NOTES ON THE RIEMANN ZETA-FUNCTION-III

BY

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Abstract: For a good Dirichlet series (see definition in §1) $F(s)$ which is a quotient of some products of the translates of the Riemann zeta-function, we prove that there are infinitely many poles $p_1 + ip_2$ in $\Im(s) > C$ for every fixed $C > 0$. Also we study the gaps between the ordinates of the consecutive poles of $F(s)$.

§1. **INTRODUCTION.** This is a continuation of the earlier paper -II [KR, AS] with the same title (where the stress in the first five sections was on "Explicit formula" and therefore the conditions on $F(s)$ therein were very restrictive namely the condition $\sum^* < 1$) and has relavence to §6 of that paper. It includes all the results of §6 there (even without assuming RH etc etc) and includes many more functions $F(s)$ as will be seen. At the same time the result of the present paper will be sharper than those of §6 of our earlier paper mentioned above. We begin with a definition.

DEFINITION. A Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad (s = \sigma + it)$$

is said to be good if it converges in $\sigma \geq C_0$ (for some positive constant $C_0 \geq 100$) and there exist positive constants C_1, C_2, C_3 such that for all $x \geq C_1$ there holds

$$\max_{x \leq n \leq x^{C_2}} |a_n| > x^{-C_3}$$

REMARK 1. Our method allows us to treat the case where $a_n's$ and C_0 may depend on a parameter $T(\geq 10)$ (see remark 5 below theorem 1). But we do not carry out such details.

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REMARK 2. A Dirichlet series being good is equivalent to $F(s)$ satisfying the hypothesis §6 of our earlier paper.

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We begin by stating a theorem which follows from the general result whose proof will be explained in §3 onwards. (Our method consists of estimates on 'Ingham lines' and then applying maximum modulus principle with a suitable kernel twice and then applying a theorem of R.Balasubramanian and K.Ramachandra.)

THEOREM 1. *Let N and D be two finite sets of complex numbers defined by*

$$N \equiv \{\alpha_1, \alpha_2, ..., \alpha_k\}, \quad D \equiv \{\beta_1, \beta_2, ..., \beta_l\}$$

where $0 \leq k \leq l,\ l \geq 1$, *(we stress here that neither the α's nor β's need be distinct). Let $P(s)$ be any fixed Dirichlet polynomial (ie a terminating Dirichlet series). Let $F(s)$ be defined by*

$$F(s) \equiv \left(P(s) \prod_{\alpha \epsilon N} \zeta(s+\alpha)\right) \left(\prod_{\beta \epsilon D} \zeta(s+\beta)\right)^{-1} \tag{10}$$

$$\tag{11}$$

$$\equiv F_1(s)(F_2(s))^{-1}$$

say (in an obvious sense; empty product being defined as 1*) be a good Dirichlet series. Then it has infinitely many poles $p_1 + ip_2$ in $\Im(s) \geq C$ for every fixed $C > 0$. Also for every $Y \geq 1000$ there exists a $p_2 = p_2(Y)$ which lies in the interval*

$$Y, Y + a \log\ Y$$

where $a > 0$ is a constant. Further more the result is still true if we replace $F(s)$ by any finite sum of functions of the type $F(s)$ (for various $P(s), N$ and D) provided that such a sum is a good Dirichlet series (though the individual terms of the sum need not be).

REMARK 1. In our theorem some (or all) of $\zeta(s+\alpha)$ can be replaced by their derivatives of bounded order.

REMARK 2. Here some (or all) of $\zeta(s+\beta)$ can be replaced by the corresponding ordinary L-functions. At the same time some (or all) of $\zeta(s+\alpha)$ can be replaced by derivatives of bounded order of $L(s+\alpha)$ for ordinary L-functions or those of $\sum_{n=1}^{\infty}(a_1 n + b)^{-s}$ where a_1 and b are positive integers.

