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To cite this version:
Shanta Laishram. Baker’s Explicit abc-Conjecture and Waring’s problem. Hardy-Ramanujan Journal, Hardy-Ramanujan Society, 2015, 38. hal-01253665
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Shanta Laishram

Abstract. The conjecture of Masser-Oesterlé, popularly known as abc-conjecture has many consequences. We show that Waring’s problem is a consequence of an explicit version of abc—conjecture due to Baker.

Keywords. ABC Conjecture, Waring’s problem.

2010 Mathematics Subject Classification. 11D41, 11D75, 11E76, 11P05

1. Introduction

For any positive integer \( i > 1 \), let \( N = N(i) = \prod_{p| i} p \) be the radical of \( i \), \( P(i) \) be the greatest prime factor of \( i \) and \( \omega(i) \) be the number of distinct prime factors of \( i \) and we put \( N(1) = 1, P(1) = 1 \) and \( \omega(1) = 0 \). The well known conjecture of Masser-Oesterlé states that

**Conjecture 1.1. abc-conjecture of Masser and Oesterlé:** For any given \( \epsilon > 0 \) there exists a computable constant \( c_\epsilon \) depending only on \( \epsilon \) such that if

\[
a + b = c
\]

where \( a, b \) and \( c \) are coprime positive integers, then

\[
c \leq c_\epsilon \left( \prod_{p| abc} p \right)^{1+\epsilon}.
\]

This is popularly known as abc—conjecture. The abc—conjecture has already become well known for the number of interesting consequences it entails. Many famous conjectures and theorems in number theory would follow immediately from the abc—conjecture. An explicit version of this conjecture due to Baker [Bak94] is the following:

**Conjecture 1.2. Explicit abc—conjecture:** Let \( a, b \) and \( c \) be pairwise coprime positive integers satisfying (1.1). Then

\[
c < \frac{6}{5} N \frac{(\log N)^\omega}{\omega!}
\]

where \( N = N(abc) \) and \( \omega = \omega(N) \).

We observe that \( N = N(abc) \geq 2 \) whenever \( a, b, c \) satisfy (1.1). We shall refer to Conjecture 1. as abc—conjecture and Conjecture 1. as explicit abc—conjecture. We have

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Theorem 1. (Laishram and Shorey [LaSh12] ) Assume Conjecture 1. Let $a, b$ and $c$ be pairwise coprime positive integers satisfying (1.1) and $N = N(abc)$. Then we have

$$c < N^{1 + \frac{3}{4}}.$$  

Further for $0 < \epsilon \leq \frac{3}{4}$, there exists $\omega_\epsilon$ depending only $\epsilon$ such that when $N = N(abc) \geq N_\epsilon = \prod_{p \leq \omega_\epsilon} p$, we have

$$c < \kappa_\epsilon N^{1 + \epsilon}$$

where

$$\kappa_\epsilon = \frac{6}{5\sqrt{2\pi \max(\omega, \omega_\epsilon)}} \leq \frac{6}{5\sqrt{2\pi \omega_\epsilon}}$$

with $\omega = \omega(N)$. Here are some values of $\epsilon, \omega_\epsilon$ and $N_\epsilon$.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\frac{3}{4}$</th>
<th>$\frac{7}{12}$</th>
<th>$\frac{5}{8}$</th>
<th>$\frac{3}{4}$</th>
<th>$\frac{5}{8}$</th>
<th>$\frac{1}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_\epsilon$</td>
<td>14</td>
<td>$\omega_\epsilon$</td>
<td>127</td>
<td>175</td>
<td>548</td>
<td>6460</td>
</tr>
<tr>
<td>$N_\epsilon$</td>
<td>$e^{0.1101}$</td>
<td>$e^{0.75}$</td>
<td>$e^{335.71}$</td>
<td>$e^{679.585}$</td>
<td>$e^{1004.763}$</td>
<td>$e^{3894.57}$</td>
</tr>
</tbody>
</table>

Thus $c < N^{2}$ which was conjectured in Granville and Tucker [GrTu02].

