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# On a method of Davenport and Heilbronn, I

by

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*Dedicated to Professor A. Schinzel on his sixtieth birthday*

## 1. Introduction

This paper arose out of a table of RAJA RAMANNA et al. (see [RR,AS], [RR, BVS] and [RR] papers on nuclear physics) which gave values of the number $\log_{10}\left(\frac{2^n}{n}\right)$ where $n$ is a positive integer. The authors observed that sometimes this is very close to a prime. I first state a result of I.M. VINOGRADOV (see [IMV]) and then a result of S.D. CHOWLA (see [SDC]) and then go on to state the course of developments of a deep method (see [HD,HH]) due to H. DAVENPORT and H. HEILBRONN. One of the corollaries to a deep theorem of I.M. VINOGRADOV runs as follows. Let $\lambda > 0$ be any fixed irrational number. Then as $n$ runs through all positive integers and $p$ through all primes the numbers $\lambda p - n$ are everywhere dense on the real line. (Actually VINOGRADOV proves that the numbers $\lambda p$ are "uniformly distributed modulo 1".) By using a theorem of V. JARNIK and A. WALFISZ [VJ,AW] on the number of lattice points in a five-dimensional ellipsoid, S.D. CHOWLA deduced ingeniously that the set of values of $\sum_{j=1}^{9} \lambda_j x_j^2$ ($\lambda_j$ fixed non-zero real numbers, not all of the same sign and $\lambda_1\lambda_2^{-1}$ irrational) as the nine number tuple $(x_j)$ runs independently over all positive integer entries, is dense on the real line. The result of DAVENPORT and HEILBRONN states that here 9 can be reduced to 5. (The latest result in this direction is due to G.A. MARGULIS who by a different method [GAM] showed that the values of any real indefinite quadratic form in 3 variables (which is not a multiple of a form with rational coefficients), as the variables run through all triplets of positive integers, is dense everywhere on the real line.)

Now we come to the method of DAVENPORT and HEILBRONN. Their method was modified by R.P. BAMBAH [RPB] who showed that the values of $\sum_{j=1}^{5}\lambda_j x_j^2$ with the same conditions as before is dense on the real line even when we insist that one of the co-ordinates say $x_5$ be the ${k_0}^{\text{th}}$ powers of positive integers ($k_0 \geq 2$ being a fixed integer). The problem of considering prime quintuplets $(x_j)$ was considered by K. RAMACHANDRA [$\text{KR}_1$], and earlier W. SCHWARZ [WS], both of whom considered also the corresponding linear problem $\sum_{j=1}^{3}\lambda_j x_j$ with prime triplets $(x_1, x_2, x_3)$. Also A. BAKER [AB] considered the linear problem by a different method. The result of K. RAMACHANDRA was sharper than those of W. SCHWARZ and A. BAKER, and his method like the method of W. SCHWARZ was an adaptation of the fundamental method of DAVENPORT and HEILBRONN. Next R.C. VAUGHAN [$\text{RCV}_1$], [$\text{RCV}_2$] improved remarkably the result of K. RAMACHANDRA by introducing into the method, expressions involving the zeros of the Riemann zeta-function. (It should be mentioned that much later after R.C. VAUGHAN, S. SRINIVASAN [SS] has also worked out some results by the DAVENPORT-HEILBRONN fundamental method (by using VAUGHAN's ideas)).

In this paper it is our object to raise new questions which arise in connection with our new results which run as follows. (We state a few theorems and make some remarks.)

**Theorem 1.** *Let $a > 1$, $c > 1$ be any two fixed real numbers such that $\frac{\log a}{\log c}$ is irrational. Then for any two fixed real numbers $b$ and $\eta$, we have the inequality*

$$\left|\log_c\left(\frac{a^n}{n^b}\right) - p - \eta\right| < \operatorname{Exp}\left(-(\log(np))^{1/2}\right)$$

*for infinitely many pairs $(n, p)$ where $n$ is a positive integer and $p$ is prime.*

**Remark 1.** Let $\lambda$ be any positive irrational and $\varphi(u)$ $(u \geq 3)$ be any continuously differentiable function for which $\varphi'(u)$ is monotonic and $= O(\frac{1}{u})$. (We can relax this condition to some extent.) Then the numbers $\lambda p - n + \varphi(n)$ are dense (everywhere) on the real line as $n$ runs through all positive integers and $p$ through primes.

