DISPROOF OF SOME CONJECTURES OF K.RAMACHANDRA

BY

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1. INTRODUCTION

In a recent paper [9] K.Ramachandra states some conjectures, and gives consequences in the theory of the Riemann zeta function. In this paper I will will present two different disproofs of them. The first will be an elementary application of the Szasz-Müntz theorem. The second will depend on a version of the Voronin universality theorem, and is also slightly stronger in the sense that it disprove a weaker conjecture. An elementary(but more complicated) disproof has been given by Rusza-Lazkovich [11].

2. DISPROOF OF SOME CONJECTURES

2.1 The conjectures. I will first state the three conjectures as given by Ramachandra [9], and Ramachandra-Balasubramanian [5].

Conjecture 1. For all $N \geq H \geq 1000$ and all $N-$tuples $a_1 = 1, a_2, \ldots, a_N$ of complex numbers we have

$$\frac{1}{H}\int_0^H \left|\sum_{n=1}^N a_n n^{it}\right| dt \geq 10^{-1000}$$

*

Conjecture 2. For all $N \geq H \geq 1000$ and all $N-$tuples $a_1 = 1, a_2, \ldots, a_N$ of complex numbers we have when $M = H(\log H)^{-2}$

$$\frac{1}{H}\int_0^H \left|\sum_{n=1}^N a_n n^{it}\right|^2 dt \geq (\log H)^{-1000} \sum_{n=1}^M |a_n|^2$$

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Conjecture 3. There exist a constant $c > 0$ such that

$$\int_0^T \left| \sum_{n=1}^N a_n n^{it} \right|^2 dt \geq c \sum_{n \leq cT} |a_n|^2$$

2.2 The Szász-Müntz theorem To disprove these conjectures, we first consider the following classic result of Szász

Lemma 1. (Szász) If we have that

$$\sum_{n=1}^{\infty} \frac{1 + 2Re(\lambda_n)}{1 + |\lambda_n|^2} = +\infty$$

where $\Re(\lambda_n) \geq 0$ then the set of finite linear combinations of x^{λ_n} is dense in $L^2(0,1)$

Proof See Szász [12], theorem A.

I will now state a theorem that will effectively disprove the above conjectures:

Theorem 1. For each $D \geq 0$ and $\varepsilon > 0$ there exists an $N \geq 0$ and complex numbers $a_2, \ldots, a_N$, such that

$$\int_0^D \left| 1 + \sum_{n=2}^N a_n n^{it} \right|^2 dt \leq \varepsilon$$

Proof Since $-1 \in L^2(0,1)$ and

$$\sum_{n=2}^{\infty} \frac{1 + 2\Re(-i \log n)}{1 + |-i \log n|^2} = +\infty$$

we have by the lemma that for each $\delta > 0$ there exists an $N > 0$ and complex numbers $a_2, \ldots, a_N$ such that

$$\int_0^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx < \delta$$

We obtain

$$\delta > \int_0^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx \geq \int_{e^{-D}}^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx =$$

$$= (\text{Substituting } t = -\log x) =$$

$$\int_0^D e^{-t}\left|1+\sum_{n=2}^N a_n n^{it}\right|^2 dt \geq e^{-D}\int_0^D \left|1+\sum_{n=2}^N a_n n^{it}\right|^2 dt$$

By choosing $\delta = e^{-D}\varepsilon$ we obtain the theorem.

It is now an easy task to falsify the conjectures.

Proposition. *Conjectures 1, 2 and 3 are false.*

Proof. For conjecture 1 , choose $\varepsilon < 10^{-2000}H$ and $D = H$ in Theorem 1 and apply the Cauchy-Schwarz inequality

$$\int_0^H \left|\sum_{n=1}^N a_n n^{it}\right| dt \leq \sqrt{H}\sqrt{\int_0^H \left|\sum_{n=1}^N a_n n^{it}\right|^2 dt}$$

For conjecture 2, choose $\varepsilon < (\log H)^{-1000}H$, and $D = H$ in Theorem. To disprove Conjecture 3, chose e.g. $a_1 = T = 1$, and $\epsilon = c/2$.

