{B}_{m+1}}{m+1} \\ \quad +2^{2k-2}3^k\widetilde{F}_m \pmod{n} & \textit{for } k \textit{ odd.} \end{cases}$$

Proof. Apply congruences (2.16),(2.18),(2.20), resp. (2.17),(2.19),(2.21) and formula (2.14). □

Corollary 3.12. *Let* $n > 11$ *be odd not divisible by 3. Then:*
(i)

$$\begin{aligned}
T_{12,1}(n) \equiv & \frac{3}{2}Q_2(n) + Q_3(n) + 3\widetilde{F}_{n^2\varphi(n)-1} - \frac{5}{4}n\widetilde{D}_{n^2\varphi(n)-2} - \frac{5}{3}n\widetilde{E}_{n^2\varphi(n)-2} \\
& - \frac{91}{72}n^2\widetilde{B}_{n^2\varphi(n)-2} + 3n^2\widetilde{F}_{n^2\varphi(n)-3} \qquad \pmod{n^3}, \\
T_{12,1}(n) \equiv & \frac{3}{2}Q_2''(n) + Q_3(n)'' + 3\widetilde{F}_{n\varphi(n)-1} - \frac{5}{4}n\widetilde{D}_{n\varphi(n)-2} - \frac{5}{3}n\widetilde{E}_{n\varphi(n)-2} \pmod{n^2}, \\
T_{12,1}(n) \equiv & \frac{3}{2}Q_2'(n) + Q_3'(n) + 3\widetilde{F}_{\varphi(n)-1} \qquad \pmod{n}.
\end{aligned}$$

(ii)

$$
\begin{aligned}
T_{12,2}(n) \equiv & 15\widetilde{D}_{n^2\varphi(n)-2} + 20\widetilde{E}_{n^2\varphi(n)-2} + \frac{91}{3}n\widetilde{B}_{n^2\varphi(n)-2} - 72n\widetilde{F}_{n^2\varphi(n)-3} \\
& + \frac{153}{4}\widetilde{D}_{n^2\varphi(n)-4} + \frac{164}{3}n^2\widetilde{E}_{n^2\varphi(n)-4} \pmod{n^3} for 5 \not| n, \\
T_{12,2}(n) \equiv & 15\widetilde{D}_{n\varphi(n)-2} + 20\widetilde{E}_{n\varphi(n)-2} + \frac{91}{3}n\widetilde{B}_{n\varphi(n)-2} - 72n\widetilde{F}_{n\varphi(n)-3} \pmod{n^2}, \\
T_{12,2}(n) \equiv & 15\widetilde{D}_{\varphi(n)-2} + 20\widetilde{E}_{\varphi(n)-2} \qquad \pmod{n}.
\end{aligned}
$$

Proof. (i) This is a particular case of Theorems 3.11 for $k = 1$. Then $m + 1 = n^s\varphi(n)$ and, by virtue of $2^{\varphi(n)} = nq_2(n) + 1, 3^{\varphi(n)} = nq_3(n) + 1$ and (3.22), we have

$$
\begin{aligned}
& 3(1 - 2^m - 3^m + 6^m - 4 \cdot 12^m)\frac{\widetilde{B}_{m+1}}{m+1} \\
& = \Big(\frac{3}{2}(1 - (1 + nq_2(n))^{n^s}) + (1 - (1 + nq_3(n))^{n^s}) \\
& \quad - \frac{1}{2}(1 - (1 + nq_2(n))^{2n^s}(1 - (1 + nq_3(n))^{n^s}) \\
& \quad + (1 - (1 + nq_2(n))^{2n^s}(1 + nq_3(n))^{n^s})\Big) \frac{\widetilde{B}_{n^s\varphi(n)}}{n^s\varphi(n)} \\
& \equiv \left(\frac{3}{4}Q_2(n) + \frac{2}{3}Q_3(n) - \frac{1}{4}Q_2(n) - \frac{1}{3}Q_3(n) + Q_2(n) + \frac{2}{3}Q_3(n) + \eta n^3\right) \frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)} \\
& \equiv \frac{3}{2}Q_2(n) + Q_3(n) \qquad \pmod{n^{s+1}}
\end{aligned}
$$

because $\eta \in \mathbb{Z}$ and $s \leq 2$. The rest of the proof is straightforward. $\square$

7. Case $r = 24$

Theorem 3.13. *Given an odd $n > 23$ not divisible by 3 and $1 \leq k < n^s\varphi(n) - s$, write $m = n^s\varphi(n) - k$. Then:*
(i) *In the case $s = 2$*

$$
\begin{aligned}
T_{24,k}(n) \quad \equiv & 2^{2k-3}3^{k-1}(2^k + 1)\widetilde{D}_m + 2^{3k-4}(3^k + 1)\widetilde{E}_m + 2^{3k-3}3^k\widetilde{H}_m \\
& + 2^{3k-3}(3^k - 1)\widetilde{C}_m + \frac{2^{2k-4}}{3}(2^{k+1} - 1)(3^{k+1} - 1)n\widetilde{B}_m - 2^{3k-3}3^k kn\widetilde{F}_{m-1} \\
& - 2^{3k-3}3^k kn\widetilde{G}_{m-1} - \frac{2^{3k-3}}{3}(3^{k+1} + 1)kn\widetilde{A}_{m-1} \\
& + 2^{3k-5}3^{k-1}\binom{k+1}{2}(2^{k+2} + 1)n^2\widetilde{D}_{m-2} + \frac{2^{3k-4}}{9}\binom{k+1}{2}(3^{k+2} + 1)n^2\widetilde{E}_{m-2} \\
& + 2^{3k-3}3^k\binom{k+1}{2}n^2\widetilde{H}_{m-2} + \frac{2^{3k-3}}{9}\binom{k+1}{2}(3^{k+2} - 1)n^2\widetilde{C}_{m-2} \\
& \qquad \pmod{n^3} \qquad \textit{for } k \textit{ even} \quad n \in I(k,2),
\end{aligned}
$$

$$\begin{aligned}T_{24,k}(n)\ \equiv&2^{3k-3}3^k(1-2^m-3^m+6^m-8\cdot 24^m)\frac{\widetilde{B}_{m+1}}{m+1}+2^{3k-3}3^k\widetilde{F}_m\\&+2^{3k-3}3^k\widetilde{G}_m+2^{3k-3}(3^k+1)\widetilde{A}_m-2^{2k-4}3^{k-1}(2^{k+1}+1)kn\widetilde{D}_{m-1}\\&-\frac{2^{3k-4}}{3}(3^{k+1}+1)kn\widetilde{E}_{m-1}-2^{3k-3}3^k kn\widetilde{H}_{m-1}-\frac{2^{3k-3}}{3}(3^{k+1}-1)kn\widetilde{C}_{m-1}\\&-\frac{2^{3k-6}}{9}(2^{k+2}-1)(3^{k+2}-1)kn^2\widetilde{B}_{m-1}-2^{3k-3}3^k\binom{k+1}{2}n^2\widetilde{F}_{m-2}\\&+2^{3k-3}3^k\binom{k+1}{2}n^2\widetilde{G}_{m-2}+\frac{2^{3k-3}}{9}\binom{k+1}{2}(3^{k+2}+1)n^2\widetilde{A}_{m-2}\\&\qquad(\bmod n^3)\quad \textit{for } k \textit{ odd.}\end{aligned}$$