Actually we prove the following more general theorem and remark about some new questions.

**Theorem 2.** *Let $\lambda_1 > 0$, $\lambda_3 < 0$ be any two constants such that $\lambda_1\lambda_3^{-1}$ is irrational and let $\lambda_2$ be any non-zero real constant. Let $\varphi_1(u)$, $\varphi_2(u)$ and $\varphi_3(u)$ be any three continuously differentiable real valued functions (defined for $u \geq 3$) with the properties: $\varphi_j'(u)$ monotonic and $\sum_{j=1}^{3}|\varphi_j'(u)| \leq u^{-1}(\log u)^{k_0}$ where $k_0$ is any fixed positive constant. Then the inequality*

$$\left|\sum_{j=1}^{3}\lambda_j(p_j + \varphi_j(p_j))\right| < \operatorname{Exp}\left(-(\log(p_1p_2p_3))^{1/2}\right)$$

*holds for infinitely many triplets* $(p_1, p_2, p_3)$ *of primes (in fact, even with the restriction* $p_1^{-1}p_2 \le \mathrm{Exp}(-(\log(p_1p_2p_3))^{1/2})$.)

**Remark 2.** By employing R.C. VAUGHAN's method (which is a sharpening of RAMACHANDRA's method) we may be able to replace $\mathrm{Exp}(-(\log(p_1p_2p_3))^{1/2})$ which occurs in the above theorem by $(p_1p_2p_3)^{-\delta}$ where $\delta > 0$ is an absolute constant.

**Remark 3.** The precise generalisations of Theorem 2 (with for example $p_j^k + \varphi_j(p_j)$, $k \ge 1$, in place of $p_j + \varphi_j(p_j)$) will form the subject matter of a forthcoming paper.

**Remark 4.** One of the attractive problems is to consider (with any indefinite quadratic form $f(x_1, x_2, x_3)$ in three variables which is not proportional to a rational form) the problem regarding $f(x_1 + \varphi_1(x_1), x_2 + \varphi_2(x_2), x_3 + \varphi_3(x_3))$ which corresponds to the result of G.A. MARGULIS. Here $\varphi_1(u)$, $\varphi_2(u)$ and $\varphi_3(u)$ are suitable real valued functions.

## 2. Notation and sketch of the method

We denote by $A, B, C$ quantities greater than 1, which will depend on the variable $x$ ($x \ge 10$) (but all of them are $\le \exp(100\sqrt{\log x})$). We will choose them later. Very soon we will choose $B = A$ (since $B \ne A$ is not of much use). The letter $D$ will denote the constant $(3\lambda_1 + 2|\lambda_2|)|\lambda_3|^{-1}$. We write $e(u) = e^{2\pi i u}$,

$$\begin{aligned} S_1 &= \sum_{x \le p \le 2x} (\log p) e(B\lambda_1\alpha(p + \varphi_1(p))) \\ S_2 &= \sum_{2 \le p \le xC^{-1}} (\log p) e(B\lambda_2\alpha(p + \varphi_2(p))) \\ S_3 &= \sum_{2 \le p \le Dx} (\log p) e(B\lambda_3\alpha(p + \varphi_3(p))), \end{aligned}$$

where $\alpha$ is a real variable.

Then by using

$$\int_{-\infty}^{\infty} e(\beta\alpha)\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha = \max(0, 1 - |\beta|) \quad (\beta \text{ real}),$$

we see that the quantity $J$ defined by

$$J = \int_{-\infty}^{\infty} S_1S_2S_3\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha$$

satisfies

$$J = \sum\sum\sum (\log p_1)(\log p_2)(\log p_3) \max\Big(0, 1 - \Big|B\sum_{j=1}^{3}\lambda_j(p_j + \varphi(p_j))\Big|\Big)$$