2.3. The Voronin Theorem. In private correspondence, Ramachandra asked whether the conjectures hold under the additional growth assumption $|a_k| \ll (Hk)^{100}$. Ramachandra and Balasubramanian have proved conjectures 1 and 2 under the this and the additional further assumption that $N < exp(exp(cH))$. This shows that N must be very large compared to H for the conjectures to be false. However, they are in fact still false, although their proof requires a deeper result. A version of the Voronin universality theorem for the Riemann zeta-function. We will state the theorem that shows that the conjectures are still false below:

Theorem 2. *Suppose that $H, \varepsilon > 0$, $0 < \delta < \frac{1}{2}$. Then there exists $|a_k| \leq k^{\delta-1}$ such that*

$$\max_{t\in[0,H]}\left|1+\sum_{n=2}^N a_n n^{it}\right| dt < \varepsilon$$

The idea goes as follows: We use the following version of the Voronin universality theorem

Theorem 3. *(Voronin-Bagchi) For any compact subset K of the complex numbers such that $x \in K \implies \frac{1}{2} < Re(x) < 1$, non vanishing analytic function f on K and $\varepsilon > 0$, we have a real t such that $|\zeta(z+it) - f(z)| \leq \epsilon$, for all $z \in K$*

Proof. See Bagchi [1].

and the following version of the approximate functional equation for the Riemann zeta-function

$$\left|\zeta(\sigma+it)-\sum_{k=1}^{N}k^{-\sigma-it}\right|\leq CN^{\sigma-1},\text{ for e.g. } t<N<2t,\sigma\geq\frac{1}{2}$$

Proof (of Theorem 2) Choose T_0 so that when $T>T_0$ then

$$\left|\zeta(1-\delta+it)-\sum_{k<2T}k^{\delta-1-it}\right|<\frac{\varepsilon}{3}$$

when $T\leq t\leq T+H$. Now chose $T>T_0$ so that

$$|\zeta(1-\delta+it)|<\frac{\varepsilon}{3}$$

for $T\leq t\leq T+H$ (This is possible by applying Theorem 3 to $f(z)=\varepsilon/3$, and $K=[1-\delta,1-\delta+iH]$). Using the triangle inequality we get

$$\left|\sum_{k<2T}k^{\delta-1-it}\right|<\varepsilon$$

for all $T<t<T+H$. Now we can chose $N=T$ and $a_n=n^{\delta-1-iT}$ in the theorem.

3. Summary

In [8] K. Ramachandra stated a similar conjecture to conjectures 1 and 2. Although it was more general in the sense that it considered Dirichlet series of form $A(s)=1+\sum a_n\lambda_n^{-s}$, for certain λ_n generalizing $\lambda_n=n$ it was much weaker as it had three further conditions. H depended on N, $|a_n|$ were bounded from above and A was bounded in certain regions in the complex plane. Under these additional assumptions, $A(s)$ is called a Titchmarsh-series and for these, analogues of conjectures 1 and 2, and similar conjectures were proved in Ramachandra [7], and Balasubramanian and Ramachandra [2], [4] and [3]. It would certainly be interesting to see if all these additional assumptions are needed, or if a certain subset of them implies the truth of conjectures 1 and 2. For a disproof of the $L^p(0, D)$, $p > 2$ version of conjectures 1 and 2, or for considering much more sparse sequences than $\log n$ in (1), we need a stronger version of the Lemma, the reader is referred to the literature on the theory of completeness of complex exponentials, see e.g. the classic by Levinson [6] or Redheffer [10] for a more recent survey. It should be noticed that problems similar to conjectures 1 and 2 are also studied in Turán power sum theory (although there it is essentially the λ_n's which vary, instead of the a_n's), see Turán [13] for a thorough treatment.

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