(ii) *In the case* $s=1$

$$\begin{aligned}T_{24,k}(n)\equiv&2^{2k-3}3^{k-1}(2^k+1)\widetilde{D}_m+2^{3k-4}(3^k+1)\widetilde{E}_m+2^{3k-3}3^k\widetilde{H}_m\\&+2^{3k-3}(3^k-1)\widetilde{C}_m+\frac{2^{2k-4}}{3}(2^{k+1}-1)(3^{k+1}-1)n\widetilde{B}_m-2^{3k-3}3^k kn\widetilde{F}_{m-1}\\&-2^{3k-3}3^k kn\widetilde{G}_{m-1}-\frac{2^{3k-3}}{3}(3^{k+1}+1)kn\widetilde{A}_{m-1}\ (\bmod n^2)\quad \textit{for } k \textit{ even,}\\T_{24,k}(n)\equiv&2^{3k-3}3^k(1-2^m-3^m+6^m-8\cdot 24^m)\frac{\widetilde{B}_{m+1}}{m+1}+2^{3k-3}3^k\widetilde{F}_m+2^{3k-3}3^k\widetilde{G}_m\\&+2^{3k-3}(3^k+1)\widetilde{A}_m-2^{2k-4}3^{k-1}(2^{k+1}+1)kn\widetilde{D}_{m-1}-\frac{2^{3k-4}}{3}(3^{k+1}+1)kn\widetilde{E}_{m-1}\\&-2^{3k-3}3^k kn\widetilde{H}_{m-1}-\frac{2^{3k-3}}{3}(3^{k+1}-1)kn\widetilde{C}_{m-1}\ (\bmod n^2)\quad \textit{for } k \textit{ odd}, n\in I(k,1).\end{aligned}$$

(iii) *In the case* $s=0$

$$T_{24,k}(n)\equiv\begin{cases}2^{2k-3}3^{k-1}(2^k+1)\widetilde{D}_m+2^{3k-4}(3^k+1)\widetilde{E}_m\\+2^{3k-3}3^k\widetilde{H}_m+2^{3k-3}(3^k-1)\widetilde{C}_m & (\bmod n)\qquad \textit{for } k \textit{ even}\quad n\in I(k,0),\\2^{3k-3}(1-2^m-3^m+6^m-8\cdot 24^m)\frac{\widetilde{B}_{m+1}}{m+1}+2^{3k-3}3^k\widetilde{F}_m\\+2^{3k-3}3^k\widetilde{G}_m+2^{3k-3}(3^k+1)\widetilde{A}_m & (\bmod n)\qquad \textit{for } k \textit{ odd.}\end{cases}$$

Proof. This follows from congruences (2.16),(2.18),(2.20), resp. (2.17),(2.19),(2.21) if k is even, resp. odd with the use of (2.15). □

Corollary 3.14. *Let* $n>23$ *be odd and not divisible by 3. Then:*
(i)

$$\begin{aligned}T_{24,1}(n)\equiv&2Q_2(n)+Q_3(n)+3\widetilde{F}_{n^2\varphi(n)-1}+3\widetilde{G}_{n^2\varphi(n)-1}+4\widetilde{A}_{n^2\varphi(n)-1}-\frac{5}{4}n\widetilde{D}_{n^2\varphi(n)-2}\\&-\frac{5}{3}n\widetilde{E}_{n^2\varphi(n)-2}-3n\widetilde{H}_{n^2\varphi(n)-2}-\frac{8}{3}n\widetilde{C}_{n^2\varphi(n)-2}-\frac{91}{72}n^2\widetilde{B}_{n^2\varphi(n)-2}\\&+3n^2\widetilde{F}_{n^2\varphi(n)-3}+3n^2\widetilde{G}_{n^2\varphi(n)-3}+\frac{28}{9}n^2\widetilde{A}_{n^2\varphi(n)-3}\qquad(\bmod n^3),\\T_{24,1}(n)\equiv&2Q''_2(n)+Q''_3(n)+3\widetilde{F}_{n\varphi(n)-1}+3\widetilde{G}_{n\varphi(n)-1}+4\widetilde{A}_{n\varphi(n)-1}\\&-\frac{5}{4}n\widetilde{D}_{n\varphi(n)-2}-\frac{5}{3}n\widetilde{E}_{n\varphi(n)-2}-3n\widetilde{H}_{n\varphi(n)-2}-\frac{8}{3}n\widetilde{C}_{n\varphi(n)-2}\ (\bmod n^2),\\T_{24,1}(n)\equiv&2Q'_2(n)+Q'_3(n)+3\widetilde{F}_{\varphi(n)-1}+3\widetilde{G}_{\varphi(n)-1}+4\widetilde{A}_{\varphi(n)-1}\qquad(\bmod n).\end{aligned}$$

