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# Effective irrationality measures for quotients of logarithms of rational numbers

Yann Bugeaud

**Abstract.** We establish uniform irrationality measures for the quotients of the logarithms of two rational numbers which are very close to 1. Our proof is based on a refinement in the theory of linear forms in logarithms which goes back to a paper of Shorey.

**Keywords.** Linear forms in logarithms, Baker's theory, irrationality measure.

**2010 Mathematics Subject Classification.** 11J82, 11J86

## 1. Introduction and result

Let  $\xi$  be an irrational real number. The real number  $\mu$  is an irrationality measure of  $\xi$  if, for every positive  $\varepsilon$ , there are a positive number  $C(\xi, \varepsilon)$  and at most finitely many rational numbers  $p/q$  with  $q \geq 1$  and

$$\left| \xi - \frac{p}{q} \right| < \frac{C(\xi, \varepsilon)}{q^{\mu+\varepsilon}}.$$

If, moreover, the constant  $C(\xi, \varepsilon)$  is effectively computable for every positive  $\varepsilon$ , then  $\mu$  is an effective irrationality measure of  $\xi$ . We denote by  $\mu(\xi)$  (resp.  $\mu_{\text{eff}}(\xi)$ ) the infimum of the irrationality measures (resp. effective irrationality measures) of  $\xi$ . It follows from the theory of continued fractions that  $\mu(\xi) \geq 2$  for every irrational real number  $\xi$  and an easy covering argument shows that there is equality for almost all  $\xi$ , with respect to the Lebesgue measure.

The following statement is a straightforward consequence of Baker's theory of linear forms in logarithms (see e.g. [Wa00] and the references therein). By definition, two positive rational numbers are multiplicatively independent if the quotient of their logarithms is irrational.

**Theorem 1.1** *Let  $a_1, a_2, b_1, b_2$  be positive integers with  $a_1 > a_2$  and  $b_1 > b_2$ . Assume that  $a_1/a_2$  and  $b_1/b_2$  are multiplicatively independent. There exists an absolute, effectively computable, constant  $C$  such that*

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq C(\log a_1)(\log b_1).$$

The purpose of this note is to show how a known refinement in the theory of linear forms in logarithms in the special case where the rational numbers are very close to 1, which goes back to Shorey's paper [Sho74], allows one to considerably improve Theorem 1.1 in this special case. Several spectacular applications to Diophantine problems and to Diophantine equations of this idea of Shorey have already been found; see for example [Wa78, Wa00] and the survey [Bu08]. Quite surprisingly, it seems that it has not yet been noticed that it can be used to give uniform upper bounds for irrationality measures of roots of rational numbers (see [Bu16]) and of quotients of logarithms of rational numbers, under some suitable assumptions.

Our main result is the following.

**Theorem 1.2** *Let  $a_1, a_2, b_1, b_2$  be positive integers such that*

$$\max\{16, a_2\} < a_1 < 6a_2/5 \quad \text{and} \quad \max\{16, b_2\} < b_1 < 6b_2/5. \quad (1.1)$$

*Define  $\eta$  by  $a_1 - a_2 = a_1^{1-\eta}$  and  $\nu$  by  $b_1 - b_2 = b_1^{1-\nu}$ . If  $a_1/b_1$  and  $a_2/b_2$  are multiplicatively independent, then we have*

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq 1 + 27638 \frac{(\log a_1)(\log b_1)}{\min\{\eta \log a_1, \nu \log b_1\}^2}. \quad (1.2)$$

We display an immediate corollary of Theorem 1.2 which deals with the case  $\eta > 1/2$  and  $\nu > 1/2$ . It illustrates the strength of the theory of linear forms in logarithms.

**Corollary 1.3** *Let  $a_1, a_2, b_1, b_2$  be positive integers such that*

$$36 \leq a_2 < a_1 < a_2 + \sqrt{a_1}, \quad 36 \leq b_2 < b_1 < b_2 + \sqrt{b_1}, \quad \text{and} \quad \sqrt{b_1} < a_1 < b_1^2.$$

*If  $a_1/b_1$  and  $a_2/b_2$  are multiplicatively independent, then we have*

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq 221105. \quad (1.3)$$

It is apparent from the proof of Theorem 1.2 that the numerical constants in (1.2) and (1.3) can be reduced (roughly, divided by 3) if  $a_1$  and  $b_1$  are sufficiently large. No particular significance has to be attached to the numerical constant  $6/5$  in (1.1).

Let  $a, b$  and  $d$  be positive integers with  $a \neq b$  and  $\max\{a, b\} < d$ . Under certain conditions, Rhin [Rh83] (see also [RhTo86]) obtained explicit upper bounds for

$$\mu_{\text{eff}}\left(\frac{\log(1 + a/d)}{\log(1 + b/d)}\right).$$

His approach, which gives better numerical results than ours, heavily uses the fact that the two rational numbers  $a/d$  and  $b/d$  have the same denominator. It seems to us that Theorem 1.2, which applies without any specific restriction on the denominators  $a_2$  and  $b_2$  of the rational numbers, is new and cannot be straightforwardly derived from the methods of [Rh83, RhTo86].

