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Algebraic independence results on the generating Lambert series of the powers of a fixed integer

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Abstract. In this paper, the algebraic independence of values of the function $G_d(z) := \sum_{h\geq 0} z^{d^h}/(1-z^{d^h})$, $d>1$ a fixed integer, at non-zero algebraic points in the unit disk is studied. Whereas the case of multiplicatively independent points has been resolved some time ago, a particularly interesting case of multiplicatively dependent points is considered here, and similar results are obtained for more general functions. The main tool is Mahler's method reducing the investigation of the algebraic independence of numbers (over $\mathbb{Q}$) to the one of functions (over the rational function field) if these satisfy certain types of functional equations.

Keywords. Algebraic independence of numbers, Mahler's method, algebraic independence of functions.

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1. Introduction and main results

A series of type

$$\sum_{k=1}^{\infty} \lambda_k \frac{z^k}{1-z^k}$$

with $(\lambda_k) \in \mathbb{C}^{\mathbb{N}}$ is called a Lambert series. Denoting by d the 'integer' in the title, *supposing always* $d \geq 2$, and taking λ_k to be 1 or 0 depending on whether k is a power of d or not, our above series reduces to

$$G_d(z) := \sum_{h=0}^{\infty} \frac{z^{d^h}}{1-z^{d^h}}. \tag{1.1}$$

This series converges exactly on the open unit disk $\mathbb{D}$ and defines there a holomorphic function.

The similar-looking series

$$F_d(z) := \sum_{h=0}^{\infty} \frac{z^{d^h}}{1+z^{d^h}} \tag{1.2}$$

has the same analytic properties and, indeed, we have $F_d(z) = -G_d(-z)$ in $\mathbb{D}$ if d is odd. Thus, in this case, both functions F_d, G_d are very closely related.

The aim of the present paper is to study the algebraic independence of the values of the functions $G_d(z)$ and $F_d(z)$ at certain multiplicatively dependent points $\alpha_1, \dots, \alpha_n \in \mathbb{D}$. The arithmetical nature of the values of these functions, being typical examples of Mahler functions, has been studied in several works (see [BV15c], [Coo12], [Coo13], [Mah69], [Sch67]). In particular, it is known that $G_d(\alpha_1), \dots, G_d(\alpha_n)$ are algebraically independent (over $\mathbb{Q}$) if $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$ are multiplicatively independent, and the same holds for F_d instead of G_d. Here and in the sequel, $\overline{\mathbb{Q}}$ denotes the field of all complex algebraic numbers.

In this work, we *suppose always*

$$\alpha_i := \alpha^{m_i} \quad (i = 1, \dots, n)$$

with $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$ and all $m_i \in \mathbb{N}$. The following result was established in [BV15c, Theorem 3.1].

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Theorem 1. *Let $m_1, \ldots, m_n$ be $n \geq 2$ positive integers, and let $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$. Then $G_d(\alpha_1), \ldots, G_d(\alpha_n)$ are algebraically independent if and only if*

$$\frac{m_j}{m_i} \notin d^{\mathbb{Z}} \tag{1.3}$$

holds for any pair (i, j) with $i \neq j$.

To state our first new result, we define, for any function f, the notation $f(\pm\beta)$ to mean either $f(\beta)$ or $f(-\beta)$.

Theorem 2. *Let $m_1, \ldots, m_n$ be $n \geq 2$ positive integers satisfying the condition (1.3), and let $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$. Then, for each choice of n signs, the values $G_d(\pm\alpha_1), \ldots, G_d(\pm\alpha_n)$ are algebraically independent, and the same holds for $F_d(\pm\alpha_1), \ldots, F_d(\pm\alpha_n)$.*

Note that, as an immediate corollary of this result, we obtain the analogue of Theorem 1 for F_d. To state our next result, we define

$$\mathcal{G}_d(z) := \sum_{h=0}^{\infty} \frac{a_h z^{d^h}}{1 - z^{d^h}}, \quad \mathcal{F}_d(z) := \sum_{h=0}^{\infty} \frac{b_h z^{d^h}}{1 + z^{d^h}}, \tag{1.4}$$

where $(a_h), (b_h)$ are non-zero periodic sequences of algebraic numbers.