(ii)

$$
\begin{aligned}
T_{24,2}(n) \equiv & 30\widetilde{D}_{n^2\varphi(n)-2} + 40\widetilde{E}_{n^2\varphi(n)-2} + 72\widetilde{H}_{n^2\varphi(n)-2} + 64\widetilde{C}_{n^2\varphi(n)-2} \\
& + \frac{182}{3} n\widetilde{B}_{n^2\varphi(n)-2} - 36n\widetilde{F}_{n^2\varphi(n)-3} - 144n\widetilde{G}_{n^2\varphi(n)-3} \\
& - \frac{448}{3} n\widetilde{A}_{n^2\varphi(n)-3} + \frac{153}{2} n^2\widetilde{D}_{n^2\varphi(n)-4} + \frac{328}{3} n^2\widetilde{E}_{n^2\varphi(n)-4} \\
& + 216n^2\widetilde{H}_{n^2\varphi(n)-4} + \frac{640}{3} n^2\widetilde{C}_{n^2\varphi(n)-4} \qquad \pmod{n^3} \quad \textit{if} \;\; 5 \nmid n, \\
T_{24,2}(n) \equiv & 30\widetilde{D}_{n\varphi(n)-2} + 40\widetilde{E}_{n\varphi(n)-2} + 72\widetilde{H}_{n\varphi(n)-2} + 64\widetilde{C}_{n\varphi(n)-2} \\
& + \frac{182}{3} n\widetilde{B}_{n\varphi(n)-2} - 36n\widetilde{F}_{n\varphi(n)-3} - 144n\widetilde{G}_{n\varphi(n)-3} \\
& - \frac{448}{3} n\widetilde{A}_{n\varphi(n)-3} \qquad \pmod{n^2}, \\
T_{24,2}(n) \equiv & 30\widetilde{D}_{\varphi(n)-2} + 40\widetilde{E}_{\varphi(n)-2} + 72\widetilde{H}_{\varphi(n)-2} + 64\widetilde{C}_{\varphi(n)-2} \qquad \pmod{n}.
\end{aligned}
$$

Proof. (i) This is a particular case of Theorems 3.13 for $k = 1$. Then $m + 1 = n^s\varphi(n)$ and, in view of $2^{\varphi(n)} = nq_2(n) + 1, 3^{\varphi(n)} = nq_3(n) + 1$ and (3.22), we have

$$
\begin{aligned}
& 3(1 - 2^m - 3^m + 6^m - 8 \cdot 24^m) \frac{\widetilde{B}_{m+1}}{m+1} \\
& = \Big(\frac{3}{2}(1 - (1 + nq_2(n))^{n^s}) + (1 - (1 + nq_3(n))^{n^s}) \\
& \quad - \frac{1}{2}(1 - (1 + nq_2(n))^{n^s}(1 - (1 + nq_3(n))^{n^s}) \\
& \quad + (1 - (1 + nq_2(n))^{3n^s}(1 + nq_3(n))^{n^s}) \Big) \frac{\widetilde{B}_{n^s\varphi(n)}}{n^s\varphi(n)} \\
& \equiv \left(\frac{3}{4}Q_2(n) + \frac{2}{3}Q_3(n) - \frac{1}{4}Q_2(n) - \frac{1}{3}Q_3(n) + \frac{3}{2}Q_2(n) + \frac{2}{3}Q_3(n) + \omega n^3 \right) \frac{n\widetilde{B}_{n^s\varphi(n)}}{\varphi(n)} \\
& \equiv 2Q_2(n) + Q_3(n) \qquad \pmod{n^{s+1}}
\end{aligned}
$$

because $\omega \in \mathbb{Z}$ and $s \leq 2$. This proves the theorem. (ii) It is trivial. ☐

4. Further congruences of E. Lehmer's type (by T. Kuzumaki and J. Urbanowicz)

In Theorem 4.1 below we find some congruences for $U_r(n)$ modulo n^{s+1} for $s \in \{0, 1, 2\}$ in each of the seven cases $r = 2, 3, 4, 6, 8, 12$ or 24. Some of these congruences for $s \in \{0, 1\}$ and $r \in \{2, 3, 4, 6\}$ were proved in [CFZ07] and [CP09]. The remaining ones are new. Three of them for $s = 1$ and $r \in \{8, 12, 24\}$ were omitted both in [Leh38] and in [CFZ07], [CP09].

Write $\rho_i(r) = 1 - \delta_{\mathrm{ord}_i(r),0}$ $(i = 2, 3)$ where, as usual, $\delta_{X,Y}$ denotes the Kronecker delta function. Given odd $n > r$, we set

$$
\begin{aligned}
EQ_r(n) = & \alpha_2(r)q_2(r) + \alpha_3(r)q_3(r) + \beta_2(r)nq_2^2(n) + \beta_3(r)nq_3^2(n) \\
& + \gamma_2(r)n^2q_2^3(n) + \gamma_3(r)n^2q_3^3(n),
\end{aligned} \tag{4.23}
$$

where

$$\alpha_i(r) = \rho_i(r)\left(\frac{\mathrm{ord}_i(r)}{r} + \frac{1}{i\varphi(r)} - \frac{\rho_{5-i}(r)}{6\varphi(r)}\right),$$
$$\beta_i(r) = \rho_i(r)\left(-\frac{\mathrm{ord}_2(r)}{2r} - \frac{1}{2i\varphi(r)} + \frac{\rho_{5-i}(r)}{12\varphi(r)}\right),$$
$$\gamma_i(r) = \rho_i(r)\left(\frac{\mathrm{ord}_i(r)}{3r} + \frac{1}{3i\varphi(r)} - \frac{\rho_{5-i}(r)}{18\varphi(r)}\right)$$

and

$$B_r(n) = -\frac{n^2}{2r^3}\widetilde{B}_{n^2\varphi(n)-2}.$$

Set $EQ'_r(n) = \alpha_2(r)q_2(r)+\alpha_3(r)q_3(r)$ and $EQ''_r = \alpha_2(r)q_2(r)+\alpha_3(r)q_3(r)+\beta_2(r)nq_2^2(n)+\beta_3(r)nq_3^2(n)$. Obviously, we have $EQ_r(n) \equiv EQ'_r(n) \pmod n$ and $EQ_r(n) \equiv EQ''_r(n) \pmod{n^2}$. Note that $B_r(n) \equiv 0 \pmod n$, and $B_r(n) \equiv 0 \pmod{n^2}$ if n is not divisible by 3.

It was shown in Section 3 that the sums $T_{r,1}(n)$ are congruent to linear combinations of Euler's quotients $\widehat{EQ}_r(n)$ plus some generalized Bernoulli numbers. In view of Proposition 4.2 below we have $EQ_r(n) = -\frac{1}{r}\widehat{EQ}_r(n)$.

Theorem 4.1. *Assume that $s \in \{0,1,2\}$ and $r \mid 24$. Let $n > r$ be odd and not divisible by 3 if $s = 1$ or $3 \mid r$. Then, in the above notation:*

$$U_r(n) \equiv \begin{cases} EQ_r(n) + B_r(n) \pmod{n^{s+1}} & \textit{for } r \le 6,\\ EQ_r(n) + B_r(n) - \frac{1}{4}\widetilde{A}_{n^s\varphi(n)-1} \pmod{n^{s+1}} & \textit{for } r = 8,\\ EQ_r(n) + B_r(n) - \frac{1}{4}\widetilde{F}_{n^s\varphi(n)-1} \pmod{n^{s+1}} & \textit{for } r = 12,\\ EQ_r(n) + B_r(n) - \frac{1}{6}\widetilde{A}_{n^s\varphi(n)-1} - \frac{1}{8}\widetilde{F}_{n^s\varphi(n)-1} - \frac{1}{8}\widetilde{G}_{n^s\varphi(n)-1} \pmod{n^{s+1}} & \textit{for } r = 24. \end{cases}$$

Here $EQ_r(n) \equiv EQ'_r(n) \pmod n, EQ_r(n) \equiv EQ''(n) \pmod{n^2}, B_r(n) \equiv 0 \pmod n$ if n is not divisible by 3 and $B_r(n) \equiv 0 \pmod n$.