REMARK 3. Here some (or all) of $\zeta(s+\beta)$ can be replaced by $L_K^*(s+\beta)$ where L_K^* is any L-function of a number field K of degree $n(K)$. At the same time some (or all) of $L_{K^*}^*(s+\alpha)$ or $\zeta_{K^*}^*(s+\alpha)$ (where $\zeta_{K^*}^*$ are zeta-functions of ray classes of any number field K^* of degree $n(K^*)$) or derivatives of these functions of any bounded order. The only condition that is necessary is $\sum n(K^*) \leq \sum n(K)$ (in place of the condition $0 \leq k \leq l$). More general results of this type will be treated in the paper IV [RB, KR, AS, KS] mentioned above.

REMARK 4. The constants a and C are effective. They depend on other constants involved namely $\alpha's,\ \beta's,\ P(s),\ C_1,\ C_2,\ C_3$ and C_0.

REMARK 5. In all these problems mentioned above we may treat the localisation where we restrict t by $T \leq\ t\ \leq\ 2T$(or to a sub-interval of this) and make $\alpha's$ and $\beta's$ depend on the parameter T.

REMARK 6. The results of this paper are self contained except where we refer to the works whose references will be given explicitly.

§2.**NOTATION**. The notation is standard and we follow the same notation as in the earlier paper mentioned in the begining of the introduction.

§3. **STATEMENT OF THE GENERAL RESULT.** In what follows T will exceed a large positive constant. The letters ϕ, ψ and H will denote positive functions of T bounded below by large positive constants. They are assumed to satisfy

$$H = o(T)\ \ ,\ \ \log\log\phi = O(H)$$

and

$$\log\psi = O(H)$$

where the two $O-$ constants are assumed to be sufficiently small.

(A) Let $F_1(s)$ and $F_2(s)(s = \sigma + it)$ be two Dirichlet series (which may depend on a parameter T and we consider only the interval $T - H \leq t \leq T + H$) convergent absolutely in $\sigma \geq C_0(\geq\ 100)$ and bounded there. The letter $g > 0$ will denote a large absolute constant and we assume that $F_1(s)$ and $F_2(s)$ can be continued analytically in $\sigma \geq -G$, where $G = g(\log\psi)(\log\ \log\psi)^{-1}$. Also we assume that $F_2(\sigma) \to 1$ as $\sigma \to \infty$ and that in $\sigma \geq C_0$ the function $\log F_2(s)$ is bounded.

(B) Let

$$|F_2(s)| < \phi^g\ \ \text{in}\sigma \geq -20g.$$

(C) Let $F_2(s) \neq 0$ in $-g \geq \sigma \geq -G$ and also in the same region we have

$$|F_1(s)(F_2(s))^{-1}| \leq C_4^{\sigma}\psi, (C_4 > 0),$$

where C_4 is a constnat. For convenience we assume that the constant C_4 is bounded below and also above.

(D) Let $|F_1(s)| \leq \exp((g^2 \log \phi)^3)$ in $\sigma \geq -g$. Under these conditions

THEOREM 2. *We have a pole $p_1 + ip_2$ with*

$$T - H \leq p_2 \leq T + H,$$

provided $F(s) \equiv F_1(s)(F_2(s))^{-1}$ is good.

REMARK 1. We can treat also the case where the coefficients of the Dirichlet series for $F(s)$ depend on T and C_0 and the upper bound for $|F(s)|$ in $\sigma \geq C_0$ depend on T. For simplicity we do not carry out these investigations.

REMARK 2. The deduction of theorem 1 (and the remarks below it) from theorem 2 is not difficult and will be left as an exercise to the reader.

The proof will be split up into a few sections for convenience. Whenever it may be necessary we will give estimates which are uniform in g. Also we assume that for some H satisfying the above conditions ,the theorem is false and arrive at a contradition.

§4. STEP I (INGHAM LINES).

LEMMA 4.1. *The number of zeros of $F_2(s)$ in any $t-$ interval of length 1, contained in $\{\sigma \geq -g,\ T-H \leq t \leq\ T+H\}$ is $O(g\ \log\ \phi)$ provided we exclude $O(g)$ number of $t-$intervals of unit length at each extremity.*

PROOF. The proof is essentially the same as that in [RB, KR] page 285 (except for notation).