Theorem 3. *Let $m_1, \ldots, m_n$ be $n \geq 2$ positive integers such that*

$$\frac{m_j}{m_i} \notin \mathbb{N} \tag{1.5}$$

holds for any pair (i, j) with $i \neq j$, and let $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$. Then, for each choice of n signs, the values $\mathcal{G}_r(\pm\alpha_i)$ $(i = 1, \ldots, n; r \in \mathbb{N} \setminus \{1\})$ are algebraically independent. In particular, the numbers $G_r(\pm\alpha_i)$ $(i = 1, \ldots, n; r \in \mathbb{N} \setminus \{1\})$ are algebraically independent. The same holds if $\mathcal{G}_r, G_r$ are replaced by $\mathcal{F}_r, F_r$.

In the remaining results, both functions G_d, F_d are studied simultaneously. But here the case $d = 2$ has to be excluded in a natural way since, using (1.1) and (1.2), we find for $z \in \mathbb{D}$

$$G_2(z) + F_2(z) = 2\sum_{h=0}^{\infty} \frac{z^{2^h}}{1 - z^{2^{h+1}}} = 2\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} z^{(1+2k)2^h} = 2\sum_{n=1}^{\infty} z^n = \frac{2z}{1-z}. \tag{1.6}$$

Theorem 4. *Suppose $d \geq 3$, and let $m_1, \ldots, m_n$ be $n \geq 2$ positive integers satisfying (1.3) and*

$$\frac{m_j}{m_i} \notin 2d^{\mathbb{Z}} \tag{1.7}$$

for any pair (i, j) with $i \neq j$. If $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, then the numbers $G_d(\alpha_1), \ldots, G_d(\alpha_n)$, $F_d(\alpha_1), \ldots, F_d(\alpha_n)$ are algebraically independent.

It may be of some interest to see a typical example of an application of Theorem 4 involving reciprocal sums of the usual Fibonacci numbers Φ_n (or Lucas numbers Λ_n, respectively). To this purpose, we first deduce from (1.1) and (1.2) that $G_d(\beta^m) + F_d(\beta^m)$, up to an algebraic summand, equals to $(2/\sqrt{5})\sum_{h>0} 1/\Phi_{md^h}$ for any $d, m \in \mathbb{N}$ with $2\,|\,d$, where we put $\beta := (1-\sqrt{5})/2$. Assuming even $4\,|\,d$, Theorem 4 tells us the algebraic independence of all numbers

$$\sum_{h\geq 0} \frac{1}{\Phi_{md^h}} \quad (m \in 2\mathbb{N}-1).$$

Similarly we establish, for odd $d > 2$, the algebraic independence of all $\sum_{h\geq 0} \frac{1}{\Lambda_{md^h}}$ with odd m not divisible by d.

Theorem 5. *Let $m_1, \dots, m_n$ be $n \geq 2$ positive integers such that*

$$\frac{2m_j}{m_i} \notin \mathbb{N} \tag{1.8}$$

holds for any pair (i, j) with $i \neq j$. If $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, then the numbers $\mathcal{G}_r(\alpha_i), \mathcal{F}_r(\alpha_i)$ $(i = 1, \dots, n; r \in \mathbb{N}\setminus\{1\}, r \notin 2^{2\mathbb{N}-1})$ are algebraically independent. In particular, the numbers $G_r(\alpha_i), F_r(\alpha_i)$ $(i = 1, \dots, n; r \in \mathbb{N}\setminus\{1\}, r \notin 2^{2\mathbb{N}-1})$ are algebraically independent.

Remark. Assuming that $(d_h) \in \mathbb{N}^{\mathbb{N}_0}$ satisfies a linear recurrence $d_{h+t} = c_1 d_{h+t-1} + \dots + c_t d_h$ with certain conditions on t and $(c_1, \dots, c_t) \in \mathbb{N}_0^t \setminus \{\underline{0}\}$ excluding, in particular, the case of (d_h) being a geometric progression, Tanaka [Tan05] settled the algebraic independence problem for the values of the Lambert series $\sum_{h\geq 0} z^{d_h}/(1-z^{d_h})$ at distinct points $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$. Thus, our investigations on the Lambert series $G_d(z)$ just concern the important remaining case, where (d_h) reduces to the geometric progression (d^h).