4.A. Some useful observations

We deduce Theorem 4.1 from Propositions 4.2,4.3 and congruences for the sums $T_{r,k}(n)$ given in Section 3. First we find some useful congruences modulo powers of n between the sums $U_r(n)$ and some linear combinations of $T_{r,1}(n)$, $T_{r,2}(n)$ and $T_{r,3}(n)$.

Proposition 4.2. (*cf.* [Leh38].) *Assume that $n > 1$ is odd and r $(1 < r < n)$ is coprime to n. Then:*

$$U_r(n) \equiv \begin{cases} -\frac{1}{r}T_{r,1}(n) - \frac{n}{r^2}T_{r,2}(n) - \frac{n^2}{r^3}T_{r,3}(n) & \pmod{n^3},\\ -\frac{1}{r}T_{r,1}(n) - \frac{n}{r^2}T_{r,2}(n) & \pmod{n^2},\\ -\frac{1}{r}T_{r,1}(n) & \pmod{n}. \end{cases}$$

Proof. Obviously, $(n,i) = 1$ if and only if $(n - ri, n) = 1$. Consequently,

$$U_r(n) \equiv \sum_{0<i<\frac{n}{r}} \chi_n(i)(n-ri)^{n^s\varphi(n)-1}$$
$$= \sum_{0<i<\frac{n}{r}} \chi_n(i) \sum_{j=0}^{n^s\varphi(n)-1} \binom{n^s\varphi(n)-1}{j} n^j(-ri)^{n^s\varphi(n)-1-j} \pmod{n^{s+1}}$$

and hence, since $r^{n^s\varphi(n)-j} \equiv r^{-j} (\text{mod } n^{s+1})$ and $\binom{n^s\varphi(n)-1}{2} n^2 \equiv n^2 \pmod{n^3}$,

$$U_r(n) \equiv \begin{cases} -\frac{1}{r}S_{r,1,2}(n) - \frac{n}{r^2}S_{r,2,2}(n) - \frac{n^2}{r^3}S_{r,3,2}(n) & \pmod{n^3}, \\ -\frac{1}{r}S_{r,1,1}(n) - \frac{n}{r^2}S_{r,2,1}(n) & \pmod{n^2}, \\ -\frac{1}{r}S_{r,1,0}(n) & \pmod{n}. \end{cases}$$

Now Proposition 4.2 follows from (1.1) at once. □

In Section 3 some formulae for $\widehat{EQ}_r(n)$ are determined. Since, by Proposition 4.1, we have $EQ_r(n) = -\frac{1}{r}\widehat{EQ}_r(n)$, the formulae imply corresponding formulae for $EQ_r(n)$. In the next proposition, we present the formulae in a slightly different form.

Proposition 4.3. *In the above notation, if* $r \mid 24$*, then (4.23) holds.*

Proof. Following (2.17) and Proposition 4.2 we know that

$$EQ_r(n) = -\frac{1}{r}\widehat{EQ}_r(n) \equiv -\frac{\widetilde{B}_{m+1}}{r(m+1)}\left(-1 + \frac{1}{\varphi(r)r^m}\prod_{q|r}(1-q^m)\right) \pmod{n^{s+1}}$$

where $m = n^s\varphi(n) - 1$. Consequently,

$$EQ_r(n) \equiv \frac{X\widetilde{B}_{m+1}}{r^{m+1}(m+1)} \pmod{n^{s+1}} \tag{4.24}$$

where

$$X = r^m - \frac{1}{\varphi(r)}\prod_{q|r}(1-q^m).$$

Thus, in view of (4.24) and (3.22), to obtain (4.23) it suffices to determine $X \pmod{n^{s+4}}$.

Indeed, we have

$$X = \frac{1}{r}(r^{\varphi(n)})^{n^s} - \frac{1}{\varphi(r)}\left(1 - \frac{\rho_2(r)}{2}(2^{\varphi(n)})^{n^s}\right)\left(1 - \frac{\rho_3(r)}{3}(3^{\varphi(n)})^{n^s}\right),$$

and by virtue of $i^{\varphi(n)} = 1 + nq_i(n)$ $(i = 2, 3)$

$$\begin{aligned} X = &\frac{1}{r}(1+nq_2(n))^{\mathrm{ord}_2(r)n^s}(1+nq_3(n))^{\mathrm{ord}_3(r)n^s} - \frac{1}{\varphi(r)} + \frac{\rho_2(r)}{2\varphi(r)}(1+nq_2(n))^{n^s} \\ &+ \frac{\rho_3(r)}{3\varphi(r)}(1+nq_3(n))^{n^s} - \frac{\rho_2(r)\rho_3(r)}{6\varphi(r)}(1+nq_2(n))^{n^s}(1+nq_3(n))^{n^s}. \end{aligned}$$

Thus,

$$\begin{aligned} X \equiv \frac{1}{r} &+ \frac{1}{r}\sum_{i=2,3}\left(n^{s+1}\mathrm{ord}_i(r)q_i(n) - \frac{1}{2}n^{s+1}\mathrm{ord}_i(r)nq_i^2(n) + \frac{1}{3}n^{s+1}\mathrm{ord}_i(r)n^2q_i^3(n)\right) \\ &- \frac{1}{\varphi(r)} + \sum_{i=2,3}\frac{\rho_i(r)}{i\varphi(r)}\left(1 + n^{s+1}q_i(n) - \frac{1}{2}n^{s+1}nq_i^2(n) + \frac{1}{3}n^{s+1}n^2q_i^3(n)\right) \\ &- \frac{\rho_2(r)\rho_3(r)}{6\varphi(r)} - \frac{\rho_2(r)\rho_3(r)}{6\varphi(r)}\sum_{i=2,3}\left(n^{s+1}q_i(n) - \frac{1}{2}n^{s+1}nq_i^2(n) + \frac{1}{3}n^{s+1}n^2q_i^3(n)\right) \pmod{n^{s+4}}, \end{aligned}$$