LEMMA 4.2. *In any unit t-interval (refered to in Lemma 4.1)there exists a line ("Ingham line") $t = t_0$ which is seperated away from all the zeros of $F_2(s)$ in the unit t-interval by an amount greater than or equal to a positive constant times $(g\ \ \log \phi)^{-1}$.*

PROOF. The proof follows from lemma 4.1 by applying pigeon-hole principle.

LEMMA 4.3. *If $t = t_0$ is any Ingham line for $F_2(s)$ (in any unit interval) then*

$$\left|\frac{F_2'(\sigma + it)}{F_2(\sigma + it)}\right| = O(g^2(\log\ \ g)(\log \phi)^2),$$

uniformly in $\sigma \geq -g$.

PROOF. The proof is essentially the same as that in [RB, KR] pages 285 -286 (except for notation).

LEMMA 4.4. *If $t = t_0$ is any Ingham line for $F_2(s)$ (in any unit t-interval) then*

$$|\log F_2(\sigma + it_0)| \leq O(g^3(\log g)(\log \phi)^2)$$

and so

$$|F_2(\sigma + it_0)|^{-1} \leq \mathrm{Exp}(O(g^3(\log g)(\log \phi)^2)),$$

and hence $|F(\sigma + it_0)| \leq Exp(g^{10}(\log \phi)^3)$, uniformly in $\sigma \geq -g$.

PROOF. Follows from lemma 4.3. on noting our upper bound assumption for $|F_1(s)|$.

REMARK 1. We take this opportunity to say that (by oversight) the condition $a_2 \neq 0$ (essential in theorem 4 of [RB, KR]) is omitted there. However this condition does not affect the results of the present paper.

REMARK 2. So far we have not assumed the falsity of theorem 2.

¿From now on we do assume the falsity.

§5. **STEP II (MAXIMUM MODULUS PRINCIPLE WITH A DECAYING KERNEL).** The main object of this section is to prove that if we omit t-intervals of length $O(\log \log \phi)$ on both extremities of $(T - H, T + H)$ then we have the estimate

$$F(s) = O(\psi^{\frac{3}{2}})$$

valid uniformly in $\sigma \geq -g$ (and hence by our assumption on $F(s)$, in $-g \geq \sigma \geq -G$).

LEMMA 5.1. *Let $z = x + iy$ be a complex variable with $|x| \leq 1/4$. Then we have,* (a).

$$|\exp((\sin z)^2)| \leq 2 \quad \text{for all} \quad y$$

and

(b). *If $|y| \geq 2$, then*

$$|\exp((\sin z)^2)| \leq 2(\exp \exp |y|)^{-1}.$$

PROOF. See lemma 2.1 on page 38 of [K.R].

LEMMA 5.2. *Let $H_1 = H - C_5 g$ and $H_2 = H_1 - C_6 g \log \log \phi$, where $C_5 > 0$ and $C_6 > 0$ are constants. Let $s_1 = \sigma_1 + it_1$ where $-g \leq \sigma_1 \leq C_0 + 10$ and t_1 lies between $T - H_2$ and $T + H_2$. Then uniformly we have*

$$|F(s_1)| = O(\psi^{\frac{3}{2}}).$$

PROOF. Consider the function

$$F(s)\ \operatorname{Exp}((\sin\frac{s-s_1}{10g})^2).$$

Apply maximum modulus principle to the rectangle bounded by the vertical lines $\sigma = -g, \sigma = C_0 + 10$ and the horizontal lines which are Ingham lines just below $T + H_1$ and just above $T - H_1$. Note that on the left vertical line

$$|F(s)| = O(\psi)$$

and on the right vertical line $|F(s)| = O(1)$. On the horizontal lines choose C_6 so large that the factor multiplying $F(s)$ makes the product very small. Thus $|F(s_1)|$ is as asserted. This leads to Lemma 5.2.

LEMMA 5.3. *Let $C_7 > 0$ be a very large constant, $H_3 = H_2 - gC_7 \log\psi,\ s_2 = -A + it_2$ where $A > 0$ is a large constant and t_2 lies in $(T - H_3,\ T + H_3)$. Then*

$$|F(s_1)| \leq \psi^{C_8 A G^{-1}}$$

where $C_8 > 0$ is a constant independent of g.