2. The main lemma

The main tool in the proof of [BV15c, Theorem 3.1] was the following auxiliary result.

Lemma 1. *Let $m_1, \dots, m_n$ be $n \geq 2$ positive integers satisfying condition (1.3). Then the functions $G_d(z^{m_1}), \dots, G_d(z^{m_n})$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$.*

Combining the proof of this lemma with some new ideas, we are now able to generalize Lemma 1. To state this generalization, we introduce, for fixed $a \in \mathbb{C}^{\times}$, the functions

$$G_d(a, z) := \sum_{h=0}^{\infty} \frac{a^h z^{d^h}}{1 - z^{d^h}}, \quad F_d(a, z) := \sum_{h=0}^{\infty} \frac{a^h z^{d^h}}{1 + z^{d^h}}. \tag{2.9}$$

Lemma 2. *Let $m_1, \dots, m_n$ be $n \geq 2$ positive integers satisfying condition (1.3). Assume that I_1 and I_2 are (possibly empty) disjoint sets of positive integers satisfying $I_1 \cup I_2 = \{1, \dots, n\}$. Then, for any root of unity ζ, the functions $G_d(\zeta, z^{m_i})$ $(i \in I_1)$, $F_d(\zeta, z^{m_i})$ $(i \in I_2)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$.*

Proof. We first note that the functions

$$g_i(z) := G_d(\zeta, z^{m_i}), f_i(z) := F_d(\zeta, z^{m_i}) \quad (i = 1, \dots, n) \tag{2.10}$$

satisfy the functional equations

$$\zeta g_i(z^d) = g_i(z) + \frac{z^{m_i}}{z^{m_i} - 1}, \; \zeta f_i(z^d) = f_i(z) - \frac{z^{m_i}}{z^{m_i} + 1} \quad (i = 1, \dots, n). \tag{2.11}$$

Assume now, contrary to Lemma 2, that there exists some $\underline{c} := (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{\underline{0}\}$ such that

$$r(z) := \sum_{i \in I_1} c_i g_i(z) + \sum_{i \in I_2} c_i f_i(z)$$

is a rational function. For $i \in \{1, \ldots, n\}$, we now write $m_i = d^{t(i)} k_i$ with integers $t(i) \geq 0, k_i > 0$ such that $d \nmid k_i$. Then condition (1.3) is equivalent to the distinctness of $k_1, \ldots, k_n$. By using a suitable permutation of $\{1, \ldots, n\}$, we may assume without loss of generality that, for some $m \in \{1, \ldots, n\}$, the conditions $c_1 \cdots c_m \neq 0, c_{m+1} = \cdots = c_n = 0$ hold and, moreover, $k_1 > \cdots > k_m$. If, under this permutation, $\{i \in I_j : c_i \neq 0\}$ changes to J_j $(j = 1, 2)$, then $J_1 \cup J_2 = \{1, \ldots, m\}$ and we may write

$$r(z) := \sum_{i \in J_1} c_i g_i(z) + \sum_{i \in J_2} c_i f_i(z).$$

From (2.11) we see

$$\zeta r(z^d) = r(z) + \sum_{i \in J_1} c_i \frac{z^{m_i}}{z^{m_i} - 1} - \sum_{i \in J_2} c_i \frac{z^{m_i}}{z^{m_i} + 1}.$$

With $\underline{c}$ and r as above, we define the rational function s by

$$s(z) := r(z) - \sum_{i \in J_1} \sum_{\tau=0}^{t(i)-1} \frac{c_i z^{d^\tau k_i}}{\zeta^{t(i)-\tau}(z^{d^\tau k_i} - 1)} + \sum_{i \in J_2} \sum_{\tau=0}^{t(i)-1} \frac{c_i z^{d^\tau k_i}}{\zeta^{t(i)-\tau}(z^{d^\tau k_i} + 1)}.$$