and so,

$$\begin{aligned}X \equiv & Y + \frac{1}{r} n^{s+1} \sum_{i=2,3} \left(\mathrm{ord}_i(r) q_i(n) - \frac{1}{2} \mathrm{ord}_i(r) n q_i^2(n) + \frac{1}{3} \mathrm{ord}_i(r) n^2 q_i^3(n) \right) \\ & + \sum_{i=2,3} \frac{\rho_i(r)}{i\varphi(r)} n^{s+1} \left(q_i(n) - \frac{1}{2} n q_i^2(n) + \frac{1}{3} n^2 q_i^3(n) \right) \\ & - \frac{\rho_2(r)\rho_3(r)}{6\varphi(r)} n^{s+1} \sum_{i=2,3} \left(q_i(n) - \frac{1}{2} n q_i^2(n) + \frac{1}{3} n^2 q_i^3(n) \right) \pmod{n^{s+4}},\end{aligned}$$

where

$$Y = \frac{1}{r} - \frac{1}{\varphi(r)} + \frac{\rho_2(r)}{2\varphi(r)} + \frac{\rho_3(r)}{3\varphi(r)} - \frac{\rho_2(r)\rho_3(r)}{6\varphi(r)}.$$

An easy verification shows that $Y = 0$. To check it we consider the cases. If $\rho_2(r) = 0$ and $\rho_3(r) = 1$; then $r = 3$ and obviously $Y = 0$. If $\rho_2(r) = 1$ and $\rho_3(r) = 0$; then $r = 2, 4, 8$ and we have $Y = \dfrac{1}{r} - \dfrac{1}{2\varphi(r)} = 0$ since $r = 2\varphi(r)$ for these r. Finally, if $\rho_2(r) = \rho_3(r) = 1$; then $r = 6, 12, 24$ and $Y = \dfrac{1}{r} - \dfrac{1}{3\varphi(r)} = 0$ since $r = 3\varphi(r)$ in these cases. This completes the proof of Proposition 4.3.
□

4.B. Proof of Theorem 4.1

The proof of Theorem 4.1 falls naturally into seven cases $r = 2, 3, 4, 6, 8, 12$ or 24. In view of Proposition 4.2, in each of the cases, it suffices to determine:

(i) the sums $T_{r,1}(n) \pmod{n^{s+1}}$ for $s \in \{0, 1, 2\}$, which are determined in (i) of Corollaries 3.2, 3.4, 3.6, 3.8, 3.10, 3.12 or 3.14;

(ii) the congruences for $nT_{r,2}(n) \pmod{n^{s+1}}$ for $s \in \{1, 2\}$, which follow immediately from parts (ii) of Corollaries 3.2, 3.4, 3.6, 3.8, 3.10, 3.12 or 3.14;

(iii) the congruences for $n^2 T_{r,3}(n) \pmod{n^3}$, which follow easily from parts (i) of Theorems 3.1, 3.3, 3.5, 3.7, 3.9, 3.11 or 3.13 for $k = 3$[14].

Set $Q_i'(n) \equiv Q_i(n) \pmod{n}$ and $Q_i''(n) \equiv Q_i(n) \pmod{n^2}$ $(i = 2, 3)$. We consider the cases:

1. If $r = 2$, Theorem 4.1 is a consequence of Proposition 4.2, Theorems 3.1 and Corollary 3.2; then for $n > 1$ odd and $s = 2$ we have

$$\begin{aligned} T_{2,1}(n) \equiv & Q_2(n) - \frac{7}{8} n^2 \widetilde{B}_{n^2\varphi(n)-2} \qquad \pmod{n^3}, \\ nT_{2,2}(n) \equiv & \frac{7}{2} n^2 \widetilde{B}_{n^2\varphi(n)-2} \qquad \pmod{n^3}, \\ n^2 T_{2,3}(n) \equiv & -3n^2 \widetilde{B}_{n^2\varphi(n)-2} \qquad \pmod{n^3}. \end{aligned}$$

The first of these congruences is the same as that Section 3 and the second one is an immediate consequence of that in Section 3. The third congruence follows immediately from Theorem 3.1 for $k = 3$; then

$$n^2 T_{2,3}(n) \equiv \frac{6n^2 \widetilde{B}_{n^2\varphi(n)-2}}{n^2\varphi(n) - 2} \qquad \pmod{n^3}.$$

[14] More precisely, we need to determine $T_{r,1}(n), nT_{r,2}(n), n^2T_{r,3}(n) \pmod{n^3}$ if $s = 2$, $T_{r,1}(n), nT_{r,2}(n) \pmod{n^2}$ if $s = 1$ and $T_{r,1}(n) \pmod{n}$ if $s = 0$.

On the other hand,

$$\frac{n^2\widetilde{B}_{n^2\varphi(n)-2}}{n^2\varphi(n)-2} \equiv -\frac{1}{2}n^2\widetilde{B}_{n^2\varphi(n)-2} \pmod{n^3}, \tag{4.25}$$

which completes the proof in this case. For $s=1$

$$\begin{aligned} T_{2,1}(n) \equiv & Q_2''(n) - \frac{7}{8}n^2\widetilde{B}_{n\varphi(n)-2} \pmod{n^2}, \\ nT_{2,2}(n) \equiv & \frac{7}{2}n^2\widetilde{B}_{n\varphi(n)-2} \pmod{n^2}. \end{aligned}$$

If we assume that $3 \nmid n$, then $\widetilde{B}_{n\varphi(n)-2}$ is p-integral for any $p \mid n$ and so

$$T_{2,1}(n) \equiv Q_2''(n) \pmod{n^2}, \quad nT_{2,2}(n) \equiv 0 \pmod{n^2}$$

as claimed. If $s=0$ $T_{2,1}(n) \equiv Q_2'(n) \pmod{n}$.