where the three summations are as in $S_1$, $S_2$ and $S_3$ respectively. Clearly it suffices to prove $J \neq 0$ (with a suitable $B$) in order to prove Theorem 2. This is done as follows. First of all we show that the contribution $J_1$ (to $J$) from the interval $|\alpha| \leq x^{-1}A$ is the dominant term for $J$ (we may call $|\alpha| \leq x^{-1}A$ as the basic interval). Next we prove that the contribution $J_3$ (to $J$) from $|\alpha| \geq B(\log x)^2$ is small (we may call this the supplementary interval). The contribution $J_2$ (to $J$) from the remaining interval will have to be shown to be small for a sequence $x = x_\nu \to \infty$ (that will be done by using the irrationality of $\lambda_1\lambda_3^{-1}$, this (remaining) interval may be called the intermediary interval). Though this reminds us of the famous "CIRCLE METHOD", this method is somewhat different and is an ingenious method due to H. DAVENPORT and H. HEILBRONN. We introduce some more notation.

$$I_1 = \int_x^{2x} e(B\lambda_1\alpha(u + \varphi_1(u))du,$$

$$I_2 = \int_2^{xC^{-1}} e(B\lambda_2\alpha(u + \varphi_2(u))du,$$

$$I_3 = \int_2^{Dx} e(B\lambda_3\alpha(u + \varphi_3(u))du,$$

and

$$E_j = S_j - I_j \quad (j = 1, 2, 3).$$

We assume the prime number theorem in the form $\vartheta(u) - u = O(uE^{-1}(u))$ $(u \geq 3)$, where $E(u) = \operatorname{Exp}((\log u)^{\frac{11}{20}})$, and $\vartheta(u) = \sum_{2\leq p\leq u} \log p$. The letters $K_1$ and $K_2$ will denote certain positive constants independent of $x$. The Vinogradov symbols $\ll$ and $\gg$ have the usual meaning.

## 3. Treatment of the basic interval

Consider the portion $J_1$ of $J$, where the integration is restricted by $|\alpha| \leq x^{-1}A$. In $J_1$ we define $J^*$ to be the same as $J_1$ but with $S_1$, $S_2$ and $S_3$ being replaced by $I_1$, $I_2$ and $I_3$, respectively. Next we define $J^{**}$ to be the same as $J^*$ with the integration limits $-\infty \leq \alpha \leq \infty$. Our first object is to show that $|J_1 - J^*|$ is small and that $|J^* - J^{**}|$ is also small, but $J^{**}$ is big. This will complete the object of this section namely that $J_1$ is big. We begin with the following lemma.

**Lemma 3.1.** *We have*

$$|S_1S_2S_3 - I_1I_2I_3| \leq \prod_{j=1}^{3}(|I_j| + |E_j|) - \prod_{j=1}^{3}|I_j|.$$

□

*Proof.* Follows from $S_j = I_j + E_j$

**Lemma 3.2.** *We have*

$$|E_1| + |E_3| \ll xE^{-1}(x) + (|B\lambda_1\alpha| + |B\lambda_3\alpha|)x^2E^{-1}(x)$$

*and*

$$|E_2| \ll xC^{-1}E^{-1/2}(x) + |B\lambda_2\alpha|C^{-2}x^2E^{-1/2}(x)$$

*where the implied constants depend only on* $\lambda_1$, $\lambda_2$ *and* $\lambda_3$.

*Proof.* We have

$$\begin{aligned} S_1 &= \sum_{x\le p\le 2x} (\log p)e(B\lambda_1\alpha(p+\varphi_1(p))) \\ &= \int_{x-0}^{2x+0} e(B\lambda_1\alpha(u+\varphi_1(u)))(d(\vartheta(u)-u)+du) \\ &= I_1 + (\vartheta(u)-u)e(B\lambda_1\alpha(u+\varphi_1(u)))\Big]_{x-0}^{2x+0} \\ &\quad +O\Big(\int_x^{2x} |B\lambda_1\alpha(1+\varphi_1'(u))uE^{-1}(u)e(B\lambda_1\alpha(u+\varphi_1(u)))|du\Big) \end{aligned}$$

Hence

$$|E_1| \ll xE^{-1}(x) + |B\lambda_1\alpha|x^2E^{-1}(x).$$

Similarly we can prove estimates for $|E_3|$ and $|E_2|$. □

From now on we choose $A = B$.