This new function satisfies

$$\begin{aligned} \zeta s(z^d) &= s(z) + \sum_{i \in J_1} \frac{c_i z^{k_i}}{\zeta^{t(i)}(z^{k_i} - 1)} - \sum_{i \in J_2} \frac{c_i z^{k_i}}{\zeta^{t(i)}(z^{k_i} + 1)} \qquad (2.12) \\ &= s(z) + \sum_{i \in J_1} \frac{c_i}{\zeta^{t(i)}} - \sum_{i \in J_2} \frac{c_i}{\zeta^{t(i)}} + \sum_{i \in J_1} \frac{c_i}{\zeta^{t(i)}(z^{k_i} - 1)} + \sum_{i \in J_2} \frac{c_i}{\zeta^{t(i)}(z^{k_i} + 1)}. \end{aligned}$$

Since all polynomials $z^{k_i} - 1, z^{k_i} + 1$ $(i = 1, \ldots, m)$ divide $z^L - 1$, where $L := 2 \operatorname{lcm}(k_1, \ldots, k_m)$, it follows from [Nis97, Lemma 1] that s must be of the form

$$s(z) = \frac{a(z)}{z^L - 1}$$

with some $a \in \mathbb{C}[z]$. By considering poles on the right-hand side of the first line of (2.12), one easily concludes $a \neq 0$. Moreover, considering the same line near ∞, we obtain $\deg a \leq L$, whence $\sigma := s(\infty) \in \mathbb{C}$ and $s_1(z) := s(z) - \sigma$ tends to 0 as $z \to \infty$. All in all, we conclude from the second line of (2.12)

$$\begin{aligned} \zeta s_1(z^d) &= s_1(z) + \sum_{i \in J_1} \frac{c_i}{\zeta^{t(i)}(z^{k_i} - 1)} + \sum_{i \in J_2} \frac{c_i}{\zeta^{t(i)}(z^{k_i} + 1)} \\ &= s_1(z) + \sum_{i=1}^{m} \frac{c_i}{\zeta^{t(i)}(z^{k_i} - 1)} - \sum_{i \in J_2} \frac{2c_i}{\zeta^{t(i)}(z^{2k_i} - 1)}. \end{aligned}$$

Let now $d | 2k_i$ exactly for $i \in \{i(1), \ldots, i(p)\}$ $(i \in J_2$, clearly such i can exist only if d is even), and write $2k_i = d\ell_i$ for these i. Then $d \nmid \ell_i$ and the ℓ_i's are distinct. Assume that $j(1), \ldots, j(q)$ are those values $i \in J_2$ with $d \nmid 2k_i$. Since an equation $\ell_{i(u)} = 2k_{j(v)}$ leads to the contradiction $d | k_{i(u)}$, the intersection $\{\ell_{i(1)}, \ldots, \ell_{i(p)}\} \cap \{2k_{j(1)}, \ldots, 2k_{j(q)}\}$ is empty. It follows that the rational function

$$S(z) := s_1(z) + \sum_{u=1}^{p} \frac{2c_{i(u)}}{\zeta^{t(i(u))+1}(z^{\ell_{i(u)}} - 1)}$$

satisfies

$$\zeta S(z^d) - S(z) = \sum_{i=1}^{m} \frac{c_i}{\zeta^{t(i)}(z^{k_i}-1)} - \sum_{u=1}^{p} \frac{2c_{i(u)}}{\zeta^{t(i(u))+1}(z^{\ell_{i(u)}}-1)} - \sum_{v=1}^{q} \frac{2c_{j(v)}}{\zeta^{t(j(v))}(z^{2k_{j(v)}}-1)}, \tag{2.13}$$

where, without loss of generality, we may assume $\ell_{i(1)} > \cdots > \ell_{i(p)}$ and $2k_{j(1)} > \cdots > 2k_{j(q)}$.

The possible poles on both sides of (2.13) are roots of unity. If there ever is a pole, let a primitive Nth root of unity be one of these on the left-hand side with maximal N. Crucial to our final reasoning will be the fact $d|N$.