2. Ideal Waring’s Conjecture

For each integer $k \geq 2$, denote by $g(k)$ the smallest integer $g$ such that any positive integer is the sum of at most $g$ integers of the form $x^k$. A result of Euler implies that a lower bound for $g(k)$ is $2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2$. The so-called Ideal Waring’s Conjecture is the following conjecture, dating back to 1853:

Conjecture 2.1. For any $k \geq 2$, the equality $g(k) = 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2$ holds.

Theorem 2. Assume Conjecture 1. Then Conjecture 2. is true.

Conjecture 2. has a long and interesting history. We refer to Waldschmidt [Mic00, pp 12] for further details. We prove Theorem 2. in the next section.

3. Proof of Theorem 2.

We write

$$3^k = 2^k q + r$$

with $0 < r < 2^k$ and $q = \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor$.

L. E. Dickson and S.S. Pillai (see for instance [HaWr54, Chap. XXI] or [Nar86, p. 226 Chap. IV]) proved independently in 1939 that the ideal Waring’s Conjecture(Conjecture 2.) holds provided that the remainder $r = 3^k - 2^k q$ satisfies

$$r \leq 2^k - q - 3.$$  

(3.3)

The condition (3.3) is satisfied for $3 \leq k \leq 471600000$ as well as for sufficiently large $k$, as shown by K. Mahler [Mah57] in 1957 by means of Ridout’s extension of the Thue-Siegel-Roth theorem.
Therefore we may now suppose that \( k > 471600000 \) and further (3.3) does not hold, i.e.,

\[
r \geq 2^k - q - 2
\]

(3.4)

Let \( \gcd(3^k, 2^k(q + 1)) = 3^v \) and set

\[
a = 3^{k-v}, c = 3^{-v}2^k(q + 1) \text{ and } b = c - a = 3^{-v}(2^k - r).
\]

Then \( a, b, c \) are relatively prime positive integers satisfying \( a + b = c \) and

\[
b = 3^{-v}(2^k - r) \leq 3^{-v}(q + 3)
\]

by (3.4). Then

\[
N = N(abc) = N \left( \frac{3^{k-v} \cdot 2^k(q + 1)}{3^v} \cdot b \right) \leq \frac{6b(q + 1)}{3^v} \leq \frac{6(q + 1)(q + 3)}{3^{2v}}.
\]

(3.5)

First assume that \( N < e^{63727} \). Then by (1.2), we have

\[
2^k \leq \frac{2^k(q + 1)}{3^v} < N^{\frac{7}{12}} < e^{63727^{\frac{7}{4}}}
\]

implying

\[
k < 63727 \cdot 7^{\frac{7}{4}} \cdot \log 2 < 160893.
\]

This is a contradiction since \( k > 471600000 \). Therefore we may suppose that \( N \geq e^{63727} \). By Theorem 1. with \( \epsilon = \frac{1}{3} \) and (3.5), we have

\[
\frac{2^k(q + 1)}{3^v} < \frac{6}{5\sqrt{2\pi} \cdot 6460} \left( \frac{6(q + 1)(q + 3)}{3^{2v}} \right)^{\frac{4}{3}}.
\]

implying

\[
2^k < \frac{6\sqrt{2\pi}}{5\sqrt{12920\pi}} q^{\frac{3}{2}} (1 + \frac{3}{q})^{\frac{5}{4}}.
\]

Since \( 3^k > 2^kq \), we have \( q < \left( \frac{3}{2} \right)^k \). Also \( 1 + \frac{3}{q} < 2 \) since \( k \geq 3 \). Therefore

\[
2^k < \frac{6\sqrt{2\pi}}{5\sqrt{12920\pi}} \left( \frac{3}{2} \right)^{\frac{6k}{3}} < \left( \frac{3}{2} \right)^{\frac{k}{3}} < 2^k.
\]

This is a contradiction. Hence the assertion. \( \Box \)

**Acknowledgments**

The author would like to thank Michel Waldschmidt for pointing out his paper [Mic00, p. 12] to the author which led to this paper. We would like to also thank the referee for careful reading and suggestions and remarks on an earlier draft of this paper.
References


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