**Lemma 3.3.** *For* $|\alpha| \le x^{-1}A$, *we have the inequality*

$$|E_1| + |E_2| + |E_3| \ll xE^{-1/3}(x).$$

*Proof.* Follows from the condition $A = B$ imposed already. □

**Lemma 3.4.** *We have*

$$\int_{|\alpha|\le x^{-1}A} |S_1S_2S_3 - I_1I_2I_3|d\alpha \ll x^2(E(x))^{-1/4}.$$

*Proof.* The proof follows from Lemmas 3.1 and 3.3 (on using the trivial estimates for $|I_1|$, $|I_2|$ and $|I_3|$). □

**Lemma 3.5.** *We have*

$$|I_1 I_2 I_3| \ll |B\alpha|^{-3}$$

*for all real $\alpha$ (where the implied constant depends only on $\lambda_1$, $\lambda_2$ and $\lambda_3$).*

*Proof.* The integrals $I_1$, $I_2$, $I_3$ are of the form

$$\int e(B\lambda_j\alpha(u+\varphi_j(u)))du \quad \text{(over suitable limits of integration)}$$

$$= \int \frac{B\lambda_j\alpha(1+\varphi_j'(u))e(B\lambda_j\alpha(u+\varphi_j(u)))du}{B\lambda_j\alpha(1+\varphi_j'(u))} = O(|B\alpha|^{-1})$$

on using the monotonicity of $\varphi_j'(u)$. □

**Lemma 3.6.** *We have*

$$\int_{|\alpha|\geq x^{-1}A} |I_1 I_2 I_3| d\alpha \ll A^{-5}x^2.$$

*Proof.* LHS is

$$\ll \int_{|\alpha|\geq x^{-1}A} |B\alpha|^{-3} d\alpha \ll B^{-3}(x^{-1}A)^{-2} = A^{-5}x^2.$$

□

**Lemma 3.7.** *We have*

$$\int_{-\infty}^{\infty} I_1 I_2 I_3 \Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha \gg x^2 A^{-1} C^{-1},$$

where the implied constant depends only on $\lambda_1$, $\lambda_2$ and $\lambda_3$.

*Proof.* LHS is the same as

$$\int_x^{2x}\int_2^{xC^{-1}}\int_2^{Dx} \max\Big(0, 1 - \Big|B\sum_{j=1}^{3}\lambda_j(u_j+\varphi_j(u_j))\Big|\Big) du_1 du_2 du_3.$$

Let $u_1$ and $u_2$ run freely over their ranges. For every pair $(u_1, u_2)$ the integrand is $\geq \frac{1}{2}$ provided

$$|B\lambda_3(u_3+\varphi_3(u_3))|$$

(as a function of $u_3$) varies over an interval of length $\frac{1}{2}$. For this $u_3$ has to be in an interval of length $\gg B^{-1} = A^{-1}$. This proves Lemma 3.7. □

Collecting we have the following main lemma of this section.

**Lemma 3.8.** *We have*

$$J_1 = \int_{|\alpha|\leq x^{-1}A} S_1 S_2 S_3 \Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha$$

$$\geq x^2 A^{-1} C^{-1}(K_1 - K_2 A^{-4} C - K_2 AC(E(x))^{-1/4}).$$

## 4. Treatment of supplementary interval

Denote by $J_3$ the portion $|\alpha|$ exceeding a positive quantity (see Lemma 4.2) of the integral

$$\int S_1 S_2 S_3 \Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha.$$

We will show that $J_3$ is small.

**Lemma 4.1.** *Let $y \geq 0$. Then $S_2 \ll xC^{-1}$ and*

$$\int_{y\leq\alpha\leq y+1} |S_1 S_3| d\alpha \ll x\log x.$$

*Proof.* The first assertion is trivial. Next we have $|S_1 S_3| \leq |S_1|^2 + |S_3|^2$ and

$$\int_{y\leq|\alpha|\leq y+1} \Big| \sum_{x\leq p\leq 2x} (\log p) e(B\lambda_1\alpha(p+\varphi_1(p)))\Big|^2 d\alpha$$
$$\leq \sum_p (1+O(1))(\log p)^2 \ll x\log x.$$

Here we have used a famous theorem due to H.L. MONTGOMERY and R.C. VAUGHAN (see for instance [KR$_2$]). We may prove what is required for our purposes by simpler arguments but it is convenient to use this.