Indeed, this divisibility property follows from the proof of [BV15c, Lemma 3.3], but, for the sake of completeness, we briefly explain here the reasoning. For this purpose, we may write down the partial fraction decomposition of S as

$$S(z) = \sum_{\delta|L} s_\delta(z) \;\text{ with }\; s_\delta(z) := \sum_{\substack{j=0\\(j,\delta)=1}}^{\delta-1} \frac{s_{\delta,j}}{z-\zeta_\delta^j},$$

where the $s_{\delta,j}$'s are complex constants and $\zeta_\delta := e^{2\pi i/\delta}$.

From this definition of s_δ for positive divisors δ of L we obtain

$$s_\delta(z^d) = \sum_{\substack{j=0\\(j,\delta)=1}}^{\delta-1} \frac{s_{\delta,j}}{z^d-\zeta_\delta^j} = \sum_{\substack{j=0\\(j,\delta)=1}}^{\delta-1} s_{\delta,j} \sum_{\kappa=0}^{d-1} \frac{1}{d\zeta_{d\delta}^{(j+\kappa\delta)(d-1)}(z-\zeta_{d\delta}^{j+\kappa\delta})}. \tag{2.14}$$

Suppose, from now on, $p_1^{\nu(1)}\cdot\ldots\cdot p_\omega^{\nu(\omega)}$ to be the canonical factorization of d. Assume that $p_1,\ldots,p_\sigma$ are not divisors of δ but $p_{\sigma+1},\ldots,p_\omega$ are, where we have to consider the cases $\sigma = 0,\ldots,\omega$. Then we have the following equivalence

$$(j,\delta)=1 \iff (j+\kappa\delta,\delta) = (j+\kappa\delta, d\delta/\textstyle\prod_{i=1}^{\sigma} p_i^{\nu(i)}) = 1$$

with the usual convention here and later that empty products (or sums) equal 1 (or 0, respectively). Now, any positive divisor D of $p_1\cdot\ldots\cdot p_\sigma$ is relatively prime to δ, whence there are precisely $\frac{d}{D}$ numbers $\kappa \in \{0,\ldots,d-1\}$ satisfying $D|(j+\kappa\delta)$. Thus, by the well-known inclusion-exclusion principle, we can say that, for fixed coprime j,δ, the number of $\kappa \in \{0,\ldots,d-1\}$ such that $j+\kappa\delta$ is prime to $p_1\cdot\ldots\cdot p_\sigma$ (or equivalently to $\prod_{i=1}^{\sigma} p_i^{\nu(i)}$) equals $d\prod_{i=1}^{\sigma}(1-1/p_i)$. Therefore we can note that, for fixed coprime j,δ, there are exactly $d\prod_{i=1}^{\sigma}(1-1/p_i)$ values $\kappa \in \{0,\ldots,d-1\}$ such that $(j+\kappa\delta, d\delta)=1$ holds. Hence we conclude

$$s_\delta(z^d) = \sum_{\substack{j=0\\(j,d\delta)=1}}^{d\delta-1} \frac{s_{\delta,j-[j/\delta]\delta}}{d\zeta_{d\delta}^{j(d-1)}(z-\zeta_{d\delta}^j)} + \Sigma_\delta(z) \tag{2.15}$$

from the double sum in (2.14). The rational function Σ_δ in (2.15) vanishes identically in case $\sigma=0$, whereas, in the cases $1 \le \sigma \le \omega$, it may have poles at certain primitive ρth roots of unity but with $\rho < d\delta$ only. Since $s_\delta \ne 0$ is equivalent to the fact that not all $s_{\delta,j-[j/\delta]\delta}$, $j \in \{0,\ldots,d\delta-1\}$ and prime to $d\delta$, vanish, we conclude from (2.15) that, in this case of δ, the difference $s_\delta(z^d) - s_\delta(z)$ has poles at $(d\delta)$th roots of unity. Thus, the number N defined after (2.13) must be of the form $d\delta$, whence $d|N$ holds.

If the set J_2 is empty $(p+q=0)$, then we obtain $N=k_1$ from the right-hand side of (2.13), hence $d\,|\,k_1$, a contradiction, and it suffices to subsequently consider only the case $p+q \ge 1$.