2. If $r=3$, Theorem 4.1 is an immediate consequence of Proposition 4.2, Theorems 3.3 and Corollary 3.4; then for odd $n>1, 3 \nmid n$ and $s=2$ we have

$$\begin{aligned} T_{3,1}(n) &\equiv Q_3(n) - \frac{1}{2}n\widetilde{D}_{n^2\varphi(n)-2} - \frac{13}{18}n^2\widetilde{B}_{n^2\varphi(n)-2} && \pmod{n^3}, \\ nT_{3,2}(n) &\equiv \frac{3}{2}n\widetilde{D}_{n^2\varphi(n)-2} + \frac{13}{3}n^2\widetilde{B}_{n^2\varphi(n)-2} && \pmod{n^3}, \\ n^2T_{3,3}(n) &\equiv -6n^2\widetilde{B}_{n^2\varphi(n)-2} && \pmod{n^3}. \end{aligned}$$

Again the first congruence is the same as that in Section 3 and the second one is an easy consequence of that in that section. The third congruence follows from Theorem 3.3 (i) for $k=3$ and (4.25); then

$$n^2T_{3,3}(n) \equiv \frac{12n^2\widetilde{B}_{n^2\varphi(n)-2}}{n^2\varphi(n)-2} \qquad \pmod{n^3}.$$

For $s=1$

$$T_{3,1}(n) \equiv Q_3''(n) - \frac{1}{2}n\widetilde{D}_{n\varphi(n)-2} \pmod{n^2}, \; nT_{2,2}(n) \equiv \frac{3}{2}n\widetilde{D}_{n\varphi(n)-2} \pmod{n^2}.$$

Likewise, if $s=0$, $T_{3,1}(n) \equiv Q_3'(n) \pmod{n}$.

3. If $r=4$, Theorem 4.1 follows from Proposition 4.2 and Theorems 3.5 and Corollary 3.6; then for $n>3$ odd and $s=2$ we have

$$\begin{aligned} T_{4,1}(n) &\equiv \frac{3}{2}Q_2(n) - n\widetilde{E}_{n^2\varphi(n)-2} - \frac{7}{8}n^2\widetilde{B}_{n^2\varphi(n)-2} && \pmod{n^3}, \\ nT_{4,2}(n) &\equiv 4n\widetilde{E}_{n^2\varphi(n)-2} + 7n^2\widetilde{B}_{n^2\varphi(n)-2} && \pmod{n^3}, \\ n^2T_{4,3}(n) &\equiv -\frac{27}{2}n^2\widetilde{B}_{n^2\varphi(n)-2} && \pmod{n^3}. \end{aligned}$$

The first congruence is the same as that in Section 3 and the second one is an immediate consequence of that in that section. The third congruence follows immediately from Theorem 3.5 for $k=3$ and (4.25); then

$$n^2T_{4,3}(n) \equiv \frac{27n^2\widetilde{B}_{n^2\varphi(n)-2}}{n^2\varphi(n)-2} \qquad \pmod{n^3}.$$

For s = 1

$$\begin{aligned} T_{4,1}(n) &\equiv \frac{3}{2}Q_2''(n) - n\widetilde{E}_{n\varphi(n)-2} - \frac{7}{8}n^2\widetilde{B}_{n\varphi(n)-2} \pmod{n^2}, \\ nT_{4,2}(n) &\equiv 4n\widetilde{E}_{n\varphi(n)-2} + 7n^2\widetilde{B}_{n\varphi(n)-2} \pmod{n^2} \end{aligned}$$

and so

$$T_{4,1}(n) \equiv \frac{3}{2}Q_2''(n) - n\widetilde{E}_{n\varphi(n)-2} \pmod{n^2}, \;\; nT_{4,2}(n) \equiv 4n\widetilde{E}_{n\varphi(n)-2} \pmod{n^2}$$

if $3 \not| n$. If $s = 0$ $T_{4,1}(n) \equiv \frac{3}{2}Q_2'(n) \pmod{n}$.

4. If $r = 6$, Theorem 4.1 is an immediate consequence of Proposition 4.2, Theorems 3.7 and Corollary 3.8; then for odd $n > 5, 3 \not| n$ and $s = 2$ we have

$$\begin{aligned} T_{6,1}(n) &\equiv Q_2(n) + Q_3(n) - \frac{5}{4}n\widetilde{D}_{n^2\varphi(n)-2} - \frac{91}{72}n^2\widetilde{B}_{n^2\varphi(n)-2} \quad \pmod{n^3}, \\ nT_{6,2}(n) &\equiv \frac{15}{2}n\widetilde{D}_{n^2\varphi(n)-2} + \frac{91}{6}n^2\widetilde{B}_{n^2\varphi(n)-2} \qquad \pmod{n^3}, \\ n^2T_{6,3}(n) &\equiv -45n^2\widetilde{B}_{n^2\varphi(n)-2} \qquad \pmod{n^3}. \end{aligned}$$

The first congruence is the same as that in Section 3 and the second one is an immediate consequence of that in that section. The third congruence follows from (4.25) and the congruence

$$n^2T_{6,3}(n) \equiv \frac{90n^2\widetilde{B}_{n^2\varphi(n)-2}}{n^2\varphi(n)-2} \pmod{n^3}.$$

For $s = 1$

$$\begin{aligned} T_{6,1}(n) \equiv& Q_2''(n) + Q_3''(n) - \frac{5}{4}n\widetilde{D}_{n\varphi(n)-2} \pmod{n^2}, \\ nT_{6,2}(n) \equiv& \frac{15}{2}n\widetilde{D}_{n\varphi(n)-2} \qquad \pmod{n^2}. \end{aligned}$$

If $s = 0$ $T_{6,1}(n) \equiv Q_2'(n) + Q_3'(n) \pmod{n}$.

5. If $r = 8$, Theorem 4.1 follows from Proposition 4.2, Theorems 3.9 and Corollary 3.10 ; then for $n > 7$ odd and $s = 2$ we have

$$\begin{aligned} T_{8,1}(n) \equiv& 2Q_2(n) + 2\widetilde{A}_{n^2\varphi(n)-1} - n\widetilde{E}_{n^2\varphi(n)-2} - 2n\widetilde{C}_{n^2\varphi(n)-2} \\ &- \frac{7}{8}n^2\widetilde{B}_{n^2\varphi(n)-2} + 2n^2\widetilde{A}_{n^2\varphi(n)-3} \qquad \pmod{n^3}, \\ nT_{8,2}(n) \equiv& 8n\widetilde{E}_{n^2\varphi(n)-2} + 16n\widetilde{C}_{n^2\varphi(n)-2} + 14n^2\widetilde{B}_{n^2\varphi(n)-2} - 32n^2\widetilde{A}_{n^2\varphi(n)-3} \pmod{n^3}, \\ n^2T_{8,3}(n) \equiv& -\frac{111}{2}n^2\widetilde{B}_{n^2\varphi(n)-2} + 128n^2\widetilde{A}_{n^2\varphi(n)-3} \pmod{n^3}. \end{aligned}$$