Similarly

$$\int_{y\leq|\alpha|\leq y+1} \Big| \sum_{2\leq p\leq Dx} (\log p) e(B\lambda_3\alpha(p+\varphi_3(p)))\Big|^2 d\alpha \ll x\log x.$$

□

**Lemma 4.2.** *For all $y \geq 0$ we have*

$$\int_{|\alpha|\geq y} |S_1 S_2 S_3| \Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha \ll \frac{x^2(\log x)C^{-1}}{y+1}$$

*and choosing $y = A(\log x)^2$, we have*

$$\int_{|\alpha|\geq A(\log x)^2} |S_1 S_2 S_3| \Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha \ll A^{-1}C^{-1}x^2(\log x)^{-1}.$$

*Proof.* The proof follows from Lemma 4.1. □

## 5. Treatment of intermediary interval (namely $I : x^{-1}A \leq |\alpha| \leq A(\log x)^2$)

We show that the integral

$$J_2 = \int_I S_1 S_2 S_3 \Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha$$

over this interval is small compared with $A^{-1}C^{-1}x^2$ (for a sequence $x = x_\nu \to \infty$). Let $M = \max_{\alpha\in I}\min_{j=1,3}|S_j|$. Then

$$\int_I |S_1S_2S_3|\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha$$
$$\leq \int_{(\alpha)_1} M|S_2S_3|\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha + \int_{(\alpha)_3} M|S_1S_2|\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha$$

(where $(\alpha)_1$ denotes those $\alpha \in I$ for which $M = |S_1|$ and $(\alpha)_3$ denotes those $\alpha \in I$ for which $M = |S_3|$)

$$\ll M\Big(\int_I |S_2|^2\frac{d\alpha}{\alpha^2+1}\Big)^{\frac{1}{2}}\Big(\int_I |S_3|^2\frac{d\alpha}{\alpha^2+1}\Big)^{\frac{1}{2}}$$
$$+ M\Big(\int_I |S_1|^2\frac{d\alpha}{\alpha^2+1}\Big)^{\frac{1}{2}}\Big(\int_I |S_2|^2\frac{d\alpha}{\alpha^2+1}\Big)^{\frac{1}{2}} \ll MC^{-1/2}x\log x.$$

We adopt a similar procedure at some places below. Finally we let $x = x_\nu \to \infty$ ($\nu = 1, 2, 3. \dots$).

**Lemma 5.1.** *Let $Q \geq 1$ and $\beta$ be any real numbers. Then there exist integers $a, q$ with $(a, q) = 1$, $1 \leq q \leq Q$ and $|\beta - \frac{a}{q}| \leq (qQ)^{-1}$. Hence we have*

$$\Big|\lambda_1 B\alpha - \tfrac{a_1}{q_1}\Big| \leq (q_1Q)^{-1}, \quad 1 \leq q_1 \leq Q, \quad (a_1, q_1) = 1,$$
$$\Big|\lambda_3 B\alpha - \tfrac{a_3}{q_3}\Big| \leq (q_3Q)^{-1}, \quad 1 \leq q_3 \leq Q, \quad (a_3, q_3) = 1,$$

*and*

$$\Big|\frac{\lambda_1}{\lambda_3} - \frac{a_0}{q_0}\Big| \leq (q_0Q_0)^{-1} (\leq q_0^{-2}), \quad 1 \leq q_0 \leq Q_0, \quad (a_0, q_0) = 1,$$

*where $a_1, a_3, q_1, q_3, Q, Q_0$ are integers and $Q$ and $Q_0$ are at our choice.*

*Proof.* The first assertion is well-known and follows by a simple application of the Dirichlet box principle. The rest of the assertions are special corollaries. □

Hereafter we impose $x$ and $Q_0$ to be large enough and $\lambda_1\lambda_3^{-1}$ irrational and so $q_0$ can be assumed to run through a sequence which tends to infinity.