In case $d=2$, we have $q=0$ and $k_i=\ell_i$ for all $i \in J_2$. Thus, the right-hand side of (2.13) is of the form

$$\sum_{i=1}^{m} \frac{c_i}{\zeta^{t(i)}(z^{k_i}-1)} - \sum_{i\in J_2} \frac{2c_i}{\zeta^{t(i)+1}(z^{k_i}-1)},$$

whence again $N = k_1$, a contradiction.

We next suppose $d \geq 3$. If $k_1 \neq \max\{\ell_{i(1)}, \dots, \ell_{i(p)}, 2k_{j(1)}, \dots, 2k_{j(q)}\}$, we are led to the same contradiction as before. Hence, let k_1 be equal to that maximum. This implies $k_1 = 2k_{j(1)}$ observing that $\ell_{i(1)} < k_1$ follows from $d \geq 3$. If $c_1/\zeta^{t(1)} \neq 2c_{j(1)}/\zeta^{t(j(1))}$, we have our contradiction, whence $c_1 = 2c_{j(1)}\zeta^{t(1)-t(j(1))}$ must hold. We may now continue in the same way and obtain a contradiction unless $\{k_1, \dots, k_{p+q}\} = \{\ell_{i(1)}, \dots, \ell_{i(p)}, 2k_{j(1)}, \dots, 2k_{j(q)}\}$. Moreover, for each $i \in \{1, \dots, p+q\}$, there must exist a unique $\mu(i) \in \{i(1), \dots, i(p), j(1), \dots, j(q)\}$ such that $c_i = 2c_{\mu(i)}\zeta^{t(i)-t(\mu(i))-\delta}$ with $\delta = 1$ if $\mu(i) \in \{i(1), \dots, i(p)\}$ and $\delta = 0$ otherwise. If $p + q < m$, then $N = k_{p+q+1}$ holds under the preceding conditions, and we end up at our 'standard' contradiction. Therefore, the only remaining possibility is that $p+q = m, J_1 = \emptyset$. In this case, the above conditions lead to $c_1 = 2^j \zeta^\nu c_1$ with some $j \in \{1, \dots, m\}, \nu \in \mathbb{Z}$, and this contradiction completes the proof of Lemma 2. □

We next apply some ideas introduced in [Nis02],[NTT99] (see also [BV14],[BV15b]) to the functions

$$g_{i,d}(z) := \mathcal{G}_d(z^{m_i}), \ \ f_{i,d}(z) := \mathcal{F}_d(z^{m_i}) \quad (i = 1, \dots, n),$$

the $\mathcal{G}_d, \mathcal{F}_d$ as defined in (1.4). We obtain from Lemma 2 and [BV14, Lemma 5] the following

Lemma 3. *Under the assumptions of Lemma 2, the functions*

$$g_{i,d^j}(z) = \sum_{h=0}^{\infty} \frac{a_h z^{m_i d^{jh}}}{1 - z^{m_i d^{jh}}} \ \ (i \in I_1, j \in \mathbb{N}), \ \ f_{i,d^j}(z) = \sum_{h=0}^{\infty} \frac{a_h z^{m_i d^{jh}}}{1 + z^{m_i d^{jh}}} \ \ (i \in I_2, j \in \mathbb{N})$$

are algebraically independent over $\mathbb{C}(z)$.

Similarly to the proof of [BV14, Theorem 3], and noting that condition (1.5) implies (1.3), this lemma leads to the following result.