The first congruence is the same as that in Section 3, the second one follows from that in Section 3 and the third one is an immediate consequence of Theorem 3.9 for $k = 3$ and (4.25); then

$$n^2T_{8,3}(n) \equiv \frac{111n^2\widetilde{B}_{n^2\varphi(n)-2}}{n^2\varphi(n)-2} + 128n^2\widetilde{A}_{n^2\varphi(n)-3} \qquad \pmod{n^3}.$$

For $s = 1$

$$\begin{aligned} T_{8,1}(n) \equiv& 2Q_2''(n) + 2\widetilde{A}_{n\varphi(n)-1} - n\widetilde{E}_{n\varphi(n)-2} - 2n\widetilde{C}_{n\varphi(n)-2} - \frac{7}{8}n^2\widetilde{B}_{n\varphi(n)-2} \pmod{n^2}, \\ nT_{8,2}(n) \equiv& 8n\widetilde{E}_{n\varphi(n)-2} + 16n\widetilde{C}_{n\varphi(n)-2} + 14n^2\widetilde{B}_{n\varphi(n)-2} \qquad \pmod{n^2} \end{aligned}$$

and so

$$\begin{aligned} T_{8,1}(n) &\equiv 2Q_2''(n) + 2\widetilde{A}_{n\varphi(n)-1} - n\widetilde{E}_{n\varphi(n)-2} - 2n\widetilde{C}_{n\varphi(n)-2} \pmod{n^2},\\ nT_{8,2}(n) &\equiv 8n\widetilde{E}_{n\varphi(n)-2} + 16n\widetilde{C}_{n\varphi(n)-2} \pmod{n^2} \end{aligned}$$

if $3 \not| n$. If $s = 0$, $T_{8,1}(n) \equiv 2Q_2'(n) + 2\widetilde{A}_{\varphi(n)-1} \pmod{n}$.

6. If $r = 12$, Theorem 4.1 follows at once from Proposition 4.2, Theorems 3.11 and Corollary 3.12; then for $n > 11$ odd and $s = 2$ we have

$$\begin{aligned} T_{12,1}(n) \equiv& \frac{3}{2}Q_2(n) + Q_3(n) + 3\widetilde{F}_{n^2\varphi(n)-1} - \frac{5}{4}n\widetilde{D}_{n^2\varphi(n)-2} - \frac{5}{3}n\widetilde{E}_{n^2\varphi(n)-2}\\ &- \frac{91}{72}n^2\widetilde{B}_{n^2\varphi(n)-2} + 3n^2\widetilde{F}_{n^2\varphi(n)-3} \qquad \pmod{n^3},\\ nT_{12,2}(n) \equiv& 15n\widetilde{D}_{n^2\varphi(n)-2} + 20n\widetilde{E}_{n^2\varphi(n)-2} + \frac{91}{3}n^2\widetilde{B}_{n^2\varphi(n)-2} - 72n^2\widetilde{F}_{n^2\varphi(n)-3} \pmod{n^3},\\ n^2T_{12,3}(n) \equiv& -\frac{363}{2}n^2\widetilde{B}_{n^2\varphi(n)-2} + 432n^2\widetilde{F}_{n^2\varphi(n)-3} \qquad \pmod{n^3}. \end{aligned}$$

The first congruence is the same as that in Section 3, the second one is implied by that in that section and the third one follows from Theorem 3.11 for $k = 3$ and (4.25); then

$$n^2T_{12,3}(n) \equiv \frac{363n^2\widetilde{B}_{n^2\varphi(n)-2}}{n^2\varphi(n) - 2} + 432n^2\widetilde{F}_{n^2\varphi(n)-3} \qquad \pmod{n^3}.$$

For $s = 1$

$$\begin{aligned} T_{12,1}(n) &\equiv \frac{3}{2}Q_2''(n) + Q_3''(n) + 3\widetilde{F}_{n\varphi(n)-1} - \frac{5}{4}n\widetilde{D}_{n\varphi(n)-2} - \frac{5}{3}n\widetilde{E}_{n\varphi(n)-2} \pmod{n^2},\\ nT_{12,2}(n) &\equiv 15n\widetilde{D}_{n\varphi(n)-2} + 20n\widetilde{E}_{n\varphi(n)-2} \pmod{n^2}. \end{aligned}$$

If $s = 0$, $T_{12,1}(n) \equiv \frac{3}{2}Q_2'(n) + Q_3'(n) + 3\widetilde{F}_{\varphi(n)-1} \pmod{n}$.

7. If $r = 24$, Theorem 4.1 follows from Proposition 4.2, Theorems 3.13 and Corollary 3.14; then for $n > 23$ odd and $s = 2$ we have

$$\begin{aligned} T_{24,1}(n) \equiv& 2Q_2(n) + Q_3(n) + 3\widetilde{F}_{n^2\varphi(n)-1} + 3\widetilde{G}_{n^2\varphi(n)-1} + 4\widetilde{A}_{n^2\varphi(n)-1}\\ &- \frac{5}{4}n\widetilde{D}_{n^2\varphi(n)-2} - \frac{5}{3}n\widetilde{E}_{n^2\varphi(n)-2} - 3n\widetilde{H}_{n^2\varphi(n)-2} - \frac{8}{3}n\widetilde{C}_{n^2\varphi(n)-2}\\ &- \frac{91}{72}n^2\widetilde{B}_{n^2\varphi(n)-2} + 3n^2\widetilde{F}_{n^2\varphi(n)-3} + 3n^2\widetilde{G}_{n^2\varphi(n)-3} + \frac{28}{9}n^2\widetilde{A}_{n^2\varphi(n)-3} \pmod{n^3},\\ nT_{24,2}(n) \equiv& 30n\widetilde{D}_{n^2\varphi(n)-2} + 40n\widetilde{E}_{n^2\varphi(n)-2} + 72n\widetilde{H}_{n^2\varphi(n)-2} + 64n\widetilde{C}_{n^2\varphi(n)-2}\\ &+ \frac{182}{3}n^2\widetilde{B}_{n^2\varphi(n)-2} - 36n^2\widetilde{F}_{n^2\varphi(n)-3} - 144n^2\widetilde{G}_{n^2\varphi(n)-3}\\ &- \frac{448}{3}n^2\widetilde{A}_{n^2\varphi(n)-3} \qquad \pmod{n^3},\\ n^2T_{24,3}(n) \equiv& -\frac{1455}{2}n^2\widetilde{B}_{n^2\varphi(n)-2} + 1728n^2\widetilde{F}_{n^2\varphi(n)-3}\\ &+ 1728n^2\widetilde{G}_{n^2\varphi(n)-3} + 1792n^2\widetilde{A}_{n^2\varphi(n)-3} \qquad \pmod{n^3}. \end{aligned}$$