## 2. Proof of Theorem 1.2

We reproduce with some simplification Corollary 2.4 of Gouillon [Gou06] in the special case where the algebraic numbers involved are rational numbers. We replace his assumption  $E \geq 2$  by  $E \geq 15$ , to avoid trouble with the quantity  $\log \log \log E$  occurring in the definition of  $E^*$  in Corollary 2.4 of [Gou06], which is not defined if  $E$  is too small.

**Theorem G** *Let  $a_1, a_2, b_1, b_2$  be positive integers such that  $a_1/a_2$  and  $b_1/b_2$  are multiplicatively independent and greater than 1. Let  $A$  and  $B$  be real numbers such that*

$$A \geq \max\{a_1, e\}, \quad B \geq \max\{b_1, e\}.$$

*Let  $x$  and  $y$  be positive integers and set*

$$X' = \frac{x}{\log A} + \frac{y}{\log B}.$$

Set

$$E = 1 + \min \left\{ \frac{\log A}{\log(a_1/a_2)}, \frac{\log B}{\log(b_1/b_2)} \right\}$$

and

$$\log X = \max\{\log X' + \log E, 265 \log E, 600 + 150 \log E\}.$$

Assume furthermore that  $15 \leq E \leq \min\{A^{3/2}, B^{3/2}\}$ . Then,

$$\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -8550(\log A)(\log B)(\log X)(4 + \log E)(\log E)^{-3}.$$

It is crucial for our proof that the dependence on  $X$  comes through the factor  $(\log X)$  and not through  $(\log X)^2$ , as in [LMN95]. Instead of Theorem G we could use an earlier result of Waldschmidt [Wa93] (see also Theorem 9.1 of [Wa00]), but the numerical constants in (1.2) and (1.3) would then be slightly larger.

*Proof of Theorem 1.2.* Our aim is to estimate from below the quantity

$$\left| \frac{\log(a_1/a_2)}{\log(b_1/b_2)} - \frac{x}{y} \right|,$$

for large positive integers  $x, y$ . We will establish a lower bound of the form

$$\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -C_1(C_2 + \log \max\{x, y\}), \quad (2.1)$$

for some quantities  $C_1, C_2$  which depend at most on  $a_1, a_2, b_1$  and  $b_2$ . This will show that  $1 + C_1$  is an irrationality measure for  $\log(a_1/a_2)/\log(b_1/b_2)$ .

We apply Theorem G and set  $A = a_1$  and  $B = b_1$ .

Observe that

$$\log(a_1/a_2) = \log(1 + ((a_1 - a_2)/a_2)) \leq (a_1 - a_2)/a_2 = a_1^{1-\eta} a_2^{-1},$$

thus

$$\frac{\log a_1}{\log(a_1/a_2)} \geq (\log a_1) \frac{a_2}{a_1} a_1^\eta \geq \max\{14, a_1^\eta\},$$

since  $a_1 \geq 17$  and  $a_1^\eta = a_1/(a_1 - a_2) \geq 6$ . Likewise, we check that

$$\frac{\log b_1}{\log(b_1/b_2)} \geq \max\{14, b_1^\nu\}.$$

Furthermore, since  $a_2 \geq 14$  and  $a_1 \geq 17$ , we get

$$1 + \frac{\log a_1}{\log(a_1/a_2)} \leq 1 + \frac{\log a_1}{\log(1 + 1/a_2)} \leq 1 + 1.1a_2 \log a_1 \leq a_1^{3/2},$$

and a similar upper bound holds for  $1 + (\log b_1)/(\log(b_1/b_2))$ , thus we have proved that

$$E := 1 + \min \left\{ \frac{\log a_1}{\log(a_1/a_2)}, \frac{\log b_1}{\log(b_1/b_2)} \right\}$$

satisfies

$$15 \leq E \leq \min\{A^{3/2}, B^{3/2}\}.$$

Note also that the quantity

$$E' = \min\{a_1^\eta, b_1^\nu\}$$

satisfies  $6 \leq E' \leq E$ , thus  $(4 + \log E')/(\log E') \leq (4 + \log 6)/(\log 6)$ . It then follows from Theorem G that

$$\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -27638(\log A)(\log B)(\log \max\{x, y\} + \log E)(\log E')^{-2},$$

when  $\max\{x, y\}$  is sufficiently large. This lower bound is of the form (2.1), with explicit values for  $C_1$  and  $C_2$ , thus we have proved that

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq 1 + 27638 \frac{(\log a_1)(\log b_1)}{\min\{\eta \log a_1, \nu \log b_1\}^2}.$$

This completes the proof of Theorem 1.2.  $\square$

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