Theorem 6. *Let $m_1, \dots, m_n$ be $n \geq 2$ positive integers satisfying (1.5), and let $\alpha \in \overline{\mathbb{Q}}^\times \cap \mathbb{D}$. Then the numbers $\mathcal{G}_r(\alpha_i)$ with $i \in I_1, r \in \mathbb{N} \setminus \{1\}$, and $\mathcal{F}_r(\alpha_i)$ with $i \in I_2, r \in \mathbb{N} \setminus \{1\}$ are algebraically independent. In particular, the numbers $G_r(\alpha_i)$ $(i \in I_1, r \in \mathbb{N} \setminus \{1\}), F_r(\alpha_i)$ $(i \in I_2, r \in \mathbb{N} \setminus \{1\})$ are algebraically independent.*

3. Proof of Theorems 2 and 3

To this end, we denote $I_1 := \{i : \alpha_i = \alpha^{m_i}\}, I_2 := \{i : \alpha_i = -\alpha^{m_i}\}$. Clearly, these I_j satisfy the conditions of Lemma 2. Further, let

$$h_i(\zeta, z) := G_d(\zeta, z^{m_i}) \ \text{for} \ \ i \in I_1, \quad h_i(\zeta, z) := G_d(\zeta, -z^{m_i}) \ \text{for} \ \ i \in I_2,$$

where $G_d(a, z)$ (and its F-analogue) is defined in (2.9). If $i \in I_2$, then $h_i(\zeta, z) = -F_d(\zeta, z^{m_i})$ for odd d, and $h_i(\zeta, z) = G_d(\zeta, z^{m_i}) + 2z^{m_i}/(z^{2m_i} - 1)$ for even d. Therefore, Lemma 2 immediately gives

Lemma 4. *Let $m_1, \dots, m_n$ be $n \geq 2$ positive integers satisfying (1.3). Then, for any root of unity ζ, the functions $h_i(\zeta, z)$ $(i = 1, \dots, n)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$.*

By using [Nis96, Theorem 3.3.11], this lemma provides us directly Theorem 2.

Lemma 4 leads also to an analogue of Lemma 3 for the functions

$$h_{i,d}(z) := g_{i,d}(z) \text{ if } i \in I_1, \quad h_{i,d}(z) := \sum_{h=0}^{\infty} \frac{a_h(-z^{m_i})^{d^h}}{1-(-z^{m_i})^{d^h}} \text{ if } i \in I_2.$$

Lemma 5. *If the assumptions of Lemma 4 are satisfied, then the functions $h_{i,d^j}(z)$ $(i = 1, \ldots, n, j \in \mathbb{N})$ are algebraically independent over $\mathbb{C}(z)$.*

Analogously to Theorem 6, this lemma implies Theorem 3.

4. Proof of Theorems 4 and 5

In the following, we want to study algebraic independence of the functions G_d and F_d. By (1.6), it is natural to suppose $d \geq 3$ for this consideration. First we note that, for any $m \in \mathbb{N}$, the functional equation

$$\zeta r(z^d) = r(z) + \frac{z^m}{z^m - 1} - \frac{2z^{2m}}{z^{2m} - 1} + \frac{z^m}{z^m + 1}.$$

has $r(z) = 0$ as a solution. Thus, the functions $G_d(\zeta, z^m), G_d(\zeta, z^{2m})$, and $F_d(\zeta, z^m)$ are linearly dependent over $\mathbb{C}$. More generally, if

$$\frac{m_j}{2m_i} \in d^{\mathbb{Z}}$$

holds, then the functions $g_j(z), g_i(z)$, and $f_i(z)$ introduced in (2.10) are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$. Indeed, if $m_j = d^t 2m_i$ with $t \geq 0$, then the rational function

$$r_1(z) := -2\sum_{\tau=0}^{t-1} \frac{z^{d^\tau 2m_i}}{\zeta^{t-\tau}(z^{d^\tau 2m_i} - 1)}$$

satisfies

$$\zeta r_1(z^d) = r_1(z) - \frac{2z^{m_j}}{z^{m_j} - 1} + \frac{z^{m_i}}{\zeta^t(z^{m_i} - 1)} + \frac{z^{m_i}}{\zeta^t(z^{m_i} + 1)}$$

implying

$$r_1(z) = -2g_j(z) + \zeta^{-t} g_i(z) - \zeta^{-t} f_i(z).$$

Furthermore, if $d^t m_j = 2m_i$ with $t > 0$, then

$$r_2(z) := 2\sum_{\tau=0}^{t-1} \frac{z^{d^\tau m_j}}{\zeta^{t-\tau}(z^{d^\tau m_j} - 1)}$$

is a solution of

$$\zeta r_2(z^d) = r_2(z) - 2\zeta^{-t}\frac{z^{m_j}}{z^{m_j} - 1} + \frac{z^{m_i}}{z^{m_i} - 1} + \frac{z^{m_i}}{z^{m_i} + 1}$$

implying

$$r_2(z) = -2\zeta^{-t} g_j(z) + g_i(z) - f_i(z).$$

This makes it evident that we have to suppose condition (1.7) for our linear independence considerations.