Again the first congruence is the same as that in Section 3, the second one follows immediately from that in that section and the third one follows from Theorem 3.13 for $k = 3$ and(4.25); then

$$\begin{aligned} n^2T_{24,3}(n) \equiv& -\frac{1455n^2\widetilde{B}_{n^2\varphi(n)-2}}{n^2\varphi(n) - 2} + 1728n^2\widetilde{F}_{n^2\varphi(n)-3}\\ &+ 1728n^2\widetilde{G}_{n^2\varphi(n)-3} + 1792n^2\widetilde{A}_{n^2\varphi(n)-3} \pmod{n^3}. \end{aligned}$$

For $s = 1$

$$
\begin{aligned}
T_{24,1}(n) \equiv & 2Q_2''(n) + Q_3''(n) + 3\widetilde{F}_{n\varphi(n)-1} + 3\widetilde{G}_{n\varphi(n)-1} + 4\widetilde{A}_{n\varphi(n)-1} \\
& - \frac{5}{4}n\widetilde{D}_{n\varphi(n)-2} - \frac{5}{3}n\widetilde{E}_{n\varphi(n)-2} - 3n\widetilde{H}_{n\varphi(n)-2} - \frac{8}{3}n\widetilde{C}_{n\varphi(n)-2} \pmod{n^2}, \\
nT_{24,2}(n) \equiv & 30n\widetilde{D}_{n\varphi(n)-2} + 40n\widetilde{E}_{n\varphi(n)-2} + 72n\widetilde{H}_{n\varphi(n)-2} + 64n\widetilde{C}_{n\varphi(n)-2} \pmod{n^2}.
\end{aligned}
$$

For $s = 0$,

$$T_{24,1}(n) \equiv 2Q_2'(n) + Q_3'(n) + 3\widetilde{F}_{\varphi(n)-1} + 3\widetilde{G}_{\varphi(n)-1} + 4\widetilde{A}_{\varphi(n)-1} \pmod{n}.$$

This completes the proof of Theorem 4.1.

5. Concluding remarks

Let $p \geq 3$ be a prime number and let r be a natural number such that $1 < r < p$. Assume that $s \in \{0, 1, 2\}$ and $r \mid 24$. In the next part of the paper we are going to prove some new congruences for the sums $T_{r,k}(p) = \sum_{i=1}^{[\frac{p}{r}]}(1/i^k)$ modulo p^{s+1} for $k \geq 1$, in particular for $k = 1$ or 2 in all the cases. Similarly we would like to derive some new congruences for the sums $U_r(p) = \sum_{i=1}^{[\frac{p}{r}]} \frac{1}{p-ri}$ modulo p^{s+1}. We shall use the congruences proved in the present paper in the case when $n = p$ is an odd prime as well as Kummer's congruences for the generalized Bernoulli numbers.

6. Acknowledgements

The authors gratefully acknowledge a discussion with Professor Nianliang Wang concerning an earlier version of Section 3 of the paper. We are also grateful to the referee for valuable comments and helpful suggestions.

References

[Cai02] T.X. Cai, A congruence involving the quotients of Euler and its appli- cations (I), *Acta Arithmetica* **103** (2002), 313-320.

[CFZ07] T.X. Cai, X.D. Fu and X. Zhou, A congruence involving the quotients of Euler and its applications (II), *Acta Arithmetica* **130** (2007), 203-214.

[CP09] H.-Q. Cao and H. Pan, Note on some congruences of Lehmer, *J. Number Theory* **129** (2009), 1813-1819.

[Car59] L. Carlitz, Arithmetic properties of generalized Bernoulli numbers, *J. Reine Angew. Math.* **202** (1959), 174-182.

[Car2] L. Carlitz, Some congruences for the Bernoulli numbers, *Amer. J. Math.*, **75** (1953), 163-172.

[Ern79] R. Ernvall, *Generalized Bernoulli numbers, generalized irregular primes, and class number*, Ann. Univ. Turku, Ser. A1, **178** (1979), 72 pp.

[FUW97] G.J. Fox, J. Urbanowicz and K.S. Williams, Gauss congruence from Dirichlet's class number formula and generalizations, in: *K. Gy'ry, H. Iwaniec and J. Urbanowicz, editors, Number Theory in Progress (Za- kopane, 1997)*, vol. II, Walter de Gruyter, Berlin-New York, 1999, 813-839.

[IR90] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd edition, Springer-Verlag, New York, 1990.

[KUW12] S. Kanemitsu, J. Urbanowicz and N.-L. Wang, *On some new congruences for generalized Bernoulli numbers*, 14 pp., accepted for publication in Acta Arithmetica, 2012.

[Kle55] H. Kleboth, *Untersuchung über Klassenzahl und Reziprozitätsgesetz im Körper der 6l-ten Einheitswurzeln und die Diophantische Gleichungo* $x^{2l} + 3^l y^{2l} = z^{2l}$ *für eine Primzahl l grosser als 3*, Dissertation, Universität Zürich, 1955, 37 pp.

[Leh38] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Annals of Math.* **39** (1938), 350-360.

[Ler05] M. Lerch, Zur Theorie des Fermatschen Quotienten $\frac{a^{p-1}-1}{p} = q(a)$ *Math. Annalen* **60** (1905), 471-490.

[SUV99] A. Schinzel, J. Urbanowicz and P. Van Wamelen, Class numbers and short sums of Kronecker symbols, *J. Number Theory* **78** (1999), 62-84.

[Sun08] Z.-H. Sun, Congruences involving Bernoulli and Euler numbers, *J. Number Theory* **128** (2008), 280-312.

[SUZ95] J. Szmidt, J. Urbanowicz and D. Zagier, Congruences among generalized Bernoulli numbers, *Acta Arithmetica* **71** (1995), 273-278.

[UW00] J. Urbanowicz and K.S. Williams, *Congruences for L-functions*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2000.

[Wash97] L.C. Washington, *Introduction to Cyclotomic Fields*, second edition, Springer-Verlag, New York, 1997.

Shigeru Kanemitsu
Graduate School of Advanced Technology
Kinki University Iizuka
Fukuoka, Japan
e-mail: kanemitu@fuk.kindai.ac.jp

Takako Kuzumaki
Applied Physics Course
Faculty of Engineering
Gifu University, Gifu
Japan
e-mail: kuzumaki@gifu-u.ac.jp

Jerzy Urbanowicz
Institute of Mathematics
Polish Academy of Sciences and Institute of Computer Science
Polish Academy of Sciences
Warsaw, Poland