**Lemma 5.2.** *If $Q_0$ is large enough then $a_0 \neq 0$. If $a_1 = 0$ then for such $\alpha$ ( note that $I : A(\log x)^2 \geq \alpha \geq x^{-1}A$ and also $|B\alpha| \ll Q^{-1}$)*

$$|S_1| \ll (B\alpha)^{-1} + xE^{-1}(x) + |B\alpha|x^2E^{-1}(x).$$

*If $a_3 = 0$ then for such $\alpha$ we have*

$$|S_3| \ll |B\alpha|^{-1} + xE^{-1}(x) + |B\alpha|x^2E^{-1}(x).$$

*Proof.* Similar to the proofs of Lemmas 3.2 and 3.5. □

Let $(\alpha)_{0,1}$ be the set of those $\alpha \in I$ for which $a_1 = 0$ and $(\alpha)_{0,3}$ the set of those $\alpha \in I$ for which $a_3 = 0$. Then

$$\begin{aligned}
&\int_{(\alpha)_{0,1}} |S_1S_2S_3|\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha \\
&\ll \int \Big\{A^{-1}B^{-1}x + xE^{-1}(x) + \frac{x^2}{Q}E^{-1}(x)\Big\}|S_2S_3|\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha \\
&\ll A^{-2}C^{-1/2}x^2\log x + C^{-1/2}x^2E^{-1}(x)\log x \\
&\quad + \frac{x^2}{Q}E^{-1}(x)\int_{\alpha > x^{-1}A} |S_2S_3|\frac{d\alpha}{\alpha^2+1} \\
&\ll A^{-2}C^{-1/2}x^2\log x + C^{-1/2}x^2E^{-1}(x)\log x + \frac{x^3}{Q}E^{-1}(x)C^{-1/2}\log x.
\end{aligned}$$

We can get the same bound for

$$\int_{(\alpha)_{0,3}} |S_1S_2S_3|\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha$$

by proceeding in a similar way.

Collecting we have the following lemma.

**Lemma 5.3.** *Contribution to the integral over the intermediary interval from those $\alpha$ for which $a_1 = 0$ or $a_3 = 0$ is*

$$\ll A^{-2}C^{-1/2}x^2\log x + C^{-1/2}x^2E^{-1}(x)\Big(1 + \frac{x}{Q}\Big)x^2\log x.$$

*From now on we consider the contribution form those $\alpha$ for which both $a_1 \neq 0$ and $a_3 \neq 0$.*

**Lemma 5.4.** *We have*

$$\frac{a_0}{q_0} - \frac{a_1 q_3}{a_3 q_1} = O(q_0^{-2} + Q^{-1}|a_3|q_3^{-1}(q_1^{-1} + q_3^{-1})).$$

*Proof.* We have

$$\left|\frac{\lambda_1}{\lambda_3}\lambda_3 B\alpha - \frac{a_1}{q_1}\right| \le (q_1 Q)^{-1}.$$

Here the LHS is

$$\left|\left(\frac{a_0}{q_0} + O(q_0^{-2})\right)\left(\frac{a_3}{q_3} + O(q_3^{-1}Q^{-1})\right) - \frac{a_1}{q_1}\right|.$$

This gives

$$\frac{a_0 a_3}{q_0 q_3} + O(|a_3|q_3^{-1}q_0^{-2}) + O(q_3^{-1}Q^{-1}) - \frac{a_1}{q_1} = O(q_1^{-1}Q^{-1}),$$

i.e.

$$\frac{a_0}{q_0} - \frac{a_1 q_3}{a_3 q_1} = O(q_0^{-2} + Q^{-1}|a_3|q_3^{-1}(q_1^{-1} + q_3^{-1})).$$

□

**Lemma 5.5** *We have $a_3 = O(A^2 q_3 (\log x)^2)$.*

*Proof.* Follows from the inequality defining $a_3$ and $q_3$. □

**Lemma 5.6.** *Let $q_1$ and $q_3$ be both $\le q_0^t$ where $t > 0$ is such that*

$$A^2(\log x)^2 q_0^{2t} = o(q_0). \tag{1}$$

*Then*

$$q_0^{-1+2t}A^2(\log x)^2 + Q^{-1}q_0^{1+t}A^4(\log x)^4 \gg 1. \tag{2}$$

*Proof.* Otherwise

$$\frac{a_0}{q_0} - \frac{a_1 q_3}{a_3 q_1} = 0$$

and hence $q_0$ divides $a_3 q_1$ and so $q_0 \le |a_3| q_1$ which is not possible by hypothesis (we have used Lemma 5.5). □