PROOF Note that by our assumption on $|F(s)|$ we have (by lemma 5.2), that in the region

$$\sigma \geq -G,\ T - H_3 \leq t \leq T + H_3$$

the estimate $|F(s)| \leq \psi^2$. We now apply maximum modulus principle to the function

$$Y(s) = F(s)X^{s-s_2}\ Exp((sin\frac{s-s_2}{100G})^2).$$

Accordingly $|Y(s_2)| = |F(s_2)|$ does not exceed the maximum modulus of $|Y(s)|$ on the boundary of the rectangle bounded by the lines

$$\sigma = -G, \sigma = C_0, t = t_2 + C_7 g \log\psi, t = t_2 - C_7 g \log\psi$$

We choose X by

$$\psi^2 X^{-G+A} = C_9 X^{C_0+A} \text{ ie by } \mathrm{X} = (\mathrm{C}_9^{-1}\psi^2)^{(\mathrm{G}+\mathrm{C}_0)^{-1}},$$

where $C_9 \geq 1$ is a constant independent of g.

On the horizontal lines the contribution from $F(s)X^{s-s_2}$ is bounded by a constant (independent of g) power of ψ. Now the decaying factor decays like

$$\left(\operatorname{Exp}\operatorname{Exp}\frac{|t-t_2|}{100G}\right)^{-1}$$

and so by choosing C_7 large the contribution of $|Y(s)|$ on the horizontal lines is negligible. The vertical portions contribute a quantity

$$\leq 2C_9 X^{C_0+A} \leq C_{10}\psi^{2C_{11}AG^{-1}} \leq \psi^{C_8 AG^{-1}}$$

This proves the lemma.

§6. **STEP III (APPLICATION OF BALASUBRAMANIAN-RAMACHANDRA THEOREM).** We now quote the corollary on page 45 of [KR] and explain the notation involved therein.

BALASUBRAMANIAN-RAMACHANDRA THEOREM. *Let* A *and* C *be as in the introduction* §2.1, $0 < \epsilon \leq \frac{1}{2}$, $r \geq [(200A+200)\epsilon^{-1}]$, $|a_n| \leq n^A H^{\frac{r\epsilon}{8}}$. *Then* $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ *is analytic in* $\sigma \geq A+2$. *Let* $K \geq 30, U_1 = H^{1-\frac{\epsilon}{2}}$. *Assume that* $K_1 = (HC)^{12A}K, H \geq (120(A+2)^2 C^{2A+4}(4rC^2)^r)^{\frac{100}{\epsilon}} + (100r(A+2))^{20} \log\log K_1$, *and that there exist* T_1, T_2 *with* $0 \leq T_1 \leq U_1, H - U_1 \leq T_2 \leq H$ *such that uniformly in* $\sigma \geq 0$ *we have*

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K,$$

where $F(s)$ *is assumed to be analytically continuable in* $(\sigma \geq 0, 0 \leq t \leq H)$. *Then*

$$\frac{1}{H}\int_0^H |F(it)|^2 dt \geq (1 - 10rC^2 H^{-\frac{\epsilon}{4}} - 100BH^{-1}\log\log K_1) \sum_{n \leq H^{1-\epsilon}} |a_n|^2.$$

REMARKS. We need only the special case $\lambda_n = n$ and for this choice $C = 1, B = A+2$. Also we can fix ϵ to be $\frac{1}{2}$ (We could have applied instead of this theorem the theorem on page 52 (due to the same two authors) of the same reference. We wish to say that on page 52 the quantity $(4C)^{9000A^2}$ could be replaced by $(4CA)^{90000000A}$).