Lemma 6. *Let $m_1, \ldots, m_n$ be $n \geq 2$ positive integers, let $d \geq 3$, and assume that conditions (1.3) and (1.7) are satisfied. Then, for any root of unity ζ, the functions $G_d(\zeta, z^{m_1}), \ldots, G_d(\zeta, z^{m_n}), F_d(\zeta, z^{m_1})$,*

$\dots, F_d(\zeta, z^{m_n})$ *are linearly independent over* $\mathbb{C}$ *modulo* $\mathbb{C}(z)$.

Proof. Assume, contrary to Lemma 6, that there exists a $\underline{C} := (b_1, \dots, b_n, c_1, \dots, c_n) \in \mathbb{C}^{2n} \setminus \{\underline{0}\}$ such that $r(z) := \sum_{i=1}^{n}(b_i g_i(z) + c_i f_i(z))$ is a rational function satisfying

$$\zeta r(z^d) = r(z) + \sum_{i=1}^{n} b_i \frac{z^{m_i}}{z^{m_i}-1} - \sum_{i=1}^{n} c_i \frac{z^{m_i}}{z^{m_i}+1}.$$

We may now argue as in the proof of Lemma 2 to obtain

$$\zeta S(z^d) - S(z) = \sum_{i=1}^{n} \frac{b_i + c_i}{\zeta^{t(i)}(z^{k_i}-1)} - \sum_{u=1}^{p} \frac{2c_{i(u)}}{\zeta^{t(i(u))+1}(z^{\ell_{i(u)}}-1)} - \sum_{v=1}^{q} \frac{2c_{j(v)}}{\zeta^{t(j(v))}(z^{2k_{j(v)}}-1)} \tag{4.16}$$

as an analogue of (2.13), where $p+q=n$ and some of b_i, c_i may vanish. An equation $k_i = 2k_{j(v)}$ is impossible, by (1.7). Thus, $k_i \neq 2k_{j(v)}$ for all pairs $(i, j(v))$, and similarly $k_i \neq \ell_{i(u)}$ for all pairs $(i, i(u))$. Further, $\{\ell_{i(1)}, \dots, \ell_{i(p)}\} \cap \{2k_{j(1)}, \dots, 2k_{j(q)}\} = \emptyset$ was shown in the proof of Lemma 2. All in all, this means that $\{k_1, \dots, k_n, \ell_{i(1)}, \dots, \ell_{i(p)}, 2k_{j(1)}, \dots, 2k_{j(q)}\}$ is a set of $2n$ distinct positive integers, each one not divisible by d. Moreover, condition $\underline{C} \neq \underline{0}$ implies $(b_1 + c_1, \dots, b_n + c_n, c_1, \dots, c_n) \neq \underline{0}$. Therefore, if we now define N for the equation (4.16) as we did it for (2.13), then we get $d|N$ from the left-hand side of (4.16), but clearly $d \nmid N$ from its right-hand side, a contradiction. □

To establish Theorem 4, we proceed similarly to our proof of Theorem 2 in the previous section. Moreover, the following analogue of Lemma 5 holds.

Lemma 7. *Let* $m_1, \dots, m_n$ *be* $n \geq 2$ *positive integers, let* $d \geq 3$*, and assume that conditions (1.3) and (1.7) are satisfied. Then the functions* $g_{i,d^j}(z), f_{i,d^j}(z)$ $(i = 1, \dots, n; j \in \mathbb{N})$ *are algebraically independent over* $\mathbb{C}(z)$.

Outline of Proof. By noting that (1.8) implies (1.3) and (1.7) for any $d \geq 3$, the proof of Theorem 5 runs very much parallel to that of Theorem 6. □

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