**Lemma 5.7.** *If $t(>0)$ is such that (1) holds and*

$$q_0^{-1+2t}A^2(\log x)^2 + Q^{-1}q_0^{1+t}A^4(\log x)^4 = o(1), \tag{3}$$

*then we have*

$$\text{either } \; q_1 \ge q_0^t \; \text{ or } \; q_3 \ge q_0^t. \tag{4}$$

*Proof.* The proof follows from Lemma 5.6. □

**Lemma 5.8.** *For $j=1,3$ we have*

$$S_j=\sum_p(\log p)e(\lambda_jB\alpha(p+\varphi_j(p)))=S_j^*+O(Q^{-1}q_j^{-1}x^2), \tag{5}$$

*where*

$$S_j^*=\sum_p(\log p)e\Big(\frac{a_j}{q_j}(p+\varphi_j(p))\Big), \tag{6}$$

*the summation over p each time being as in $S_j$.*

*Proof.* Observe that

$$\begin{aligned}&e(\lambda_jB\alpha(p+\varphi_j(p)))-e(a_jq_j^{-1}(p+\varphi_j(p)))\\&\qquad=O(|\lambda_jB\alpha-a_jq_j^{-1}|(p+\varphi_j(p)))=O(Q^{-1}q_j^{-1}p).\end{aligned}$$

Thus

$$|S_j-S_j^*|\le\sum_p p(\log p)Q^{-1}q_j^{-1}=O(Q^{-1}q_j^{-1}x^2).$$

This proves the lemma (since $\varphi_j(p)=\int_3^p\varphi_j'(u)du+O(1)=O((\log p)^{k_0+1})$). □

We now proceed to estimate $S_j^*$. We have

$$S_j^*=\int_2^{D_jx}e\Big(\frac{a_j}{q_j}\varphi_j(u)\Big)d\Big(\sum_{2\le p\le u}(\log p)e\Big(\frac{a_j}{q_j}p\Big)\Big)$$

(where $D_1$ and $D_3$ are positive constants depending only on $\lambda_1$, $\lambda_2$ and $\lambda_3$)

$$\begin{aligned}&=e\Big(\frac{a_j}{q_j}\varphi_j(u)\Big)\sum_{2\le p\le u}(\log p)e\Big(\frac{a_j}{q_j}p\Big)\Big]_{u=2-0}^{D_jx+0}\\&\quad-\int_2^{D_jx}\Big(\sum_{2\le p\le u}(\log p)e\Big(\frac{a_j}{q_j}p\Big)\Big)\frac{a_j}{q_j}\varphi_j'(u)e\Big(\frac{a_j}{q_j}\varphi_j(u)\Big)du\\&=O(1)+O\Big(\Big|\sum_{2\le p\le D_jx+0}(\log p)e\Big(\frac{a_j}{q_j}p\Big)\Big|\Big)\\&\quad+O\Big(\int_2^{D_jx}\Big|\frac{a_j}{q_j}\sum_{2\le p\le u}(\log p)e\Big(\frac{a_j}{q_j}p\Big)\Big|(\log x)^{k_0}u^{-1}du\Big)\\&=O(1)+O\Big(\Big|S\Big(\frac{a_j}{q_j},D_jx\Big)\Big|\Big)\\&\quad+O\Big(\Big|\frac{a_j}{q_j}\Big|(\log x)^{k_0}\int_2^{D_jx}\Big|S\Big(\frac{a_j}{q_j},u\Big)\Big|u^{-1}du\Big)\end{aligned} \tag{7}$$

where

$$S\Big(\frac{a_j}{q_j},u\Big)=\sum_{2\le p\le u}(\log p)e\Big(\frac{a_j}{q_j}p\Big). \tag{8}$$

**Lemma 5.9.** *We have (for $2\le u\le x$, $j=1,3$)*

$$S\Big(\frac{a_j}{q_j},u\Big)=O\Big(((q_jx)^{1/2}+x^{5/7}q_j^{3/14}+xq_j^{-1/2})(\log x)^{17}\Big), \tag{9}$$

*and so since* $|a_j q_j^{-1}| \ll A^2(\log x)^2$,

$$S_j = S_j^* + O(x^2(q_jQ)^{-1})$$
$$= O\Big(x^2(q_jQ)^{-1} + A^2(\log x)^{k_0+20}((q_jx)^{1/2} + q_j^{3/14}x^{5/7} + q_j^{-1/2}x)\Big). \qquad (10)$$