In the present situation we (get back to our notation and) wish to apply this theorem to get a lower bound for the mean value

$$M = \frac{1}{2H_4}\int_{T-H_4}^{T+H_4} |F(-A+it)|^2 \, dt$$

where $H_4 = \frac{1}{20}H_3$. We need analytic continuation of $F(s)$ in $\sigma \geq -A$. For T_1 and T_2 (of the theorem above) we choose the lines $t = T + H_4$ and $t = T - H_4$. The condition on H_4 required is

$$H_4 \geq A^{D_1 A} + 1000000A^{40} \log\ \log(H_4^{12A}\,\mathrm{Exp}(O(g^2 \log\psi))$$

$$i.e \log H_4 \geq 4000 D_1 A \log\ A$$

where D_1 is a certain (absolute) positive constant. Under this condition the lower bound for M given by the theorem reads

$$M \geq D_2 \sum_{n \leq H^{\frac{1}{2}}} |a_n|^2 \; n^{2A},$$

where $D_2 > 0$ is a certain (absolute) constant. We now use the fact that $F(s)$ is a good Dirichlet series. This gives us an n for which $|a_n| \geq n^{-C_7}$ where $C_7 > 0$) is independent of A.

Thus

$$M \geq H^{-D_3 + D_4 A}$$

for suitable constants $D_3 > 0, D_4 > 0$ independent of A. Thus if $\log H \geq D_5 A \log A$ we have $M \geq \mathrm{Exp}(D_6 A \log H)$ and so

$$\max_{T-H_4 \leq t \leq T+H_4} |F(-A+it)| \geq \mathrm{Exp}(\frac{1}{2} D_6 A \log H)$$

where $D_5 > 0,\; D_6 > 0$ are independent of A.

§7.**STEP IV (FINAL STEP)** Thus combining this with Lemma 5.3, we have

$$Exp(\frac{1}{2} D_6 A \log H) \leq Exp(C_8 A G^{-1} \log \psi) = \mathrm{Exp}(C_8 A g^{-1} \log\log \psi) \leq (\log \psi)^{\frac{1}{2} D_6 A}$$

by choosing g large enough. i.e. $H \leq \log \psi$. Hence if H is greater than $a(\log \psi + \log\log \phi)$ for a suitable constant $a > 0$, we are lead to a contradiction. This proves the theorem 2 completely.

§8.**CONDLUDING REMARKS**. In a forthcoming paper [RB, KR, AS, KS] IV with the same title we discuss more general applications of theorem 2. The utmost we can do seems to be to take

$$F_1(s) = \sum \; P(s) f_1(s) f_2(s) ... f_k(s)$$

and

$$F_2(s) = \sum \; Q(s) g_1(s) g_2(s) ... g_l(s)$$

where f_j and g_j are zeta-functions of ray classes (or any derivatives of bounded orders of these functions) in any algebraic number field. If $n(f)$ denotes the degree of the number field associated with the zeta function f we need some condition like

$$0 < \max \sum_{j \leq k} n(f_j) \leq \max \sum_{j \leq l} n(g_j)$$

we need also the condition that the limit of $F_2(\sigma)$ as $\sigma \to \infty$ is 1. (Here $P(s)$ and $Q(s)$ are fixed Dirichlet polynomials). Also the variable s appearing of various places in the definition of $F(s)$ may be replaced by $s+\alpha$ where $\alpha's$ are complex constants. Also we wish to derive handy (sufficient) conditions for $F(s)$ to be a good Dirichlet series in a suitable right half plane. Before leaving this section we mention a special cae of a result from [RB, KR, AS, KS] IV.

THEOREM 3. *Let $\alpha_1, \alpha_2 ..., \alpha_k, \beta_1, \beta_2, ..., \beta_l$ with $(1 \leq k \leq l)$ be any purely imaginary constants.*

Assume that $s = 1 - \alpha_1$ is a pole of

$$F(s) = P(s) \frac{\prod \zeta(s+\alpha_j)}{\prod \zeta(s+\beta_j)},$$

$P(s)$ being any fixed Dirichlet polynomial. Then $F(s)$ has infinitely many poles in $t \geq C$ for every fixed $C > 0$. Moreover for every $Y > 100$ there is a pole $p = p(Y) = p_1 + ip_2$ of $F(s)$ with

$$|p_2 - Y| \leq \ d \log\ \log\ Y$$

where $d > 0$ being a suitable constant.

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