*Proof.* The equation (9) is precisely Theorem 16.1 (on p. 141) of H.L. MONTGOMERY (see [HLM]). □

Collecting we have

**Lemma 5.10.** *Let*

$$A^2 q_0^{2t}(\log x)^2 = o(q_0) \text{and} (q_0^{-1+2t} + Q^{-1}q_0^{1+t}A^2)A^2(\log x)^4 = o(1). \qquad (11)$$

*Then choosing that* $j$ *for which* $q_j \geq q_0^t$, *we have*

$$S_j = O\Big(x^2(q_0^tQ)^{-1} + ((Qx)^{1/2} + Q^{3/14}x^{5/7} + q_0^{-t/2}x)A^2(\log x)^{k_0+20}\Big). \qquad (12)$$

We have still to choose $x$ in terms of $q_0$ and relate $Q$ to $q_0$. In this direction we have the following lemma.

**Lemma 5.11.** Let $1 \leq A \leq \operatorname{Exp}(100\sqrt{\log x})$, $1 \leq C \leq \operatorname{Exp}(100\sqrt{\log x})$, $t = \frac{1}{16}$, $x = q_0^2$, $Q \geq q_0^{2-(1/100)}$. Then the conditions (11) are all satisfied and we have

$$\int_{(\alpha)_1} |S_1S_2S_3|\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha + \int_{(\alpha)_3} |S_1S_2S_3|\Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha$$
$$\ll \Big\{\Big(\frac{x}{Qq_0^t}\Big) + \Big(\Big(\frac{Q}{x}\Big)^{1/2} + \Big(\frac{Q^3}{x^4}\Big)^{1/14} + q_0^{-t/2}\Big)A^2(\log x)^{k_0+20}\Big\}x^2C^{-1/2}\log x$$
$$\ll x^2C^{-1/2}\Big\{\frac{q_0^{2-t}}{Q} + \Big(\Big(\frac{Q}{q_0^2}\Big)^{1/2} + \Big(\frac{Q}{x}\Big)^{3/14}x^{-1/14} + q_0^{-t/2}\Big)A^2(\log x)^{k_0+21}\Big\}(\log x).$$

Here $(\alpha)_1$ and $(\alpha)_3$ denote respectively those $\alpha$ for which $q_1 \geq q_0^t$ and those for which $q_3 \geq q_0^t$.

## 6. Conclusion

Thus collecting Lemmas 3.8, 4.2, 5.3 and 5.11 we have the following theorem.

**Theorem 3.** *Let* $x = q_0^2$, $1 \le A \le \mathrm{Exp}(100\sqrt{\log x})$, $1 \le C \le \mathrm{Exp}(100\sqrt{\log x})$, $t = \frac{1}{16}$, $Q \ge q_0^{2-(1/100)}$. *Then we have*

$$\int_{-\infty}^{\infty} S_1 S_2 S_3 \Big(\frac{\sin(\pi\alpha)}{\pi\alpha}\Big)^2 d\alpha \ge x^2 A^{-1} C^{-1} \Lambda$$

*where*

$$\begin{aligned}\Lambda = {} & K_1 - K_2 A^{-4} C - K_2 (\log x)^{-1} - K_2 A C (E(x))^{-1/4} \\ & - K_2 A^{-1} C^{1/2} (\log x) - K_2 A C^{1/2} \Big(1 + \frac{q_0^2}{Q}\Big)(E(x))^{-1} \log x \\ & - K_2 A^3 C^{1/2} \Big\{ \frac{q_0^{2-t}}{Q} + \Big(\frac{Q}{q_0^2}\Big)^{1/2} + \Big(\frac{Q}{q_0^2}\Big)^{3/14} q_0^{-1/7} + q_0^{-t/2} \Big\} (\log x)^{k_0+22}\end{aligned}$$

*where* $K_1 > 0$ *and* $K_2 > 0$ *are constants depending only on* $\lambda_1$, $\lambda_2$ *and* $\lambda_3$.

**Remark.** The choice $A = C = \mathrm{Exp}(10\sqrt{\log x})$ and $Q = q_0^2 (E(x))^{-1}$ proves Theorem 2.


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